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Hyperplane integrals of BLD and monotone BLD functions

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1 Introduction

Let $\mathbf{D}$ denote the half space

$$\mathbf{D} = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}^1 : x_n > 0\}$$

and set

$$\mathbf{H} = \partial \mathbf{D};$$

we sometimes identify $x' \in \mathbf{R}^{n-1}$ with $(x', 0) \in \mathbf{H}$. We define the qth hyperplane integral $H_q(u)$ over $\mathbf{H}$ by

$$H_q(u) = \left(\int_{\mathbf{H}} |u(x')|^q dx' \right)^{1/q}$$

for a measurable function u on $\mathbf{H}$.

Our main aim in this note is to study the existence of limits of $H_q(U_r)$ at $r = 0$, where

$$U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} [(\partial/\partial x_n)^k u](x', 0)$$

for quasicontinuous Sobolev functions u on $\mathbf{D}$, where the vertical limits

$$(\partial/\partial x_n)^k u(x', 0) = \lim_{x_n \to 0} (\partial/\partial x_n)^k u(x', x_n)$$

exist for almost all $x' = (x', 0) \in \mathbf{H}$ and $0 \leqq k \leqq m-1$ (see [15, Theorem 2.4]). More precisely, we show (in Theorem 3.1 below) that

$$\lim_{r \to 0} r^{-\omega} H_q(U_r) = 0$$

for some $\omega > 0$.

Consider the Dirichlet problem for polyharmonic operator

$$\Delta^m u(x) = 0$$

with the boundary conditions

$$(\partial/\partial x_n)^k u(x', 0) = f_k(x') \qquad (k = 0, 1, ..., m-1).$$

We show (in Corollary 3.1 below) that if $1 < p \leqq q < \infty$, $n/p - (n-1)/q < 1$ and $u \in W^{m,p}(\mathbf{D})$ is a solution of the Dirichlet problem with $f_k(x') = (\partial/\partial x_n)^k u(x',0)$ for $0 \leqq k \leqq m-1$, then

$$\lim_{r\to 0} r^{n/p-(n-1)/q-m} H_q(U_r) = 0,$$

where $U_r(x') = u(x',r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} f_k(x')$.

To prove our results, we apply the integral representation in [12, 15]. For this purpose, we are concerned with K-potentials $U_K f$ defined by

$$U_K f(x) = \int K(x-y) f(y) dy$$

for functions f on $\mathbf{R}^n$ satisfying the weighted L^p condition :

$$\int_{\mathbf{R}^n} |f(y)|^p |y_n|^\beta \, dy < \infty, \qquad -1 < \beta < p-1.$$

In connection with our integral representation, $K(x)$ is of the form $x^\lambda |x|^{-n}$ for a multi-index λ with length m. Our basic fact is stated as follows (see Theorem 2.1 below):

$$\lim_{r\to 0} r^{n/p-(n-1)/q-m} H_q(u_r) = 0,$$

where $u_r(x') = U_K f(x',r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} [(\partial/\partial x_n)^k U_K f](x')$.

In Section 4, we give growth estimates of higher differences of Sobolev functions.

In the final section, we study the existence of limits of hyperplane integrals for monotone BLD functions u on $\mathbf{D}$ satisfying

$$\int_{\mathbf{D}} |\nabla u(x)|^p x_n^\beta \, dx < \infty, \qquad p > n-1, \tag{1.1}$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \beta < p-1$; see Section 5 for the definition of monotone functions. We here note that harmonic functions are monotone, $\mathcal{A}$-harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [4] and [20]), and thus the class of monotone functions is considerably wide.

For related results, see Gardiner [2], Stoll [24, 25, 26], the first author [11, 12, 16] and the authors [17, 18].

2 Hyperplane integrals of potentials

For a multi-index λ and $\ell > 0$, set

$$K(x) = \frac{x^\lambda}{|x|^\ell}.$$

We define the K-potential $U_K f$ by

$$U_K f(x) = \int_{\mathbf{R}^n} K(x-y) f(y) dy$$

for a measurable function f on $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{R}^n} (1+|y|)^{|\lambda|-\ell} |f(y)| dy < \infty \tag{2.1}$$

and

$$\int_{\mathbf{R}^n} |f(y)|^p |y_n|^\beta dy < \infty, \quad y = (y_1, \cdots, y_n). \tag{2.2}$$

In particular, K is the Riesz α-kernel when $\lambda = 0$ and $\ell = n - \alpha$. In this case, $U_K f$ is written as $U_\alpha f$ with $\alpha = |\lambda| - \ell + n$. Note here that (2.1) is equivalent to the condition that

$$U_\alpha |f| \not\equiv \infty. \tag{2.3}$$

Throughout this paper, let M denote various constants independent of the variables in question.

For a nonnegative integer m, consider

$$K_m(x,y) = K(x-y) - \sum_{k=0}^{m} \frac{{x_n}^k}{k!} [(\partial/\partial x_n)^k K](x'-y),$$

where $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$; we sometimes identify x' with $(x', 0)$.

LEMMA 2.1 ([19, Lemma 2.1]). *Let m be a nonnegative integer such that $|\lambda| - \ell < m+1$.*

(1) If $|x'-y| \geqq x_n/2 > 0$ and $|x-y| \geqq x_n/2 > 0$, then

$$|K_m(x,y)| \leqq M x_n^{m+1} |x'-y|^{|\lambda|-\ell-m-1}.$$

(2) If $|x-y| < x_n/2$, then $|K_m(x,y)| \leqq M(x_n^{|\lambda|-\ell} + |x-y|^{|\lambda|-\ell})$.

(3) If $|x'-y| < x_n/2$, then $|K_m(x,y)| \leqq M(x_n^{|\lambda|-\ell} + x_n^m |x'-y|^{|\lambda|-\ell-m})$.

For a point $x \in \mathbf{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball with center at x and radius r.

LEMMA 2.2 (cf. [16, Lemma 3.2]). *Let $\beta > -1$, $q > 0$ and $|\lambda| - \ell + n/q > 0$. Let m be a nonnegative integer such that*

$$m < |\lambda| - \ell + \frac{n+\beta}{q} < m+1.$$

Then

$$\left(\int |K_m(x,y)|^q |y_n|^\beta dy\right)^{1/q} \leqq M x_n^{|\lambda|-\ell+(n+\beta)/q}$$

for all $x = (x', x_n) \in \mathbf{D}$.

PROOF. For fixed $x \in \mathbf{D}$, consider the sets

$$E_1 = B(x, x_n/2), \qquad E_2 = B(x', x_n/2), \qquad E_3 = \mathbf{D} - (E_1 \cup E_2).$$

Since $|\lambda| - \ell + (n+\beta)/q - m - 1 < 0$, applying the polar coordinates about x', we have by Lemma 2.1(1)

$$\begin{aligned}
\left(\int_{E_3} |K_m(x,y)|^q |y_n|^\beta dy\right)^{1/q} &\leqq M x_n^{m+1} \left(\int_{E_3} |x'-y|^{(|\lambda|-\ell-m-1)q} |y_n|^\beta dy\right)^{1/q} \\
&\leqq M x_n^{m+1} \left(\int_{x_n/2}^{\infty} r^{(|\lambda|-\ell-m-1)q+\beta} r^{n-1} dr\right)^{1/q} \\
&= M x_n^{|\lambda|-\ell+(n+\beta)/q}.
\end{aligned}$$

Similarly, since $|\lambda| - \ell + n/q > 0$, we have by Lemma 2.1(2)

$$\begin{aligned}
\left(\int_{E_1} |K_m(x,y)|^q |y_n|^\beta dy\right)^{1/q} &\leqq M x_n^{\beta/q} \left(\int_{E_1} (x_n^{|\lambda|-\ell} + |x-y|^{|\lambda|-\ell})^q dy\right)^{1/q} \\
&= M x_n^{|\lambda|-\ell+(n+\beta)/q}.
\end{aligned}$$

Finally, since $|\lambda| - \ell + (n+\beta)/q - m > 0$, we obtain by Lemma 2.1(3)

$$\begin{aligned}
\left(\int_{E_2} |K_m(x,y)|^q |y_n|^\beta dy\right)^{1/q} &\leqq M \left(\int_{E_2} (x_n^{|\lambda|-\ell} + x_n^m |x'-y|^{|\lambda|-\ell-m})^q |y_n|^\beta dy\right)^{1/q} \\
&\leqq M x_n^{|\lambda|-\ell+(n+\beta)/q} + M x_n^m \left(\int_0^{x_n/2} r^{(|\lambda|-\ell-m)q+\beta} r^{n-1} dr\right)^{1/q} \\
&= M x_n^{|\lambda|-\ell+(n+\beta)/q}.
\end{aligned}$$

The required inequality now follows.

LEMMA 2.3 (cf. [16, Lemma 3.4]). *Let* $q > 0$ *and* m *be a nonnegative integer such that*

$$m < |\lambda| - \ell + \frac{n-1}{q} < m+1.$$

If $x = (x', x_n) \in D$ *and* $y = (y', y_n) \in R^n$, *then*

$$\left(\int_{\mathbf{R}^{n-1}} |K_m(x,y)|^q dx'\right)^{1/q} \leqq M x_n^{m+1} (x_n + |y_n|)^{|\lambda|-\ell-m-1+(n-1)/q}.$$

PROOF. Let $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$. If $|y_n| \geqq 2x_n$, then, since $|\lambda| - \ell - m - 1 + (n-1)/q < 0$, we have by Lemma 2.1(1)

$$\begin{aligned}\left(\int_{\mathbf{R}^{n-1}} |K_m(x,y)|^q dx'\right)^{1/q} &\leqq Mx_n^{m+1}\left(\int_{\mathbf{R}^{n-1}} |x'-y|^{(|\lambda|-\ell-m-1)q} dx'\right)^{1/q} \\ &= Mx_n^{m+1}\left(\int_0^\infty (r^2+y_n{}^2)^{(|\lambda|-\ell-m-1)q/2} r^{n-2} dr\right)^{1/q} \\ &= Mx_n^{m+1}|y_n|^{|\lambda|-\ell-m-1+(n-1)/q}.\end{aligned}$$

If $|y_n| < 2x_n$, then we have as in the proof of Lemma 2.1

$$\begin{aligned}&\left(\int_{\mathbf{R}^{n-1}} |K_m(x,y)|^q dx'\right)^{1/q} \leqq M\left(\int_{\{x':(x',x_n)\in E_1\}} (x_n^{|\lambda|-\ell} + |x-y|^{|\lambda|-\ell})^q dx'\right)^{1/q} \\ &+M\left(\int_{\{x':(x',x_n)\in E_2\}} (x_n^{|\lambda|-\ell} + x_n^m|x'-y|^{|\lambda|-\ell-m})^q dx'\right)^{1/q} \\ &+Mx_n^{m+1}\left(\int_{\{x':(x',x_n)\in E_3\}} |x'-y|^{(|\lambda|-\ell-m-1)q} dx'\right)^{1/q} \\ \leqq\ & Mx_n^{|\lambda|-\ell+(n-1)/q} + M\left(\int_{B(y',x_n/2)} |x'-y'|^{(|\lambda|-\ell)q} dx'\right)^{1/q} \\ &+Mx_n^m\left(\int_{B(y',x_n/2)} |x'-y'|^{(|\lambda|-\ell-m)q} dx'\right)^{1/q} \\ &+Mx_n^{m+1}\left(\int_{\mathbf{R}^{n-1}} (x_n+|x'-y'|)^{(|\lambda|-\ell-m-1)q} dx'\right)^{1/q} \\ =\ & Mx_n^{|\lambda|-\ell+(n-1)/q}.\end{aligned}$$

Therefore the required inequality now follows.

LEMMA 2.4 (cf. [1, Theorem 13.5], [15, Section 6.5]). *Let $\alpha = |\lambda| - \ell + n$, $\alpha p > 1$, $\alpha p > 1 + \beta$ and $-1 < \beta < p - 1$. If f is a measurable function on $\mathbf{R}^n$ satisfying (2.2) and (2.3), then $U_K f$ has the (ACL) property; in particular, $U_K f(x', x_n)$ is absolutely continuous on $\mathbf{R}$ for almost every $x' \in \mathbf{R}^{n-1}$. Moreover, in case m is a positive integer such that $(\alpha - m)p > 1$ and $(\alpha - m)p > 1 + \beta$,*

$$(\partial/\partial x_n)^m U_K f(x', x_n) = \int (\partial/\partial x_n)^m K(x-y) f(y)\, dy$$

is absolutely continuous on $\mathbf{R}$ for almost every $x' \in \mathbf{R}^{n-1}$.

THEOREM 2.1 (cf. [11, Theorem 2.1] and [16, Theorem 2.1]). *Let $\alpha = |\lambda| - \ell + n$ satisfy $m + 1/p < \alpha < m + n$. Let $1 < p \leqq q < \infty$, $\beta < p - 1$ and*

$$\frac{n-\alpha p}{p(n-\alpha)} < \frac{n-1}{q(n-\alpha+m)} \qquad \text{when } n - \alpha > 0.$$

Further suppose $m < \omega < m+1$, *where* $\omega = (n-1)/q - (n - \alpha p + \beta)/p$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying (2.2) and (2.3), then*

$$\lim_{r \to 0} r^{-\omega} H_q(u_r) = 0,$$

where $u_r(x') = U_K f(x', r) - \sum_{k=0}^{m} \frac{r^k}{k!}[(\partial/\partial x_n)^k U_K f](x', 0)$.

PROOF. Under the assumptions on p, α, β, q and m in Theorem 2.1, we can take (δ, γ) such that

$$\beta < \gamma < p(n - \alpha + m + 1)\delta + \beta - p(n-1)/q, \tag{2.4}$$

$$p(n - \alpha + m + 1)\delta + (\alpha - m - 1)p - n < \gamma < p(n - \alpha + m)\delta + (\alpha - m)p - n, \tag{2.5}$$

$$\beta < \gamma < p - 1, \qquad 0 < \delta < 1, \tag{2.6}$$

$$\delta p(n - \alpha) > n - \alpha p \tag{2.7}$$

and

$$\frac{n-1}{q(n - \alpha + m + 1)} < \delta < \frac{n-1}{q(n - \alpha + m)}. \tag{2.8}$$

Set $a = (1-\delta)p'$ and $b = -\gamma p'/p$. Then, by (2.6) and (2.7),

$$b > -1 \qquad \text{and} \qquad \alpha - n + \frac{n}{a} > 0.$$

Further, (2.5) implies

$$m < \alpha - n + \frac{n+b}{a} < m + 1.$$

We first note from Lemma 2.4 that

$$\begin{aligned} u_{x_n}(x') &= U_K f(x) - \sum_{k=0}^{m} \frac{x_n^k}{k!}[(\partial/\partial x_n)^k U_K f](x', 0) \\ &= \int K_m(x, y) f(y)\, dy. \end{aligned}$$

Using Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} |u_{x_n}(x')| &\leqq \left(\int |K_m(x,y)|^a |y_n|^b dy \right)^{(1-\delta)/a} \left(\int |K_m(x,y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p} \\ &\leqq M x_n^{(\alpha - n)(1-\delta) + n/p' - \gamma/p} \left(\int |K_m(x,y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p}. \end{aligned}$$

In view of Minkowski's inequality for integral we have

$$H_q(u_{x_n}) \leqq Mx_n^{(\alpha-n)(1-\delta)+n/p'-\gamma/p} \times \left\{ \int \left(\int_{R^{n-1}} |K_m(x,y)|^{\delta q} dx' \right)^{p/q} f(y)^p |y_n|^\gamma dy \right\}^{1/p}.$$

Here, noting (2.8), we have by Lemma 2.3

$$\left(\int_{\mathbf{R}^{n-1}} |K_m(x,y)|^{\delta q} dx' \right)^{p/q} \leqq M[x_n^{m+1}(x_n + |y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p}.$$

Consequently

$$H_q(u_r) \leqq Mr^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta} \times \left\{ \int [(r+|y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p} |y_n|^{\gamma-\beta} f(y)^p |y_n|^\beta dy \right\}^{1/p}.$$

Consider the function

$$k(r,y_n) = r^{p[(n-\alpha p+\beta)/p-(n-1)/q]} r^{p[(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta]} \times [(r+|y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p} |y_n|^{\gamma-\beta}.$$

Then

$$r^{-\omega} H_q(u_r) \leqq M \left\{ \int k(r,y_n) f(y)^p |y_n|^\beta dy \right\}^{1/p},$$

where $\omega = (n-1)/q - (n-\alpha p+\beta)/p$. It follows from (2.4) that

$$r^{-\omega} r^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta} = r^{(n-\alpha+m+1)\delta+(\beta-\gamma)/p-(n-1)/q} \longrightarrow 0$$

as $r \to 0$. If $r < |y_n|$, then

$$k(r,y_n) \leqq M(r/|y_n|)^{(n-\alpha+m+1)\delta p+(\beta-\gamma)-p(n-1)/q} \leqq M;$$

if $|y_n| \leqq r$, then

$$k(r,y_n) \leqq M(|y_n|/r)^{\gamma-\beta} \leqq M.$$

Hence Lebesgue's dominated convergence theorem implies that

$$\lim_{r\to 0} r^{-\omega} H_q(u_r) = 0.$$

Now the proof of Theorem 2.1 is completed.

3 Sobolev functions

For an open set $G \subset \mathbf{R}^n$, we denote by $BL_m(L^p_{loc}(G))$ the Beppo Levi space

$$BL_m(L^p_{loc}(G)) = \{u \in L^p_{loc}(G) : D^\lambda u \in L^p_{loc}(G) \ \ (|\lambda| = m)\}$$

(see [15]). In view of [15], each $u \in BL_m(L^p_{loc}(\mathbf{D}))$ satisfying

$$\text{(3.1)} \qquad \int_{\mathbf{D}} |\nabla_m u(x)|^p x_n^\beta \, dx < \infty$$

has an (m,p)-quasicontinuous representative $\tilde{u}$, where $|\nabla_m u(x)| = (\sum_{|\mu|=m} |D^\mu u(x)|^2)^{1/2}$, $1 < p < \infty$ and $-1 < \beta < p-1$. Moreover, $\tilde{u}$ is given by

$$\tilde{u}(x) = \sum_{|\lambda|=m} a_\lambda \int \tilde{K}_{\lambda,m}(x,y) D^\lambda \overline{u}(y) \, dy + P(x),$$

where $\overline{u}$ is an extention of u to $\mathbf{R}^n$, $P(x)$ is a polynomial of degree at most $m-1$, $K_\lambda(x) = x^\lambda |x|^{-n}$ and

$$\tilde{K}_{\lambda,m}(x,y) = \begin{cases} K_\lambda(x-y), & y \in B(0,1), \\ K_\lambda(x-y) - \sum_{|\mu| \leqq m-1} \frac{x^\mu}{\mu!}[(\partial/\partial x)^\mu K_\lambda](-y), & y \in \mathbf{R}^n - B(0,1). \end{cases}$$

Note further from Lemma 2.4 that for each k with $0 \leqq k \leqq m-1$ and for almost every $x' \in \mathbf{R}^{n-1}$,

$$(\partial/\partial x_n)^k \int \tilde{K}_{\lambda,m}(x,y) D^\lambda \overline{u}(y) \, dy = \int (\partial/\partial x_n)^k \tilde{K}_{\lambda,m}(x,y) D^\lambda \overline{u}(y) \, dy$$

holds for $x_n \in \mathbf{R}$, where $x = (x', x_n)$.

Since $Q(x) - \sum_{k=0}^{m-1} \frac{r^k}{k!}[(\partial/\partial x_n)^k Q](x') = 0$ for any polynomial Q of degree at most $m-1$, we have

$$\begin{aligned} U(x) &\equiv \tilde{u}(x) - \sum_{k=0}^{m-1} \frac{{x_n}^k}{k!} (\partial/\partial x_n)^k \tilde{u}(x') \\ &= \sum_{|\lambda|=m} a_\lambda \int K_{\lambda,m}(x,y) D^\lambda \overline{u}(y) \, dy = \tilde{u}(x) - P(x) \end{aligned}$$

for q.e. $x \in \mathbf{D}$, where $K_{\lambda,m}(x,y) = K_\lambda(x-y) - \sum_{k=0}^{m-1} \frac{{x_n}^k}{k!}[(\partial/\partial x_n)^k K_\lambda](x'-y)$.

Theorem 2.1 now gives the following result.

THEOREM 3.1 (cf. [19, Theorem 3.1]). *Let* $1 < p \leqq q < \infty$,

$$\frac{n-mp}{p(n-m)} < \frac{1}{q} \qquad \textit{when } n-m > 0$$

and

$$\frac{n-p+\beta}{p(n-1)} < \frac{1}{q} < \frac{n+\beta}{p(n-1)}.$$

If $u \in BL_m(L^p_{loc}(\mathbf{D}))$ satisfying (3.1) for $-1 < \beta < p-1$ is (m,p)-quasicontinuous on $\mathbf{D}$, then

$$\lim_{r\to 0} r^{(n-mp+\beta)/p-(n-1)/q} H_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!}[(\partial/\partial x_n)^k u](x', 0)$.

Consider the Dirichlet problem for polyharmonic operator :

$$\Delta^m u(x) = 0$$

with the boundary conditions

$$(\partial/\partial x_n)^k u(x', 0) = f_k(x') \qquad (k = 0, 1, ..., m-1).$$

We denote by $W^{m,p}(G)$ the Sobolev space

$$W^{m,p}(G) = \{u \in L^p(G) : D^\lambda u \in L^p(G) \ (|\lambda| \leqq m)\}$$

(see Stein [23, Chapter 6]). If $u \in W^{m,p}(\mathbf{D})$ is a solution of the Dirichlet problem, then the vertical limit $(\partial/\partial x_n)^k u(x', 0)$ exists for almost every $x' = (x', 0) \in \partial\mathbf{D}$ and $0 \leqq k \leqq m-1$ (see [12], [14]).

We also see that every function in $W^{m,p}(\mathbf{D})$ can be extended to a function in $W^{m,p}(\mathbf{R}^n)$ (see Stein [23]). Hence Theorem 3.1 gives the following result.

COROLLARY 3.1 . *Let $1 < p \leqq q < \infty$ and*

$$(\,0 <\,)\ \frac{n}{p} - \frac{n-1}{q} < 1.$$

If $u \in W^{m,p}(\mathbf{D})$ is a solution of the Dirichlet problem with $f_k(x') = (\partial/\partial x_n)^k u(x', 0)$ for $0 \leqq k \leqq m-1$, then

$$\lim_{r\to 0} r^{n/p-(n-1)/q-m} H_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} f_k(x')$.

4 Higher differences

For $r > 0$ and a function u, we define the first difference

$$\Delta_r u(t) = \Delta_r^1 u(t) = u(t+r) - u(t)$$

and the m-th difference

$$\Delta_r^m u(t) = \Delta_r^{m-1} (\Delta_r u(\cdot))(t).$$

It is easy to see that

$$\Delta_r^m u(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} u(t+kr).$$

As in Section 2, we consider

$$K(x) = \frac{x^\lambda}{|x|^\ell}$$

and define

$$u_r(x') = \Delta_r^m U_K f(x', \cdot)(0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} U_K f(x', kr).$$

THEOREM 4.1. *Let* $\alpha = |\lambda| - \ell + n$, $1 < p \leqq q < \infty$, $\beta < p - 1$ *and*

$$\frac{n - \alpha p}{p} < \frac{n-1}{q} \qquad (\text{when } n - \alpha > 0).$$

Further suppose $0 < \omega < m$, *where* $\omega = (n-1)/q - (n - \alpha p + \beta)/p$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying (2.2) and (2.3), then*

$$\lim_{r \to 0} r^{-\omega} H_q(u_r) = 0,$$

where $u_r(x') = \Delta_r^m U_K f(x', \cdot)(0)$.

To prove this, we have only to prepare the following two lemmas instead of Lemmas 2.2 and 2.3.

LEMMA 4.1 ([19, Lemma 4.1]). *Let* $\beta > -1$, $q > 0$ *and* $|\lambda| - \ell + n/q > 0$. *Let* m *be a positive integer such that*

$$0 < |\lambda| - \ell + \frac{n+\beta}{q} < m.$$

Then

$$\left(\int |K_m^*(x,y)|^q |y_n|^\beta dy \right)^{1/q} \leqq M x_n^{|\lambda| - \ell + (n+\beta)/q}$$

for all $x = (x', x_n) \in \mathbf{D}$, *where* $K_m^*(x,y) = \Delta_{x_n}^m K(x' - y', \cdot - y_n)(0)$ *for* $x = (x', x_n) \in \mathbf{D}$ *and* $y = (y', y_n) \in \mathbf{R}^n$.

Lemma 4.2 ([19, Lemma 4.2]). *Let $q > 0$ and m be a positive integer such that*

$$0 < |\lambda| - \ell + \frac{n-1}{q} < m.$$

If $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$, then

$$\left(\int_{\mathbf{R}^{n-1}} |K_m^*(x,y)|^q dx' \right)^{1/q} \leqq M x_n^m (x_n + |y_n|)^{|\lambda| - \ell - m + (n-1)/q}.$$

Theorem 4.2 . *Let $1 < p \leqq q < \infty$ and*

$$(\, 0 < \,)\ \frac{n}{p} - \frac{n-1}{q} < m.$$

If $u \in BL_m(L^p(\mathbf{R}^n))$ is (m,p)-quasicontinuous on $\mathbf{R}^n$, then

$$\lim_{r \to 0} r^{n/p-(n-1)/q-m} H_q(U_r) = 0,$$

where $U_r(x') = \Delta_r^m u(x', \cdot)(0)$ for $r > 0$.

In fact, since $\Delta_r^m Q = 0$ for any polynomial Q of degree at most $m-1$, we have

$$U(x) \ \equiv \ \Delta_{x_n}^m u(x', \cdot)(0) = \sum_{|\lambda|=m} a_\lambda \int K_{\lambda,m}^*(x,y) D^\lambda u(y) dy,$$

where $K_{\lambda,m}^*(x,y) = \Delta_{x_n}^m K_\lambda(x'-y', \cdot - y_n)(0)$ with $K_\lambda(x) = x^\lambda |x|^{-n}$.

5 Monotone functions

We say that a continuous function u is monotone in an open set G, in the sense of Lebesgue ([6]), if both

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x) \qquad \text{and} \qquad \min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set D with the closure $\overline{D} \subset G$.

The class of monotone functions is considerably wide. We give some examples of monotone functions.

Example 1. Let $f(r)$ be a non-increasing (or non-decreasing) continuous function on $(0, \infty)$. If we define

$$u(x) = f(|x - \xi|)$$

for $x \in G$ and $\xi \in \partial G$, then u is monotone in G.

EXAMPLE 2. Harmonic functions on an open set G are monotone in G. More generally, solutions of elliptic partial differential equations of second order may be monotone (see Gilbarg-Trudinger [3]).

EXAMPLE 3. Weak solutions for variational problems may be monotone; in particular, weak solutions of the p-Laplacian are monotone. Moreover, if f is a quasi-regular mapping on G, then the coordinate functions of f are monotone in G. For these facts, see Heinonen-Kilpeläinen-Martio [4], Reshetnyak [20], Serrin [21] and Vuorinen [27], [28].

A key result for monotone BLD functions is the following fact.

LEMMA 5.1 (cf. [5, Lemma 7.1], [7, Remark, p.9], [28, Sect. 16]). *Let $p > n-1$. If u is a monotone BLD function on $B(x_0, 2r)$, then*

$$|u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \qquad \text{whenever } x,\, y \in B(x_0, r).$$

This lemma is shown by the application of Sobolev's inequality on the spherical surfaces, so that the restriction $p > n-1$ is needed; for a proof of Lemma 5.1, see for example [5, Lemma 7.1] or [15, Theorem 5.2, Chap.8].

6 Hyperplane integrals of monotone functions

We define the qth integral $H_{q,N}(u)$ over $\{x' : |x'| < N\}$ by

$$H_{q,N}(u) = \left(\int_{\{x':|x'|<N\}} |u(x')|^q dx' \right)^{1/q}$$

for a measurable function u on $\{x' : |x'| < N\}$.

THEOREM 6.1 (cf. [17, Theorem 2]). *Let u be a monotone function on* **D** *satisfying (1.1). If $n-1 < p < n+\beta$, $p \leqq q < \infty$ and*

$$\frac{1}{q} < \frac{n-p+\beta}{p(n-1)},$$

then

$$\lim_{r\to 0} r^{(n-p+\beta)/p-(n-1)/q} H_{q,N}(u_r) = 0,$$

where $u_r(x') = u(x', r)$ for $r > 0$.

PROOF. Let u be a monotone function on **D** satisfying (1.1) with $n-1 < p < n+\beta$. If $|s-t| \leqq r < t/2$, then Lemma 5.1 yields

$$\begin{aligned} |H_{q,N}(u_s) - H_{q,N}(u_t)| &\leqq \left(\int_{\{x':|x'|<N\}} |u_s(x') - u_t(x')|^q \, dx' \right)^{1/q} \\ &\leqq M r^{(p-n)/p} \left(\int_{\{x':|x'|<N\}} \left(\int_{B((x',t),2r)} |\nabla u(z)|^p dz \right)^{q/p} dx' \right)^{1/q}, \end{aligned}$$

so that Minkowski's inequality for integral yields

$$|H_{q,N}(u_s) - H_{q,N}(u_t)| \leqq Mr^{(p-n)/p}(2r)^{(n-1)/q} \left(\int_{\{z=(z',z_n):t-2r<z_n<t+2r\}} |\nabla u(z)|^p \, dz \right)^{1/p}.$$

Let $r_j = 2^{-j-1}$, $t_j = r_{j-1}$ and $A_j = \{z = (z', z_n) : r_j < z_n < 3r_j\}$ for $j = 1, 2, \ldots$. Set

$$\omega = (n - p + \beta)/p - (n-1)/q > 0.$$

Then we find

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| \leqq Mr_{j+1}^{-\omega} \left(\int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for $t_j - r_{j+1} < r \leqq t_j$,

$$|H_{q,N}(u_{t_j - r_{j+1}}) - H_{q,N}(u_r)| \leqq Mr_{j+2}^{-\omega} \left(\int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for $t_j - r_{j+1} - r_{j+2} < r \leqq t_j - r_{j+1}$ and

$$|H_{q,N}(u_r) - H_{q,N}(u_{t_{j+1}})| \leqq Mr_{j+2}^{-\omega} \left(\int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for $t_{j+1} < r \leqq t_j - r_{j+1} - r_{j+2}$. Collecting these results, we have

$$\begin{aligned} |H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| &\leqq Mr_j^{-\omega} \left(\int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p} \\ &\quad + Mr_{j+1}^{-\omega} \left(\int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p} \end{aligned}$$

for $t_{j+1} < r \leqq t_j$. Hence it follows that

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_{t_{j+m}})| \leqq M \sum_{\ell=j}^{j+m} r_\ell^{-\omega} \left(\int_{A_\ell} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}.$$

Since $A_\ell \cap A_k = \emptyset$ when $\ell \geqq k+2$, Hölder's inequality gives

$$\begin{aligned} |H_{q,N}(u_{t_j}) - H_{q,N}(u_{t_{j+m}})| &\leqq M \left(\sum_{\ell=j}^{j+m} r_\ell^{-p'\omega} \right)^{1/p'} \left(\sum_{\ell=j}^{j+m} \int_{A_\ell} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p} \\ &\leqq Mr_{j+m}^{-\omega} \left(\int_{\{z=(z',z_n):r_{j+m}<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}. \end{aligned}$$

More generally, if $0 < r \leqq t_j$, then we take m such that $t_{j+m} < r \leqq t_{j+m-1}$, and establish

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| \leqq Mr^{-\omega} \left(\int_{\{z=(z',z_n):0<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p},$$

which implies that

$$\limsup_{r \to 0} \; r^\omega H_{q,N}(u_r) \leqq M \left(\int_{\{z=(z',z_n):0<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for all j. Therefore it follows that

$$\lim_{r \to 0} r^\omega H_{q,N}(u_r) = 0,$$

as required.

In case $1/q = (n-p+\beta)/p(n-1) > 0$, one might expect that $H_{q,N}(u_r)$ is bounded. In fact, we can show that this is true only in case $0 \leqq \beta < p-1$ without assuming the monotonicity. We refer the reader to the result by Yamashita [29] who showed affirmatively the case $p = 2$ and $\beta = 1$ for harmonic functions. In the hyperplane case, we refer to [16, Theorem 2.2]. The case $\beta = p-1$ remains open.

For Sobolev functions, we have a weak limit result as follows.

THEOREM 6.2 (cf. [16, Theorem 2.1]). *Let $-1 < \beta < p-1$, $1 < p \leqq q < \infty$ and*

$$\frac{n-p-1}{p(n-1)} < \frac{1}{q} < \frac{n-p+\beta}{p(n-1)}.$$

If u is a $(1,p)$-quasicontinuous function on $\mathbf{D}$ satisfying (1.1), then there exists a number A such that

$$\liminf_{r \to 0} \; r^{(n-p+\beta)/p-(n-1)/q} H_q(u_r - A) = 0,$$

where $u_r(x') = u(x',r)$ for $r > 0$.

By this together with Theorem 6.1, we can prove the follwing result.

COROLLARY 6.1. *Let u be a monotone function on $\mathbf{D}$ satisfying (1.1). If $n-1 < p < n+\beta$, $-1 < \beta < p-1$, $p \leqq q < \infty$ and*

$$\frac{1}{q} < \frac{n-p+\beta}{p(n-1)},$$

then there exists a number A such that

$$\lim_{r \to 0} r^{(n-p+\beta)/p-(n-1)/q} H_q(u_r - A) = 0,$$

where $u_r(x') = u(x',r)$ for $r > 0$.

THEOREM 6.3 (cf. [18, Theorem 1]). *Let u be a monotone function on* $\mathbf{D}$ *satisfying (1.1) with $n-1<p\leqq n+\beta$. If $p\leqq q<\infty$ and*

$$\frac{1}{q}>\frac{n-p+\beta}{p(n-1)},$$

then

$$\lim_{r\to 0} r^{(n-p+\beta)/p-(n-1)/q}H_q(U_r)=0,$$

where $U_r(x')=u(x',r)-u(x',0)$ for $r>0$.

PROOF. Let u be a monotone function on $\mathbf{D}$ satisfying (1.1) with $n-1<p\leqq n+\beta$. If $|s-t|\leqq r<t/2$, then Lemma 5.1 gives

$$\begin{aligned}|H_q(u_s-u_t)| &= \left(\int_{\mathbf{R}^{n-1}}|u_s(x')-u_t(x')|^q\,dx'\right)^{1/q}\\ &\leqq Mr^{(p-n)/p}\left(\int_{\mathbf{R}^{n-1}}\left(\int_{B((x',t),2r)}|\nabla u(z)|^p dz\right)^{q/p}dx'\right)^{1/q},\end{aligned}$$

so that Minkowski's inequality for integral yields

$$|H_q(u_s-u_t)|\leqq Mr^{(p-n)/p}(2r)^{(n-1)/q}\left(\int_{\{z=(z',z_n):t-2r<z_n<t+2r\}}|\nabla u(z)|^p\,dz\right)^{1/p}.$$

Let $r_j=2^{-j-1}$, $t_j=r_{j-1}$ and $A_j=\{z=(z',z_n):r_j<z_n<3r_j\}$ for $j=1,2,\dots$. For simplicity, set

$$\omega=(n-p+\beta)/p-(n-1)/q<0.$$

Then we find

$$|H_q(u_{t_j}-u_r)|\leqq Mr_{j+1}^{-\omega}\left(\int_{A_j}|\nabla u(z)|^p z_n^\beta dz\right)^{1/p}$$

for $t_j-r_{j+1}<r\leqq t_j$,

$$|S_q(u_r-u_s)|\leqq Mr_{j+2}^{-\omega}\left(\int_{A_j}|\nabla u(z)|^p z_n^\beta dz\right)^{1/p}$$

for $t_j-r_{j+1}-r_{j+2}<r<s\leqq t_j-r_{j+1}$, and

$$|S_q(u_s-u_{t_{j+1}})|\leqq Mr_{j+2}^{-\omega}\left(\int_{A_{j+1}}|\nabla u(z)|^p z_n^\beta dz\right)^{1/p}$$

for $t_{j+1} < s \leqq t_j - r_{j+1} - r_{j+2}$. Collecting these results, we have

$$
\begin{aligned}
|H_q(u_{t_j} - u_r)| &\leqq Mr_j^{-\omega}\left(\int_{A_j} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p} \\
&\quad + Mr_{j+1}^{-\omega}\left(\int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p}
\end{aligned}
$$

for $t_{j+1} < r \leqq t_j$. Hence it follows that

$$
|H_q(u_r - u_{t_{j+m}})| \leqq M \sum_{\ell=j}^{j+m} r_\ell^{-\omega} \left(\int_{A_\ell} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p}
$$

for $t_{j+m} < r \leqq t_j$. Since $A_\ell \cap A_k = \emptyset$ for $\ell \geqq k+2$, Hölder's inequality gives

$$
\begin{aligned}
|H_q(u_r - u_{t_{j+m}})| &\leqq M\left(\sum_{\ell=j}^{j+m} r_\ell^{-p'\omega}\right)^{1/p'} \left(\sum_{\ell=j}^{j+m} \int_{A_\ell} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p} \\
&\leqq Mr_j^{-\omega}\left(\int_{\{z=(z',z_n):r_{j+m}<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p}
\end{aligned}
$$

for $t_{j+m} < r \leqq t_j$, where $1/p + 1/p' = 1$. Now, letting $m \to \infty$, we establish

$$
|H_q(U_r)| \leqq Mr^{-\omega}\left(\int_{\{z=(z',z_n):0<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta dz\right)^{1/p}
$$

for $t_{j+1} < r \leqq t_j$, which implies that

$$
\lim_{r\to 0} r^\omega H_q(U_r) = 0,
$$

as required.

As to the condition on q, compare Theorem 6.3 with Theorem 3.1.

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