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A generalization of the Liouville theorem to polyharmonic functions

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1 Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with a point $x = (x_1, x_2, \dots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$x^\lambda = {x_1}^{\lambda_1} {x_2}^{\lambda_2} \dots {x_n}^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

We denote by rB^n the open ball centered at the origin with radius $r > 0$, whose boundary is denoted by rS^{n-1}.

A real valued function u is called polyharmonic of order m on $\mathbf{R}^n$ if $u \in C^{2m}$ and $\Delta^m u = 0$, where m is a positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $H^m(\mathbf{R}^n)$ the space of polyharmonic functions of order m on $\mathbf{R}^n$. In particular, u is harmonic on $\mathbf{R}^n$ if $u \in H^1(\mathbf{R}^n)$.

The Liouville theorem for polyharmonic functions is known in several forms (cf. [1, 3, 4]).

THEOREM A. *Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree less than s if one of the following conditions holds:*

(i) $\displaystyle \lim_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ \, dS = 0$ (see [1]);

(ii) $\displaystyle \lim_{r\to\infty} \frac{1}{r^{s+n}} \int_{rB^n} u^+ \, dx = 0$ (see [3]);

(iii) $\displaystyle \limsup_{r\to\infty} \left(\max_{x\in rS^{n-1}} \frac{u(x)}{|x|^s} \right) \le 0$ (see [4]).

Now we propose the following theorem.

THEOREM. *Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree at most s if and only if*

$$\liminf_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ \, dS < \infty. \tag{1}$$

We here note that each condition of Theorem A implies (1), so that our theorem gives an improvement of Theorem A.

2 The main lemmas

Let us begin with the following lemma, which expands polyharmonic functions to harmonic functions (cf. [2], [5]).

LEMMA 1 (THE FINITE ALMANSI EXPANSION). *A real valued function u on $\mathbf{R}^n$ belongs to $H^m(\mathbf{R}^n)$, then there exists a unique family $\{h_i\}_{i=1}^m \subset H^1(R^n)$ such that*

$$u(x) = \sum_{i=1}^m |x|^{2(i-1)} h_i(x) \tag{2}$$

for every $x \in \mathbf{R}^n$.

PROOF. We prove this lemma by induction on m. For $m = 1$ the conclusion is trivial. Suppose the conclusion is true for $m = k$, and let $u \in H^{k+1}(\mathbf{R}^n)$. Then there exists a family $\{g_i\}_{i=1}^k \subset H^1(R^n)$ such that

$$\Delta u = \sum_{i=1}^k |x|^{2(i-1)} g_i(x). \tag{3}$$

If a family $\{h_i\}_{i=1}^{k+1} \subset H^1(R^n)$ satisfies (2), then we should have

$$\begin{aligned} \Delta u &= \sum_{i=1}^{k+1} \Delta\left(|x|^{2(i-1)} h_i(x)\right) \\ &= \sum_{i=1}^{k} \Delta\left(|x|^{2i} h_{i+1}(x)\right). \end{aligned}$$

If we write $r = |x|$, then

$$\begin{aligned} \Delta\left(|x|^{2i} h_{i+1}(x)\right) &= \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^2 \left(r^{2i} h_{i+1}(x)\right) \\ &= \sum_{j=1}^n \left\{ \frac{\partial^2 r^{2i}}{\partial x_j^2} h_{i+1}(x) + 2\frac{\partial r^{2i}}{\partial x_j}\frac{\partial h_{i+1}(x)}{\partial x_j} + r^{2i}\frac{\partial^2 h_{i+1}(x)}{\partial x_j^2} \right\} \end{aligned}$$

$$\begin{aligned} &= 2ir^{2(i-1)}\left\{(2i-2+n)h_{i+1}(x)+2r\frac{\partial h_{i+1}}{\partial r}(x)\right\} \\ &= |x|^{2(i-1)}\left\{2i(2i-2+n)h_{i+1}(x)+4ir\frac{\partial h_{i+1}}{\partial r}(x)\right\}. \end{aligned}$$

From the uniqueness of Almansi expansion for Δu, it is necessary and sufficient to find a unique solution h_{i+1} for the equation

$$g_i(x) = 2i(2i-2+n)h_{i+1}(x) + 4ir\frac{\partial h_{i+1}}{\partial r}(x) \tag{4}$$

for each $i = 1, \cdots, k$. We see that the unique solution for (4) is given by

$$h_{i+1}(x) = \frac{1}{4ir^{i-1+n/2}}\int_0^r t^{i-2+n/2} g_i(tx/r)\,dt.$$

Here we have only to show that h_{i+1} is harmonic on $\mathbf{R}^n$. Actually, putting $x = r\zeta$, where $r = |x|$ and $\zeta = x/|x| = x/r$, we have

$$\begin{aligned} h_{i+1}(x) &= h_{i+1}(r\zeta) \\ &= \frac{r^{1-i-n/2}}{4i}\int_0^r t^{i-2+n/2} g_i(t\zeta)\,dt \\ &= \frac{r^{1-i-n/2}}{4i}\int_0^1 (rs)^{i-2+n/2} g_i(rs\zeta)\,rds \\ &= \frac{1}{4i}\int_0^1 s^{i-2+n/2} g_i(xs)ds. \end{aligned}$$

Since g_i is harmonic, we see that $\Delta h_{i+1}(x) = 0$ for $i = 1, \cdots, k$. Now put

$$h_1(x) = u(x) - \sum_{i=2}^{k+1} |x|^{2(i-1)} h_i(x).$$

Then $\Delta h_1(x) = 0$ by (3), and the induction is completed.

Next we prepare the following lemma, which gives a relation between spherical means and derivatives for harmonic functions.

LEMMA 2. *Suppose $u \in H^1(\mathbf{R}^n)$. For each multi-index λ, there exists a positive constant $C = C(\lambda)$ such that*

$$\int_{rS^{n-1}} ux^\lambda\, dS = Cr^{2|\lambda|+n-1}D^\lambda u(0) + P_{2|\lambda|+n-3}(r) \tag{5}$$

for every $r > 0$, where $P_k(r)$ is a polynomial of degree at most k depends on u .

PROOF. We prove this lemma by induction on the length of λ. Assume first that $\lambda_n = 1$ and $\lambda_i = 0$ $(i = 1, \ldots, n-1)$. Using Green's formula and the mean-value property for harmonic functions, we have

$$\begin{aligned}\int_{rS^{n-1}} ux^\lambda \, dS &= \int_{rS^{n-1}} ux_n \, dS \\ &= r\int_{rS^{n-1}} u\,\frac{x_n}{r}\, dS \\ &= r\int_{rB^n} \frac{\partial u}{\partial x_n}\, dx \\ &= \sigma_n r^{n+1}\frac{\partial u}{\partial x_n}(0),\end{aligned}$$

where σ_n is the n-dimensional volume of the unit ball. Hence (5) holds for $|\lambda| = 1$.

Next suppose that (5) holds for $|\lambda| \leq k$, where k is a positive integer. Let $\mu = (\mu_1, \dots, \mu_n)$ such that $|\mu| = k+1$. We may assume without loss of generality that $\mu_n \geq 2$, and set $\mu' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 1)$. Then we write

$$\int_{rS^{n-1}} ux^\mu \, dS = r\int_{rS^{n-1}} ux^{\mu'}\frac{x_n}{r}\, dS.$$

From Green's formula we obtain

$$\begin{aligned}\int_{rS^{n-1}} ux^\mu \, dS &= r\int_{rB^n} \frac{\partial(ux^{\mu'})}{\partial x_n}\, dx \\ &= r\int_{rB^n} \left(x^{\mu'}\frac{\partial u}{\partial x_n} + (\mu_n - 1)ux_1^{\mu_1}\cdots x_n^{\mu_n - 2}\right) dx = (*).\end{aligned}$$

Set $\mu'' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 2)$. Since $|\mu'| = k$ and $|\mu''| = k-1$, we find

$$\begin{aligned}(*) &= r\int_0^r \left(\int_{tS^{n-1}} \left(x^{\mu'}\frac{\partial u}{\partial x_n} + (\mu_n - 1)ux^{\mu''}\right) dS\right) dt \\ &= r\int_0^r \left(\int_{tS^{n-1}} \frac{\partial u}{\partial x_n}x^{\mu'} dS\right) dt + (\mu_n - 1)r\int_0^r \left(\int_{tS^{n-1}} ux^{\mu''} dS\right) dt \\ &= r\int_0^r \left(C(\mu')t^{2|\mu'|+n-1}D^{\mu'}\left(\frac{\partial u}{\partial x_n}\right)(0) + P_{2|\mu'|+n-3}(t)\right) dt \\ &\quad + (\mu_n - 1)r\int_0^r \left(C(\mu'')t^{2|\mu''|+n-1}D^{\mu''}u(0) + P_{2|\mu''|+n-3}(t)\right) dt \\ &= C(\mu)r^{2k+n+1}D^\mu u(0) + P_{2k+n-1}(r),\end{aligned}$$

where $C(\mu) = \dfrac{C(\mu')}{2k+n} > 0$ and P_ℓ denotes various polynomials of degree at most ℓ which may change from one occurrence to the next; throughout this note, we use this convention. Hence (5) also holds for $|\mu| = k+1$. The induction is completed.

3 Proof of the theorem

First we show that our theorem is valid under the two sided condition on spherical means for polyharmonic functions.

LEMMA 3. *Let* $u \in H^m(\mathbf{R}^n)$ *and* $s > 2(m-1)$. *Then* u *is a polynomial of degree at most* s *if*

$$\liminf_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} |u|\, dS < \infty. \tag{6}$$

PROOF. By (6) we can find a sequence $\{r_j\}_{j=1}^{\infty}$ such that $r_j \to \infty$ and

$$\sup_j \left(r_j^{-s-n+1} \int_{r_j S^{n-1}} |u|\, dS \right) < \infty. \tag{7}$$

Using (2) and Lemma 2, we have

$$\begin{aligned} \int_{rS^{n-1}} u x^\lambda \, dS &= \int_{rS^{n-1}} \left(\sum_{i=1}^m |x|^{2(i-1)} h_i(x) \right) x^\lambda \, dS \\ &= \sum_{i=1}^m r^{2(i-1)} \int_{rS^{n-1}} h_i(x) x^\lambda \, dS \\ &= \sum_{i=1}^m r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right), \end{aligned}$$

where $C_i = C_i(\lambda)$ is a positive constant and $P_{i,k}$ denotes various polynomials of degree at most k depends on h_i. Hence it follows that

$$r^{|\lambda|} \int_{rS^{n-1}} |u|\, dS \geq \left| \sum_{i=1}^m r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right) \right|,$$

so that we obtain

$$r_j^{-s-n+1} \int_{r_j S^{n-1}} |u|\, dS \geq r_j^{|\lambda|-s+2(m-1)} \left| C_m D^\lambda h_m(0) + O(r_j^{-2}) \right|$$

as $r_j \to \infty$. By (7), we find

$$D^\lambda h_m(0) = 0$$

for all $|\lambda| > s - 2(m-1)$. By analyticity of harmonic functions, we see that h_m is a polynomial of degree at most $s - 2(m-1)$. Hence we note that

$$r^{2(m-1)} \int_{rS^{n-1}} h_m(x) x^\lambda dS = O(r^{s+|\lambda|+n-1}) \qquad \text{as } r \to \infty.$$

Consequently,

$$r_j^{-s-n+1} \int_{r_j S^{n-1}} |u|\, dS \geq r_j^{|\lambda|-s+2(m-2)} \left| C_{m-1} D^\lambda h_{m-1}(0) + O(r_j^{-2}) \right| + O(1)$$

as $r_j \to \infty$. This implies that $D^\lambda h_{m-1}(0) = 0$ for $|\lambda| > s - 2(m-2)$, so that h_{m-1} is a polynomial of degree at most $s - 2(m-2)$. By repeating this arguments, we see that each h_i is a polynomial of degree at most $s - 2(i-1)$ $(i = 1, \ldots, m)$. Thus it follows that u is a polynomial. In view of (2), the degree of u is at most $2(i-1) + s - 2(i-1) = s$.

PROOF OF THE THEOREM. If $u \in H^m(\mathbf{R}^n)$, then we see from (2) that

$$\frac{1}{\omega_n r^{n-1}} \int_{rS^{n-1}} u \, dS = \sum_{i=1}^{m} r^{2(i-1)} h_i(0),$$

where ω_n denotes the surface measure of S^{n-1}.

Since $|u| = 2u^+ - u$, we have

$$\begin{aligned} &\liminf_{r\to\infty} r^{-s-n+1} \int_{rS^{n-1}} |u| \, dS \\ &= \liminf_{r\to\infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s-n+1} \int_{rS^{n-1}} u \, dS \right) \\ &= \liminf_{r\to\infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s} P_{2(m-1)}(r) \right). \end{aligned}$$

Hence (1) implies (6) since $s > 2(m-1)$, so that the present theorem follows from Lemma 3.

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