数理解析研究所講究録
1118 巻 1999 年 37-56

Dynamics of Polynomial Automorphisms of $\mathbb{C}^2$: Investigating stable and unstable manifolds using transcendental entire functions

神 貞介 (Jin, Teisuke)*

平成 11 年 9 月 30 日

概要

We study the structure of stable and unstable manifolds. Let a be a saddle point and let $W^u(a)$ be its unstable manifold. There exists a biholomorphic mapping $H:\mathbb{C}\to W^u(a)$. Then each of $H=(h_1,h_2)$ becomes a transcendental entire function. Because such a function has many interesting properties, we can investigate about $W^u(a)$. In this paper, first we inquire into the properties as functions. We show that an arbitrary algebraic variety intersects with $W^u(a)$ infinitely countable times. Secondly we examine the structure of $W^u(a)$ on $\mathbb{C}$. We prove Yoccoz inequality when $H^{-1}(K)$ is not connected. We explain the collision phenomenon.

1 Introduction

In this paper we use a notation $z=(x,y)\in\mathbb{C}^2$ and define $\pi_1(z)=x,\ \pi_2(z)=y$. Let $p_j(y)$ be monic polynomials $\deg d_j>1$ for $j=1,\ldots,m$. We call $g_j(x,y)=(y,p_j(y)-\delta_j x)$ generalized Hénon mappings, where $\delta_j\neq 0$. Moreover we define

$$F=g_m\circ\cdots\circ g_1,\quad \delta=\delta_1\cdots\delta_m,\quad d=d_1\cdots d_m.$$

For convenience, we define $F_j=g_j\circ\cdots\circ g_1$.

In [FM] Friedland and Milnor have classified polynomial automorphisms of $\mathbb{C}^2$ into three types: affine mapping, elementary mapping, composite of generalized Hénon mappings. They have investigated the former two mappings completely. So we study the last one, i.e. F which we have defined.

Easily we obtain $g_j^{-1}(x,y)=\left(\frac{1}{\delta}p_j(x)-\frac{1}{\delta}y,x\right)$. It is similar to $g_j(x,y)$ if x and y are exchanged. Therefore once we obtain a property about F, immediately we can apply it to the case of F^{-1} with a little modification.

1.1 Definitions and basic properties

We define $K^\pm=\{z\in\mathbb{C}^2\mid\{F^{\pm n}(z)\mid n\in\mathbb{N}\}\text{ is bounded}\}$, $J^\pm=\partial K^\pm$, $K=K^+\cap K^-$, $J=J^+\cap J^-$. They are closed sets and invariant under F.

Let a be a k-periodic point and let eigenvalues of $DF^k(a)$ be λ,λ' ($|\lambda|\geq|\lambda'|$). We call a

*University of Tokyo

- a source if $|\lambda|, |\lambda'| > 1$,
- a sink if $0 < |\lambda|, |\lambda'| < 1$,
- a saddle point if $0 < |\lambda'| < 1 < |\lambda|$.

Katok showed the following theorem in [Ka].

Theorem 1.1. *There exist saddle points.*

In this paper we assume a is a fixed point, since we can replace F^k by F.

Let $d(\; , \;)$ be an appropriate distance in $\mathbb{C}^2$. For arbitrary $X \subset \mathbb{C}^2$, define a stable set $W^s(X)$ and an unstable set $W^u(X)$ as follows:

$$W^s(X) = \{z \in \mathbb{C}^2 \mid d(F^n(z), F^n(X)) \to 0 \quad (n \to \infty)\},$$
$$W^u(X) = \{z \in \mathbb{C}^2 \mid d(F^n(z), F^n(X)) \to 0 \quad (n \to -\infty)\}.$$

The next theorem is well-known. See [MNTU, chapter 6] for example. The following equations act the main role in applying Nevanlinna theory to dynamical systems.

Theorem 1.2. *Assume a is a fixed point of saddle type. Then there exists a biholomorphic mapping $H : \mathbb{C} \to W^u(a)$ such that*

$$F \circ H(t) = H(\lambda t) \quad (t \in \mathbb{C}).$$

Similarly there is a biholomorphic mapping $H' : \mathbb{C} \to W^s(a)$ such that

$$F \circ H'(t) = H'(\lambda' t) \quad (t \in \mathbb{C}).$$

By the theorem we can call stable/unstable set stable/unstable manifold when a is a saddle point.

Let us recall the notion of *access.* In general, let a be a fixed point and Λ be a component of compliment of (filled) Julia set (e.g. $K^{+/-}$). Suppose $a \in \partial\Lambda$. Then we say that a is *accessible* from Λ if and only if there exists a curve $\gamma : [0,1] \to \overline{\Lambda}$ which suffices:

$$\gamma(0) = a \quad \text{and} \quad \gamma((0,1]) \subset \Lambda.$$

We call such γ an *access.* Moreover we call γ a *periodic access,* if it satisfies $F^q(\gamma) \subset \gamma$ or $F^q(\gamma) \supset \gamma$, where $q \in \mathbb{N}$ is the period of Λ.

1.2 The main theorems

At first we will show the properties of $H : \mathbb{C} \to W^u(a)$ as holomorphic mapping.

Theorem 2.1. *Each of H is a transcendental entire function. Moreover they are of mean type of order $\rho = \log d / \log |\lambda|$.*

In addition, if f is a holomorphic or rational function on $\mathbb{C}^2$, it will be shown that $f \circ H$ is also transcendental in Proposition 2.8 and 2.9. Using the fact, we can see the following.

Theorem 2.11. *Let $P(x,y)$ be a non-constant polynomial of two variables. Then $P \circ H$ has no Picard's exceptional values, i.e. an arbitrary 1-dimensional algebraic variety intersects with $W^{u/s}(a)$ infinitely countable times.*

Using the order ρ, we begin to investigate the dynamical structure on an unstable manifold. Suppose $\widetilde{K} = H^{-1}(K^+)$. The followings decide the structure of $\widetilde{K}$.

Theorem 3.1. *If $\rho < 1/2$ then any component of $\widetilde{K}$ is compact and $\mathbb{C} \setminus \widetilde{K}$ is connected.*

Theorem 3.3. *The number of components of $\mathbb{C} \setminus \widetilde{K}$ never exceeds $\max\{2\rho, 1\}$. Therefore every component of $\mathbb{C} \setminus \widetilde{K}$ is periodic.*

Corollary 3.7. *0 is periodically accessible from every component of $\mathbb{C} \setminus \widetilde{K}$. Especially each saddle point is accessible from $\mathbb{C}^2 \setminus K^{\pm}$.*

Theorem 3.11. (Yoccoz inequality). *Assume $\widetilde{K}$ is bridged i.e. the component of $\widetilde{K}$ containing 0 is not a point. Then the following holds.*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|^2} \geq \frac{Nq}{2\log d},$$

where we choose an appropriate branch of $\log\lambda$.

The above Yoccoz inequality doesn't need connectivity. Instead, we introduce the notion of *bridge*. We say that $\widetilde{K}$ is *bridged* if and only if some component of $\widetilde{K}$ is unbounded and contains 0. In Proposition 3.10, we will show that $\widetilde{K}$ is *bridged* when the component of $\widetilde{K}$ containing 0 is not a point. It seems that the notion of *bridge* is the weakest topological criterion for Yoccoz inequality. But we will see that the *bridgedness* is not nessecity criterion in Example 4.2.

In the sequel, we will proceed the relation between K^+ and $\widetilde{K}^+$. A set meeting $W^s(a)$ approaches to $W^u(a)$ by iteration. Then the structure in $W^u(a)$ reflects the original set.

Proposition 4.1. *If a point $z_0 \in W^s(a)$ is accessible from $\operatorname{int} K^+$ then $\widetilde{K}^+ = H^{-1}(K^+)$ is bridged. Therefore Yoccoz inequality holds there.*

By the argument we will show in Example 4.2 that there exists $W^s(a)$ such that any points on it are not accessible from $\operatorname{int} K^+$ though $W^s(a)$ is a dense subset of $\partial \operatorname{int} K^+$. It contrasts sharply with Corollary 3.7.

2 Transcendental entire function

We denote $H = (h_1, h_2)$ and $H' = (h_1', h_2')$.

2.1 Transcendence

We recall that the *order* ρ of $f \in \mathcal{O}(\mathbb{C})$ is:

$$\rho = \operatorname{ord} f = \limsup_{r\to\infty} \frac{\log\log \sup_{|x|=r} |f(x)|}{\log r}.$$

Moreover if ρ is finite, the *type* τ is:

$$\tau = \limsup_{r\to\infty} \frac{\log \sup_{|x|=r} |f(x)|}{r^\rho}.$$

We say f is of minimum type, mean type, maximum type of order ρ when $\tau = 0$, $0 < \tau < \infty$, $\tau = \infty$, respectively.

Theorem 2.1. *h_1, h_2, h_1', h_2' are transcendental entire functions. They are of mean type of orders:*

$$\rho = \operatorname{ord} h_1 = \operatorname{ord} h_2 = \frac{\log d}{\log|\lambda|}, \quad \rho' = \operatorname{ord} h_1' = \operatorname{ord} h_2' = \frac{\log d}{\log \frac{1}{|\lambda'|}}.$$

To prove the theorem, we quote the following.

Lemma 2.2. [BS1]. *For $R > 0$, define $V^+ = \{(x,y) \in \mathbb{C}^2 \mid |x| > R, |x| > |y|\}$, $V^- = \{(x,y) \in \mathbb{C}^2 \mid |y| > R, |y| > |x|\}$, $V = \{(x,y) \in \mathbb{C}^2 \mid |x| \le R, |y| \le R\}$. Then for sufficiently large $R > 0$,*

$$\begin{aligned}
&K^+ \subset V \cup V^+, \quad F_j(K^+) \subset V \cup V^+ \quad (j = 1, \ldots, m-1),\\
&K^- \subset V \cup V^-, \quad F_j(K^-) \subset V \cup V^- \quad (j = 1, \ldots, m-1),\\
&K \subset V,\\
&V^- \subset F^{-1}(V^-) \subset F^{-2}(V^-) \subset \cdots \nearrow \mathbb{C}^2 \setminus K^+,\\
&V^+ \subset F(V^+) \subset F^2(V^+) \subset \cdots \nearrow \mathbb{C}^2 \setminus K^-.
\end{aligned}$$

Let us proceed to prove the theorem.

Lemma 2.3. *h_2 is non-constant.*

Proof. Assume that h_2 is constant. Since H is non-constant, h_1 is not bounded. On the other hand $(h_1(t), h_2) \in W^u(a) \subset V \cup V^-$ $(t \in \mathbb{C})$, it contradicts. □

Lemma 2.4. *h_2 is of mean type of order $\rho = \log d / \log|\lambda|$.*

Proof. We assume that the order is $\rho = \log d/\log|\lambda|$ and compute the type. If it is of mean type, we see the tentative order is true.

In this proof, we define for $(\widetilde{y}_{-1}, \widetilde{y}_0) = H(t_0)$,

$$(\widetilde{y}_{-1}, \widetilde{y}_0) \xmapsto{g_1} (\widetilde{y}_0, \widetilde{y}_1) \xmapsto{g_2} \cdots \xmapsto{g_m} (\widetilde{y}_{m-1}, \widetilde{y}_m)$$

and

$$y_n = \pi_2 \circ F^n(\widetilde{y}_{-1}, \widetilde{y}_0) \quad (n = 0, 1, 2, \ldots).$$

Notice that $y_0 = \widetilde{y}_0$, $y_1 = \widetilde{y}_m$.

At first, we show that the type is greater than zero. Let $(x,y)=H(t)$. Then we can take $|y|$ large as we like. By Lemma 2.2, $|p_j(y)-x| \geq |p_j(y)|-|x| \geq |p_j(y)|-\max\{|y|,R\}$. For any $\varepsilon>0$ and sufficiently large any $|y|$, $|p_j(y)|-\max\{|y|,R\} \geq (1-\varepsilon)|y|^{d_j}$. Therefore if $|\widetilde{y}_0|$ is sufficiently large, we obtain inductively

$$|\widetilde{y}_{j+1}| = |p_{j+1}(\widetilde{y}_j)-\widetilde{y}_{j-1}| \geq (1-\varepsilon)|\widetilde{y}_j|^{d_{j+1}} \quad (j=0,\dots,m-1).$$

By repetition

$$|\widetilde{y}_m| \geq C_{-\varepsilon}|\widetilde{y}_0|^d,$$

where $C_{-\varepsilon}=(1-\varepsilon)^{d_2\cdots d_m+d_3\cdots d_m+\cdots+1}$. Therefore we have

$$|y_n| \geq C_{-\varepsilon}|y_{n-1}|^d \quad (n=1,2,\dots)$$

and obtain by repetition

$$|y_n| \geq C_{-\varepsilon}^{d^{n-1}+d^{n-2}+\cdots+1}|y_0|^{d^n} = C_{-\varepsilon}^{\frac{d^n-1}{d-1}}|y_0|^{d^n}.$$

Recall $y_0=h_2(t_0)$. By Theorem 1.2, $y_n=h_2(\lambda^n t_0)$. Therefore

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log\max_{|t|=r}|h_2(t)|}{r^\rho} &\geq \limsup_{n\to\infty}\frac{\log|h_2(\lambda^n t_0)|}{|\lambda^n t_0|^\rho}\\ \geq \limsup_{n\to\infty}\frac{\log C_{-\varepsilon}^{\frac{d^n-1}{d-1}}|y_0|^{d^n}}{d^n|t_0|^\rho} &= \frac{\frac{1}{d-1}\log C_{-\varepsilon}+\log|y_0|}{|t_0|^\rho}>0.\end{aligned}$$

In the above calculation, we employ $|\lambda|^\rho=d$.

Secondly, we show that the type is bounded. Let $(x,y)=H(t)$. Then $|p_j(y)-x| \leq |p_j(y)|+|x| \leq |p_j(y)|+\max\{|y|,R\}$. For any $\varepsilon>0$ there exists $M>1$ such that $|p_j(y)|+\max\{|y|,R\} \leq (1+\varepsilon)(\max\{|y|,M\})^{d_j}$. i.e.

$$|\widetilde{y}_{j+1}| \leq (1+\varepsilon)(\max\{|\widetilde{y}_j|,M\})^{d_{j+1}} \quad (j=0,\dots,m-1) \tag{2.1}$$

Then we obtain

$$|\widetilde{y}_m| \leq C_\varepsilon(\max\{|\widetilde{y}_0|,M\})^d,$$

where $C_\varepsilon=(1+\varepsilon)^{d_2\cdots d_m+d_3\cdots d_m+\cdots+1}$. Therefore we have

$$|y_n| \leq C_\varepsilon(\max\{|y_{n-1}|,M\})^d \quad (n=1,2,\dots)$$

and obtain by repetition

$$|y_n| \leq C_\varepsilon^{d^{n-1}+d^{n-2}+\cdots+1}(\max\{|y_0|,M\})^{d^n} = C_\varepsilon^{\frac{d^n-1}{d-1}}(\max\{|y_0|,M\})^{d^n}.$$

Then we have

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log\max_{|t|=r}|h_2(t)|}{r^\rho}\\ &\leq \limsup_{n\to\infty}\frac{\log\max_{|\lambda|^n|t_0|\leq|t|\leq|\lambda|^{n+1}|t_0|}|h_2(t)|}{(|\lambda|^n|t_0|)^\rho}\\ &= \limsup_{n\to\infty}\frac{\log\max_{|t_0|\leq|t|\leq|\lambda t_0|}|\pi_2\circ F^n\circ H(t)|}{d^n|t_0|^\rho}\\ &\leq \limsup_{n\to\infty}\frac{\log\max_{|t_0|\leq|t|\leq|\lambda t_0|}C_\varepsilon^{\frac{d^n-1}{d-1}}(\max\{|h_2(t)|,M\})^{d^n}}{d^n|t_0|^\rho}\\ &= \frac{\frac{1}{d-1}\log C_\varepsilon+\log\max_{|t_0|\leq|t|\leq|\lambda t_0|}(\max\{|h_2(t)|,M\})}{|t_0|^\rho}<\infty.\end{aligned}$$

In the calculation it is employed that $|\lambda|^\rho = d$. $\square$

Remark 2.5. *Similarly, we can compute the lower order of h_2, and it is the same as the order. Indeed,*

$$\liminf_{r\to\infty} \frac{\log\log \max_{|t|=r} |h_2|}{\log r} = \frac{\log d}{\log|\lambda|}.$$

By the following lemma, we complete the proof of Theorem 2.1.

Lemma 2.6. *Let τ, τ' be the types of h_1, h_1' respectively. Then $0 < \tau, \tau' < \infty$ and for $j = 0, 1, \ldots, m$*

$$\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H(t)|}{r^\rho} = d_{j-1} \cdots d_0 \tau,$$
$$\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho} = d_j \cdots d_0 \tau,$$
$$\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H'(t)|}{r^\rho} = \frac{\tau'}{d_j \cdots d_1},$$
$$\limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H'(t)|}{r^\rho} = \frac{\tau'}{d_{j+1} \cdots d_1},$$

where $d_0 = d_m, d_{m+1} = d_1$. Especially, all are of mean type.

Proof. For $j = 0, \ldots, m$, let

$$\alpha_j = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F_j \circ H(t)|}{r^\rho},$$
$$\beta_j = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho}.$$

We have $|y| \le (1+\varepsilon)(\max\{|x|, M\})^{d_j}$ for $(x, y) = F_j \circ H(t)$ by (2.1). Therefore

$$\begin{aligned}
\beta_j &= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho} \\
&\le \limsup_{r\to\infty} \frac{\log \max_{|t|=r} (1+\varepsilon)(\max\{|\pi_1 \circ F_j \circ H(t)|, M\})^{d_j}}{r^\rho} \\
&= d_j \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_2 \circ F_j \circ H(t)|}{r^\rho} = d_j \alpha_j.
\end{aligned}$$

Moreover by definition of g_j, we obtain $\beta_j = \alpha_{j+1}$. Therefore we have

$$d_0\alpha_0 \ge \beta_0 = \alpha_1,\ d_1\alpha_1 \ge \beta_1 = \alpha_2, \ldots, d_{m-1}\alpha_{m-1} \ge \beta_{m-1} = \alpha_m.$$

On the other hand,

$$\begin{aligned}
\alpha_m &= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |\pi_1 \circ F \circ H(t)|}{r^\rho} \\
&= \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |h_1(\lambda t)|}{r^\rho} \\
&= d \limsup_{r\to\infty} \frac{\log \max_{|t|=r} |h_1(t)|}{r^\rho} = d\alpha_0.
\end{aligned}$$

By putting the above inequalities and equations together, we have

$$\alpha_0 \ge \frac{1}{d_0}\alpha_1 \ge \cdots \ge \frac{1}{d_{m-1}\cdots d_0}\alpha_m = \frac{d}{d_{m-1}\cdots d_0}\alpha_0 = \alpha_0.$$

Lemma 2.4 implies that all of α_j are positive and bounded. This concludes the proposition. $\square$

Corollary 2.7. *There exists* $C_1 > 0$ *such that*

$$\begin{aligned}
K^+ &\subset \{(x,y)\in\mathbb{C}^2 \mid |y| \le C_1(|x|^{1/d_1}+1)\},\\
F_j(K^+) &\subset \{(x,y)\in\mathbb{C}^2 \mid |y| \le C_1(|x|^{1/d_{j+1}}+1)\} && (j=1,\dots m-1),\\
K^- &\subset \{(x,y)\in\mathbb{C}^2 \mid |x| \le C_1(|y|^{1/d_m}+1)\},\\
F_j(K^-) &\subset \{(x,y)\in\mathbb{C}^2 \mid |x| \le C_1(|y|^{1/d_j}+1)\} && (j=1,\dots m-1).
\end{aligned}$$

The degrees $1/d_j$ *are minimum.*

Proof. The former fact is used in [BS2] without proof. Let us prove the last assertion.

Assume some $1/d_j$ is not minimum. Then there exists $\gamma < 1/d_j$ where the same relation holds. Then

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\max_{|t|=r}|\pi_1\circ F_j\circ H(t)|}{r^\rho} \le \limsup_{r\to\infty} \frac{\max_{|t|=r} C_1(|\pi_2\circ F_j\circ H(t)|^\gamma+1)}{r^\rho}\\
&= \gamma \limsup_{r\to\infty} \frac{\max_{|t|=r}|\pi_2\circ F_j\circ H(t)|}{r^\rho} < \frac{1}{d_j}\limsup_{r\to\infty}\frac{|\pi_2\circ F_j\circ H(t)|}{r^\rho}.
\end{aligned}$$

It contradicts with the previous proposition. □

2.2 Compositions with functions on $\mathbb{C}^2$

We shall investigate compositions of some kinds of functions and H.

Proposition 2.8. *Let* $f \in \mathcal{O}(\mathbb{C}^2)$ *be non-constant. Then* $f\circ H$ *is a transcendental entire function.*

Proof. First we show that $f\circ H(t)$ does not have $t=\infty$ as a pole. By Picard's theorem, for some $y_0\in\mathbb{C}$ there exist infinitely many $t\in\mathbb{C}$ satisfying $h_2(t)=y_0$. By Lemma 2.2, we see that $\{h_1(t) \mid h_2(t)=y_0\}$ is bounded. Then there exists a sequence $\{t_j\}$ such that $t_j\to\infty$ and a limit of $H(t_j)$ exists. Therefore the limit of $f\circ H(t_j)$ also exists. This implies $t=\infty$ is not a pole.

Secondly we prove that $f\circ H$ is not constant. Assume $f\circ H$ is constant. By Picard's theorem, for any $y\in\mathbb{C}$ except for at most one point $y_0\in\mathbb{C}$ there exist infinitely many t satisfying $h_2(t)=y$. By Lemma 2.2, $\{h_1(t)\mid h_2(t)=y\}$ is bounded. Therefore the set has at least one limit point. On the other hand, $f(h_1(t),y)$ is constant where $h_2(t)=y$. By uniqueness theorem we obtain that $f(\cdot,y)$ is constant for each fixed y. Then $f(\cdot,h_2(t))$ becomes constant, so it is concluded that f is a constant. □

Proposition 2.9. *Let* f *be a non-constant rational function of two variables, i.e. there exist relatively prime polynomials* $P(x,y), Q(x,y)$ $(Q\ne 0)$ *which satisfy* $f(x,y)=P(x,y)/Q(x,y)$. *Then* $f\circ H$ *is a transcendental meromorphic function.*

To prove the proposition we prepare a lemma.

Lemma 2.10. *Let* $P(x,y)$ *be a non-constant polynomial. Then* $K^-\cap P^{-1}(0)$ *is compact unless empty.*

Proof. Bedford and Smillie have shown in [BS1, Proposition 4.2] that for sufficiently large any $n\in\mathbb{N}$, the terms of highest total degree of $P\circ F^n(x,y)$ consist of only power of y and some non-zero coefficient. Then we can have $\{(x,y)\mid P\circ F^n(x,y)=0\}\subset V\cup V^+$. Hence $K^-\cap\{(x,y)\mid P\circ F^n(x,y)=0\}$ is compact. Therefore

$$\begin{aligned}
K^-\cap\{(x,y)\mid P(x,y)=0\} &= K^-\cap F^n(\{(x,y)\mid P\circ F^n(x,y)=0\})\\
&= F^n\left(K^-\cap\{(x,y)\mid P\circ F^n(x,y)=0\}\right)
\end{aligned}$$

is compact, too. □

Proof of Proposition 2.9. When Q is constant it reduces to Proposition 2.8. So we assume Q is non-constant. We will show that $t = \infty$ is neither a pole nor a regular point.

At first we prove $f \circ H(t)$ doesn't have $t = \infty$ as a pole. Since $Q \circ H$ is transcendental by Proposition 2.8, there exists $q_0 \neq 0$ such that infinitely many $t \in \mathbb{C}$ satisfies $Q \circ H(t) = q_0$ by Picard's theorem. Because the image of H is included in K^-, it can be seen that $\{H(t) \mid Q \circ H(t) = q_0\}$ is bounded according to the previous lemma. $P \circ H$ is bounded on the set though t can tend to ∞. Therefore $t = \infty$ isn't a pole.

Similarly it can be shown that $t = \infty$ is not zero point of $f \circ H(t)$.

Otherwise assume that $\lim_{t \to \infty} f \circ H(t) = c$, $(c \neq 0, \infty)$. Then if we define $\widetilde{f}(x, y) = f(x, y) - c$, we see that $\widetilde{f} \circ H(t)$ has $t = \infty$ as a zero point. It contradicts with the previous statement. □

From now, we describe a property of H in Nevanlinna theory.

Theorem 2.11. *Let $P(x, y)$ be a non-constant polynomial of two variables. Then $P \circ H$ has no Picard's exceptional values, i.e. an arbitrary 1-dimensional algebraic variety intersects with $W^{u/s}(a)$ infinitely countable times. Further the intersection is bounded.*

To prove the theorem, we quote theorems in Nevanlinna theory.

Definition 2.12. [T]. *Assume $f(t)$ is a meromorphic function on complex plane. Let $n(r, a)$ be the number of zero points of $f(t) - a$ in $|t| < r$. On the other hand if $a = \infty$, $n(r, a)$ means the number of poles in $|t| < r$. We count the numbers with multiplicity. Define*

$$N(r, a) = \int_0^r \frac{n(r, a) - n(+0, a)}{r} dr + n(+0, a) \log r + const.,$$

$$T(r, f) = \int_0^r \frac{S(r)}{r} dr,$$

$$S(r) = \frac{A(r)}{\pi} = \frac{1}{\pi} \iint_{|t| \leq r} \left(\frac{|f'(t)|}{1 + |f(t)|^2} \right)^2 r dr d\theta, \quad t = re^{i\theta}.$$

The constant term is defined appropriately in the theory. We denote $N(r, \infty) = N(r, f)$ if we want express the function f explicitly.

Theorem 2.13. [O, Theorem 3.3]. *Let f be a meromorphic function on $\mathbb{C}$. f is a rational function if and only if*

$$\liminf_{r \to \infty} \frac{T(r, f)}{\log r} < \infty$$

Theorem 2.14. [O, Theorem 9.2]. *Let $f_1, \ldots, f_n$ be meromorphic functions on $\mathbb{C}$. Suppose they satisfy the followings.*

1. *$\sum_1^n c_j f_j = 0$, where c_j are constant,*
2. *f_h / f_k is not constant for any $h \neq k$,*
3. *$N(r, f_j) + N(r, 1/f_j) = o(T(r))$ when $r \notin E$, where E is a set whose length is bounded.*

Then $c_1 = \cdots = c_n = 0$. Where $T(r) = \min_{h \neq k} T(r, f_h/f_k)$.

We prove this lemma using above theorems.

Lemma 2.15. *Let $f_1, \ldots, f_n$ be meromorphic functions on $\mathbb{C}$. Assume all of them has finite zero points and finite poles. If $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$ satisfies*

$$c_1 f_1 + \cdots + c_n f_n = 0,$$

then for some $h \neq k$, f_h/f_k becomes a rational function.

Proof. Assume that any f_h/f_k are transcendental. Then by Theorem 2.13

$$\liminf_{r\to\infty} \frac{T(r, f_h/f_k)}{\log r} = \infty,$$

for any $h \neq k$. Therefore $T(r) > O(\log r)$.

On the other hand, because all of f_j have finite zero points and finite poles, we obtain for $j = 1, \ldots, n$,

$$N(r, f_j) = \int_0^r \frac{n(r,\infty) - n(+0,\infty)}{r} dr + n(+0,\infty) \log r + \text{const.} \leq O(\log r),$$
$$N(r, 1/f_j) = \int_0^r \frac{n(r,0) - n(+0,0)}{r} dr + n(+0,0) \log r + \text{const.} \leq O(\log r).$$
$$\therefore \; N(r, f_j) + N(r, 1/f_j) \leq O(\log r).$$

Hence 1., 2. and 3. in Theorem 2.14 are fulfilled. But the conclusion never holds. □

Proof of Theorem 2.11. We imitate a technique used in [Nt, Chapter 5].

First we show that at most two non-constant irreducible and relatively prime polynomials can have 0 as Picard's exceptional value when they are composed with H.

Let P_1, P_2, P_3 be non-constant, irreducible and relatively prime polynomials of two variables. Assume that $P_1 \circ H, P_2 \circ H, P_3 \circ H$ have finite zero points. Put

$$w_1 = P_1(h_1, h_2),$$
$$w_2 = P_2(h_1, h_2),$$
$$w_3 = P_3(h_1, h_2).$$

Then w_1, w_2, w_3 are entire functions which have finite zero points. On the other hand, we can utilize the polynomial ring's theory to eliminate h_1, h_2 in the above equations. In fact, by system of resultants there exists a non-constant polynomial Q which satisfies

$$Q(w_1, w_2, w_3) = 0.$$

Then we have by expanding Q,

$$Q(w_1, w_2, w_3) = \sum_{i,j,k} q_{ijk} w_1^i w_2^j w_3^k = 0.$$

Since each term has finite zero points and no poles, by the previous lemma there exist $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$ such that

$$\frac{w_1^{i_1} w_2^{j_1} w_3^{k_1}}{w_1^{i_2} w_2^{j_2} w_3^{k_2}} = \frac{P_1(H)^{i_1} P_2(H)^{j_1} P_3(H)^{k_1}}{P_1(H)^{i_2} P_2(H)^{j_2} P_3(H)^{k_2}}$$

is a rational function. But it contradicts with Proposition 2.9.

Secondly we prove that no non-constant irreducible and relatively prime polynomials can have 0 as Picard's exceptional value when they are composed with H.

Assume P is a non-constant polynomial such that $P \circ H$ has 0 as Picard's exceptional value. We can limit P to irreducible. Then

$$P \circ F^n \circ H(t) = P \circ H(\lambda^n t) \quad (n \in \mathbb{Z})$$

also have 0 as Picard's exceptional value. By the first conclusion of this proof, all of $P \circ F^n$ must be expressed by a power of two non-constant irreducible and relatively prime polynomials and suitable coefficient. One of two must be P. Let us denote another S. Hence for any $n \in \mathbb{Z}$ there exist $i, j \in \mathbb{N} \cup \{0\}$ and $c_n \neq 0$ such that

$$P \circ F^n(x, y) = c_n(P(x, y))^i (S(x, y))^j.$$

Clearly $\max\{i, j\} \to \infty$ when $n \to \pm\infty$. By the way since F is invertible, we obtain

$$P(x, y) = c_n(P \circ F^{-n}(x, y))^i (S \circ F^{-n}(x, y))^j.$$

It is clear that the degree of the right side increases when $n \to \pm\infty$. It contradicts.

The last statement is clear because of Lemma 2.10. $\square$

3 Unstable slice

We denote $\widetilde{K} = H^{-1}(K^+)$ and call it unstable slice. $\widetilde{K}$ has positive capacity near any point. In fact, let G^+ be the plurisubharmonic function describing K^+ (see [BS1]). Then $G^+ \circ H$ is subharmonic and non-negative. We can see the positivity of capacity easily. Moreover, we observe that $\widetilde{K}$ is invariant under $t \mapsto \lambda t$ by Theorem 1.2.

3.1 The case of broken $\widetilde{K}$

Let us investigate the simplest case. For $A \subset \mathbb{C}$ and $r > 0$, define $1_A(r)$ as follows:

$$1_A(r) = \begin{cases} 1 & \text{if } A \cap \{|t| = r\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.1. *If $\rho < 1/2$, for any $r_0 > 0$,*

$$\frac{1}{\log|\lambda|} \int_{r_0}^{|\lambda| r_0} \frac{1_{\widetilde{K}}(r)}{r} dr \leq 2\rho.$$

Especially any components of $\widetilde{K}$ are compact and $\mathbb{C} \setminus \widetilde{K}$ is connected.

The reverse is false, i.e. $\rho \geq 1/2$ doesn't have to imply that every component of $\widetilde{K}$ is compact. See Remark 3.12 and Example 4.2.

To prove the theorem, we quote *Precise form of Wiman's theorem* in [T].

Theorem 3.2. [T, Theorem III. 72.]. *Let $f(t)$ be an entire function of order ρ $(0<\rho<1/2)$. Then for any ε $(0<\varepsilon<\rho)$,*

$$\limsup_{r_2/r_1\to\infty}\frac{1}{\log(r_2/r_1)}\int_{r_1}^{r_2}\frac{1_{[\log\min_{|t|=r}|f(t)|>r^{\rho-\varepsilon}]}}{r}dr\geq 1-2\rho,$$

where

$$1_{[criterion]}=\begin{cases}1 & \textit{if the criterion is fulfilled,}\\ 0 & \textit{otherwise.}\end{cases}$$

Proof of Theorem 3.1. We apply the previous theorem by $f(t)=h_2(t)/R$. Let us estimate the integrand at first:

$$1_{[\log\min_{|t|=r}|h_2(t)/R|>r^{\rho-\varepsilon}]}\leq 1_{[\min_{|t|=r}|h_2(t)|>R]}\leq 1-1_{\widetilde{K}}(r).$$

hence

$$\begin{aligned}1-2\rho &\leq \limsup_{r_2/r_1\to\infty}\frac{1}{\log(r_2/r_1)}\int_{r_1}^{r_2}\frac{1_{[\log\min_{|t|=r}|h_2(t)/R|>r^{\rho-\varepsilon}]}}{r}dr\\ &\leq \limsup_{r_2/r_1\to\infty}\frac{1}{\log(r_2/r_1)}\int_{r_1}^{r_2}\frac{1-1_{\widetilde{K}}(r)}{r}dr.\end{aligned}$$

Therefore

$$2\rho\geq \liminf_{r_2/r_1\to\infty}\frac{1}{\log(r_2/r_1)}\int_{r_1}^{r_2}\frac{1_{\widetilde{K}}(r)}{r}dr$$

Let n_1,n_2 be integers satisfying $|\lambda|^{n_1}r_0\leq r_1<|\lambda|^{n_1+1}r_0$, $|\lambda|^{n_2}r_0\leq r_2<|\lambda|^{n_2+1}r_0$

$$\begin{aligned}&\geq \liminf_{n_2-n_1\to\infty}\frac{1}{\log|\lambda|^{n_2+1-n_1}}\int_{|\lambda|^{n_1+1}r_0}^{|\lambda|^{n_2}r_0}\frac{1_{\widetilde{K}}(r)}{r}dr\\ &=\liminf_{n_2-n_1\to\infty}\frac{1}{(n_2-n_1+1)\log|\lambda|}\sum_{j=n_1+1}^{n_2-1}\int_{|\lambda|^{j}r_0}^{|\lambda|^{j+1}r_0}\frac{1_{\widetilde{K}}(r)}{r}dr\\ &=\liminf_{n_2-n_1\to\infty}\frac{(n_2-n_1-1)}{(n_2-n_1+1)\log|\lambda|}\int_{r_0}^{|\lambda|r_0}\frac{1_{\widetilde{K}}(r)}{r}dr\\ &=\frac{1}{\log|\lambda|}\int_{r_0}^{|\lambda|r_0}\frac{1_{\widetilde{K}}(r)}{r}dr.\end{aligned}$$

□

3.2 Yoccoz inequality

We examine the structure of components of $\mathbb{C}\setminus\widetilde{K}$.

The number of components of $\mathbb{C}\setminus\widetilde{K}$ may be infinite. But fortunately we can obtain the following theorem.

Theorem 3.3. *The number of components of $\mathbb{C}\setminus\widetilde{K}$ never exceeds* $\max\{2\rho,1\}$. *Therefore every component of $\mathbb{C}\setminus\widetilde{K}$ is periodic.*

To prove the theorem, we prepare the followings.

Assume $\Omega \subset \mathbb{C}$ is an unbounded domain containing the origin. Let Ω_r be the component of $\{t \in \Omega \mid |t| < r\}$ which contains the origin. Then define that $r\theta(r)$ is a line measure of $\{|t| = r\} \cap \overline{\Omega_r}$. We can regard $\theta(r)$ as an angle measure of Ω_r. Further let

$$\theta^*(r) = \begin{cases} \theta(r) & \text{if } \{|t| = r\} \cap \partial\Omega_r \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Now, we quote a powerful inequality in [T] which is based on harmonic measure theory. In this paper we name it *Arima-Tsuji inequality.*

Theorem 3.4. [T, Theorem III. 68.]. *Let $\Omega \subset \mathbb{C}$ be an unbounded domain. Let $f(t)$ be holomorphic in Ω and $|f(t)| \leq 1$ on $\partial\Omega$. If there exists $t_0 \in \Omega$ such that $|f(t_0)| > 1$, then*

$$\log\log \sup_{t\in\Omega, |t|=r} |f(t)| \geq \pi \int_1^{\kappa r} \frac{dr}{r\theta^*(r)} - const. \quad (0 < \kappa < 1).$$

Define $D = \{t \in \mathbb{C} \mid |h_2(t)| > R\}$ and $D^n = \{t \in \mathbb{C} \mid |h_2(\lambda^n t)| > R\} = \{t \in \mathbb{C} \mid \lambda^n t \in D\}$ for $n = 0, 1, \ldots$. We shall prove that the number of components of D never exceeds 2ρ.

Lemma 3.5. *Every component of D, D^n is not bounded. D, D^n satisfy*

$$D = D^0 \subset D^1 \subset D^2 \subset \cdots \nearrow \mathbb{C} \setminus \widetilde{K}.$$

Therefore each component of $\mathbb{C} \setminus \widetilde{K}$ is not bounded, too.

Proof. The first assertion is clear because of maximum principle.

Secondly we show that $D^n \subset D^{n+1}$. Let $t \in D^n$, then $|h_2(\lambda^n t)| > R$. Hence $H(\lambda^n t) \in V^-$ because $H(\lambda^n t) \in V \cup V^-$. Then by Lemma 2.2,

$$H(\lambda^{n+1}t) = F \circ H(\lambda^n t) \in F(V^-) \subset V^-,$$

we obtain $|h_2(\lambda^{n+1}t)| > R$, i.e. $t \in D^{n+1}$.

If $t \in \mathbb{C} \setminus \widetilde{K}$, $H(t) \notin K^+$. Hence for some $n \in \mathbb{N}$, $F^n \circ H(t) \in V^-$ because of Lemma 2.2. Since $F^n \circ H(t) = H(\lambda^n t) = (h_1(\lambda^n t), h_2(\lambda^n t))$, we obtain $|h_2(\lambda^n t)| > R$, i.e. $t \in D^n$. □

Lemma 3.6. *Each component of $\mathbb{C} \setminus \widetilde{K}$ includes some component of D.*

Proof. Assume that some component U of $\mathbb{C} \setminus \widetilde{K}$ does not intersect with D. If $\mathbb{C} \setminus \widetilde{K}$ is connected, it is clear that it intersects with D. Hence we assume that the number of the connected components is greater than 1.

Take $t_0 \in U$. By the previous lemma, there exists $n \in \mathbb{N}$ such that $\lambda^n t_0 \in D$. Let D_1 be the component of D that contains $\lambda^n t_0$. Therefore $h_2/R \leq 1$ on ∂D_1 and $|h_2(\lambda^n t_0)/R| > 1$. On the other hand, $|h_2(t)|$ is bounded in U, thus $|h_1(t)|$ is also bounded in U because $H(t) \in V \cup V^-$. It implies $|\pi_2 \circ F^n \circ H(t)|$ is bounded in U for fixed n. Then

$$\begin{aligned} \log\log \sup_{t\in U, |t|=r} \left| \frac{\pi_2 \circ F^n \circ H(t)}{R} \right| &= \log\log \sup_{t\in U, |t|=r} \left| \frac{h_2(\lambda^n t)}{R} \right| \\ &\geq \log\log \sup_{t\in D_1, |t|=|\lambda|^n r} \left| \frac{h_2(t)}{R} \right| \end{aligned}$$

Apply Arima-Tsuji inequality (Theorem 3.4)

$$\geq \pi \int_1^{\kappa|\lambda|^n r} \frac{dr}{r\theta^*(r)} - \text{const.} \quad (0 < \kappa < 1)$$

$\theta^*(r) = \theta(r)$ holds for sufficiently large r since we assumed the number of components of $\mathbb{C} \setminus \widetilde{K}$ is greater than 1

$$\begin{aligned} &\geq \pi \int_1^{\kappa|\lambda|^n r} \frac{dr}{r\theta(r)} - \text{const.} \\ &\geq \pi \int_1^{\kappa|\lambda|^n r} \frac{dr}{r \cdot 2\pi} - \text{const.} \\ &= \frac{1}{2} \log \kappa|\lambda|^n r - \text{const.} \to \infty \quad (r \to \infty). \end{aligned}$$

It contradicts with the hypothesis that $|\pi_2 \circ F^n \circ H(t)|$ is bounded in U. Therefore U must intersect with D. □

Proof of Theorem 3.3. Assume the number of components of $\mathbb{C} \setminus \widetilde{K}$ is greater than or equal to n where $n > 1$. Note that the number of the components may be infinity. By the assumption, there exist at least n components of D. Name them $D_1, \ldots, D_n$. Since $|h_2/R|$ is ≤ 1 on ∂D_j and > 1 in D_j for $j = 1, \ldots, n$, we can apply Arima-Tsuji inequality:

$$\log\log \sup_{t \in D_j, |t|=r} \left|\frac{h_2(t)}{R}\right| \geq \pi \int_1^{\kappa r} \frac{dr}{r\theta_j^*(r)} - \text{const.} \quad (0 < \kappa < 1),$$

where θ_j is the angle measure of D_j. Sum the inequality for $j = 1, \ldots, n$,

$$\sum_{j=1}^{n} \log\log \sup_{t \in D_j, |t|=r} \left|\frac{h_2(t)}{R}\right| \geq \sum_{j=1}^{n} \pi \int_1^{\kappa r} \frac{dr}{r\theta_j^*(r)} - \text{const.}$$

$$\therefore\ n \log\log \sup_{t \in D, |t|=r} \left|\frac{h_2(t)}{R}\right| \geq \pi \int_1^{\kappa r} \left(\sum_{j=1}^{n} \frac{1}{\theta_j^*(r)}\right) \frac{dr}{r} - \text{const.}$$

Because any components of D are not bounded and the number of the components is greater than 1, we obtain $\theta_j^*(r) = \theta_j(r)$ for sufficiently large r. Further by Schwarz's inequality

$$n^2 = \left(\sum_{j=1}^{n} \frac{\sqrt{\theta_j}}{\sqrt{\theta_j}}\right)^2 \leq \left(\sum \theta_j\right)\left(\sum \frac{1}{\theta_j}\right) \leq 2\pi \left(\sum \frac{1}{\theta_j}\right).$$

Therefore we have

$$n \log\log \sup_{t \in D, |t|=r} \left|\frac{h_2(t)}{R}\right| \geq \pi \int_1^{\kappa r} \frac{n^2}{2\pi} \frac{dr}{r} - \text{const.} = \frac{n^2}{2} \log \kappa r - \text{const.}$$

Divide the left and the right sides by $\log r$ and let $r \to \infty$, we obtain $n\rho \geq n^2/2$, i.e. $n \leq 2\rho$.

Remember we have assumed $n \geq 2$. Therefore $n = 1$ when $1/2 \leq \rho < 1$. □

Corollary 3.7. 0 *is accessible from arbitrary component of* $\mathbb{C} \setminus \widetilde{K}$. *Moreover the access* γ *can be periodic, i.e. if* q *is the period of the component,* γ *satisfies* $\gamma([0,1]) \subset \lambda^q \cdot \gamma([0,1])$.

Especially, every saddle point is accessible from $\mathbb{C}^2 \setminus K^{\pm}$.

Proof. Take arbitrary $t_0 \in \mathbb{C} \setminus \widetilde{K}$ and fix it. We will show there exists the above γ such that $\gamma(1) = t_0$.

If $\rho < 1/2$, $\mathbb{C} \setminus \widetilde{K}$ is connected because of Theorem 3.1. If $\rho \geq 1/2$, Theorem 3.3 implies that all components of $\mathbb{C} \setminus \widetilde{K}$ are periodic of the same period. In any cases, let q be the period. Then t_0/λ^q is also contained in the same component. Therefore there exists a curve $\widetilde{\gamma} : [0,1] \to \mathbb{C} \setminus \widetilde{K}$ such that $\widetilde{\gamma}(1) = t_0$ and $\widetilde{\gamma}(0) = t_0/\lambda^q$.

Let us define γ as follows. The idea is to join all of $\{\frac{1}{\lambda^{q(n-1)}}\gamma([0,1])\}$ for $n \in \mathbb{N}$. For $0 \leq \xi \leq 1$,

$$\gamma(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ \frac{1}{\lambda^{q(n-1)}}\widetilde{\gamma}\left(\frac{|\lambda|^{q(n-1)}\xi - \frac{1}{|\lambda|^q}}{1 - \frac{1}{|\lambda|^q}}\right) & \text{for } n \in \mathbb{N} \text{ such that } \frac{1}{|\lambda|^q} \leq |\lambda|^{q(n-1)}\xi \leq 1. \end{cases}$$

This γ is well-defined and continuous and satisfies $\gamma([0,1]) \subset \lambda^q\gamma([0,1])$. □

Let us prove *Yoccoz inequality.* By Theorem 3.3 it is clear that all components of $\mathbb{C} \setminus \widetilde{K}$ are periodic and have the same period under $t \mapsto \lambda t$.

Definition 3.8. [BxH]. *Assume $\mathbb{C} \setminus \widetilde{K}$ has q' components. Because all components have the same period, suppose a component move to p'-th component under $t \mapsto \lambda t$, counting counterclockwise, where $0 \leq p' < q'$. Then let $p'/q' = p/q$ by reduction and let N be the greatest common divider of p' and q'.*

Definition 3.9. *Let A be a subset of $\mathbb{C}$. When an unbounded component of A contains* 0, *we call the component a* bridge. *If A has a bridge, we say that A is* bridged.

Proposition 3.10. *The following three conditions are equivalent.*

1. *$\widetilde{K}$ is bridged.*
2. *The component of $\widetilde{K}$ containing* 0 *is not a point.*
3. *Some component of $\widetilde{K}$ is unbounded.*

Proof. It is clear that 1. implies 2. and 3. Therefore we will show the reverses.

In 2., let A be the component containing 0. Then

$$\widetilde{K} \supset \overline{\bigcup_{j=0}^{\infty} \lambda^j A}.$$

The right side has a component which is not bounded. It implies 1.

In 3., let A be an unbounded component and assume A does not contain 0. If A is k-periodic, of course

$$A = \bigcup_{j=0}^{\infty} \frac{1}{\lambda^{kj}} A$$

contains 0. Therefore we suppose A is not periodic. It holds that $\frac{1}{\lambda^i}A \cap \frac{1}{\lambda^j}A = \emptyset$ when $i \neq j$, because if they intersect then $\frac{1}{\lambda^i}A \cup \frac{1}{\lambda^j}A$ becomes a component of $\widetilde{K}$. Let us regard the following:

$$\mathbb{C} \setminus \bigcup_{j=0}^{[2\rho]} \frac{1}{\lambda^j} A.$$

If we take $R > 0$ sufficiently large, the number of components of D becomes $\geq [2\rho] + 1$. It contradicts with the proof of Theorem 3.3. It reduces to 1. □

In [BxH] Buff and Hubbard have proved Yoccoz inequality on $W^u(a)$ when $\widetilde{K}$ is connected. The following theorem is slightly improved because it doesn't need the connectivity. Instead, we need the notion of *bridge.*

Theorem 3.11. (Yoccoz inequality). *Assume that $\widetilde{K}$ is bridged, i.e. the component of $\widetilde{K}$ containing 0 is not a point. Then*

$$\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|^2} \geq \frac{Nq}{2\log d}$$

holds, where we choose an appropriate branch of $\log\lambda$.

Remark 3.12. *In Theorem 3.1, we have shown a sufficiency criterion that all components of $\widetilde{K}$ become compact. The above theorem improves the criterion slightly. In fact, given d, λ. If any p, q, N cannot satisfy Yoccoz inequality, there exist no bridges, i.e. any components of $\widetilde{K}$ are compact.*

After the author had proved Theorem 3.3, Shishikura advised me to generalize the method to prove Yoccoz inequality. Therefore the following proof is similar to the proof of Theorem 3.3 and is independent of proofs by torus.

Proof. The way to prove is to transform t-plane into s-plane by logarithm and apply Arima-Tsuji inequality on s-plane.

First we classify components of $\mathbb{C}\setminus\widetilde{K}$. We say two components are equivalent when and only when they map to each other by some iteration of $t \mapsto \lambda t$. We classify the components by the equivalence relation and let $U_1, \ldots, U_N$ be their representation.

Let $D_1, \ldots, D_N$ be components of D such that D_j is a subset of U_j. Define $t = e^s$, i.e. $s = \log t$. Let D'_j be images of D_j. Since $\widetilde{K}$ is bridged, the transformation is well-defined.

Then

$$D'_j \ni s \mapsto s + q\log\lambda - 2\pi i p \in D'_j \tag{3.1}$$

is well-defined for an appropriate branch of $\log\lambda$. In fact, see Figure 1. The left figure is t-plane, the right s-plane. In the right, D'_j and its equivalent branches are illustrated. Suppose A moves to B on t-plane by multiplying by λ^q. To move A to B on s-plane, we should add $q\log\lambda$ to s. Moreover, if we subtract $2\pi i p$ from s, B moves to B' and returns to the same component involving A.

Therefore, each D'_j is a domain distributing along a line whose direction is $\log\lambda - 2\pi i p/q$, i.e.

$$\left|\operatorname{Re} s - \frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|}|s|\right| \quad \left(s \in \bigcup D'_j\right) \tag{3.2}$$

is bounded.

On the other hand, since a circle in t-plane centered at 0 is mapped to a 2π-length segment parallel to the imaginary axis in s-plane, the line measure of $\{\operatorname{Re} s = \text{const.}\} \cap \bigcup_j D'_j$ is at most $2\pi/q$ in average. Precisely speaking, if we let l be a line measure, for any $\xi \in \mathbb{R}$

$$l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup_{j=1}^{N}\bigcup_{n=0}^{q-1}\log(\lambda^n \cdot U_j)\right) \leq 2\pi,$$

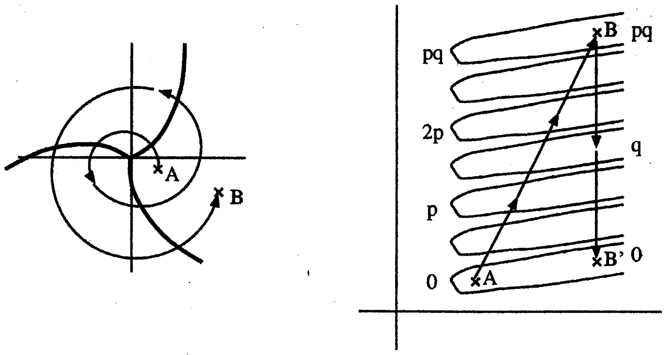

図 1: t-plane and s-plane.

because any two in $\{\lambda^n U_j\}$ never intersect for $n = 0, \dots, q-1$, $j = 1, \dots, N$. Further we employ (3.1), we can see that U_j is invariant under $\ni s \mapsto s + q\log\lambda - 2\pi i p$, so we obtain by integral

$$\begin{aligned}
2\pi \cdot q \operatorname{Re}\log\lambda &\geq \int_{\xi}^{\xi+q\operatorname{Re}\log\lambda} l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup_{j=1}^{N}\bigcup_{n=0}^{q-1} \log(\lambda^n U_j)\right) d\xi \\
&= q\int_{\xi}^{\xi+q\operatorname{Re}\log\lambda} l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup_{j=1}^{N} \log(U_j)\right) d\xi \\
&\geq q\int_{\xi}^{\xi+q\operatorname{Re}\log\lambda} l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup_{j=1}^{N} D'_j\right) d\xi.
\end{aligned}$$

Therefore

$$\frac{1}{q\operatorname{Re}\log\lambda}\int_{\xi}^{\xi+q\operatorname{Re}\log\lambda} l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup_{j=1}^{N} D'_j\right) d\xi \leq \frac{2\pi}{q}.$$

Now, we apply Arima-Tsuji inequality to $h_2(e^s)/R$ and D'_j

$$\sum_{j=1}^{N} \log\log \sup_{s\in D'_j, |s|=r}\left|\frac{h_2(e^s)}{R}\right| \geq \sum_{j=1}^{N} \pi\int_1^{\kappa r}\frac{dr}{r\theta_j^*(r)} - \text{const.} \quad (0<\kappa<1),$$

where $\theta_j(r)$ are the angle measures of D'_j, respectively. Note that $r = |s|$. We can have $\theta_j^*(r) = \theta_j(r)$ for sufficiently large r.

We compute the right side. By Schwarz's inequality, we obtain

$$N^2 = \left(\sum_{j=1}^{N}\frac{\sqrt{r\theta_j}}{\sqrt{r\theta_j}}\right)^2 \leq \left(\sum r\theta_j\right)\left(\sum\frac{1}{r\theta_j}\right),$$

$$(\kappa r - 1)^2 = \left(\int_1^{\kappa r}\frac{\sqrt{\sum r\theta_j}}{\sqrt{\sum r\theta_j}}dr\right)^2 \leq \left(\int_1^{\kappa r}\sum r\theta_j\, dr\right)\left(\int_1^{\kappa r}\frac{dr}{\sum r\theta_j}\right).$$

Therefore

$$\sum\pi\int_1^{\kappa r}\frac{dr}{r\theta_j(r)} \geq \pi N^2\int_1^{\kappa r}\frac{dr}{\sum r\theta_j(r)} \geq \frac{\pi N^2(\kappa r-1)^2}{\int_1^{\kappa r}\sum r\theta_j(r)dr}.$$

Recall (3.2) and $l\left(\{\operatorname{Re} s = \text{const.}\} \cap \bigcup D_j'\right) \le 2\pi/q$ in average. We obtain by regarding the area of $\bigcup D_j$,

$$\int_1^{\kappa r} \sum r\theta_j(r)dr \le \int_0^{\frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|}\kappa r} l\left(\{\operatorname{Re} s = \xi\} \cap \bigcup D_j'\right) d\xi + \text{const.}$$
$$\le \frac{2\pi}{q} \frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|}\kappa(r + \text{const.}).$$

Hence, the right side of Arima-Tsuji inequality can be estimated as:

$$\sum_{j=1}^{N} \pi \int_1^{\kappa r} \frac{dr}{r\theta_j^*(r)} \ge \frac{N^2 q|\log\lambda - 2\pi i p/q|}{2\operatorname{Re}\log\lambda} \frac{(\kappa r - 1)^2}{\kappa(r + \text{const.})} - \text{const.}$$

On the other hand, let us estimate the left side in the Arima-Tsuji inequality. Note that $|t| = |e^s| = e^{\operatorname{Re} s}$.

$$\sum_{j=1}^{N} \log\log \sup_{s\in D_j', |s|=r} \left|\frac{h_2(e^s)}{R}\right| \le N \log\log \sup_{s\in \bigcup D_j', |s|=r} |h_2(e^s)|$$
$$= \sup_{s\in \bigcup D_j', |s|=r} N \operatorname{Re} s \frac{\log\log|h_2(e^s)|}{\log|e^s|}.$$

We put the above inequalities together and obtain

$$\sup_{s\in \bigcup D_j', |s|=r} N \operatorname{Re} s \frac{\log\log|h_2(e^s)|}{\log|e^s|} \ge \frac{N^2 q|\log\lambda - 2\pi i p/q|}{2\operatorname{Re}\log\lambda} \frac{(\kappa r - 1)^2}{\kappa(r + \text{const.})} - \text{const.}$$

Divide the both sides by r and let $r \to \infty$, we have

$$N \frac{\operatorname{Re}\log\lambda}{|\log\lambda - 2\pi i p/q|}\rho \ge \frac{N^2 q|\log\lambda - 2\pi i p/q|}{2\operatorname{Re}\log\lambda}\kappa.$$

We employ that $\rho = \frac{\log d}{\operatorname{Re}\log\lambda}$ and that κ is arbitrary $(0 < \kappa < 1)$, we obtain

$$\frac{N\log d}{|\log\lambda - 2\pi i p/q|} \ge \frac{N^2 q|\log\lambda - 2\pi i p/q|}{2\operatorname{Re}\log\lambda}.$$

It reduces to Yoccoz inequality. □

4 Collision

Suppose a connected closed subset of K^+ meets $W^s(a)$. Then by iteration the set runs to a along $W^s(a)$ and collides with $W^u(a)$. Marks of the set will be left on the unstable manifold. The marks are subsets of K^+. In this section we investigate how the set collides. But the set does not have to intersect with $W^u(a)$ in finite time. We study the situation after infinite time passed.

4.1 Explanation

Let us describe precisely. Assume $z_0 \in W^s(a)$ is accessible from $\operatorname{int} K^+$, i.e. there exists a curve $\gamma : [0,1] \to K^+$ such that

$$\gamma(0) = z_0 \quad \text{and} \quad \gamma((0,1]) \subset \operatorname{int} K^+.$$

The z_0 runs to a along $W^s(a)$ by iteration.

On the other hand, F can be regularized at a as follows. Refer to [MNTU, Chapter 6] for example. There exists a local biholomorphic mapping Φ such that $\Phi(0) = a$ and it satisfies

$$\widetilde{F}(x,y) = \Phi^{-1} \circ F \circ \Phi(x,y) = (\lambda' x + xy\alpha(x,y), \lambda y + xy\beta(x,y)) \tag{4.1}$$

in a neighborhood of 0, where λ, λ' $(0 < |\lambda'| < 1 < |\lambda|)$ are eigenvalues of $DF(a)$ and α, β are holomorphic functions near 0. We may assume $\widetilde{F}$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}^2_{r_0}$ for some $r_0 > 0$, where $\mathbb{D}_{r_0} = \{x \in \mathbb{C} \mid |x| < r_0\}$.

It is clear that $\lim_{n\to\infty} \widetilde{F}^n(x,0) = 0$ and $\lim_{n\to\infty} \widetilde{F}^{-n}(0,y) = 0$. Therefore we have $\Phi(\{(x,0) \mid x \in \mathbb{D}_{r_0}\}) \subset W^s(a)$ and $\Phi(\{(0,y) \mid y \in \mathbb{D}_{r_0}\}) \subset W^u(a)$.

Let us study the behavior of $F^n(\gamma)$. For some $n_0 \in \mathbb{N}$, $F^{n_0}(z_0) \in \Phi(\mathbb{D}^2_{r_0})$. Define $L_j \subset \overline{\mathbb{D}}^2_{r_0}$ $(j = 0, 1, \ldots)$ as follows.

$$L_0 = \text{the component of } \Phi^{-1}(F^{n_0}(\gamma) \cap \Phi(\overline{\mathbb{D}}^2_{r_0})) \text{ containing } \Phi^{-1} \circ F^{n_0}(z_0),$$
$$L_{j+1} = \text{a component of } \widetilde{F}(L_j) \cap \overline{\mathbb{D}}^2_{r_0} \text{ intersecting with } x\text{-axis} \quad (j = 0, 1, \ldots).$$

Suppose $r_0 > 0$ is sufficiently small. By the regular form (4.1), it can be seen that L_j approaches y-axis uniformly when j tends to ∞. In fact, choose small $\varepsilon > 0$ and $r_0 > 0$ so that $|\lambda'| + r_0|\alpha(x,y)| < 1 - \varepsilon$ holds on $\overline{\mathbb{D}}^2_{r_0}$. Then

$$|\lambda' x + xy\alpha(x,y)| \leq (|\lambda'| + r_0|\alpha(x,y)|)|x| < (1-\varepsilon)|x|.$$

It reduces to the assertion. Furthermore L_j stretches, i.e. there is j_0 for any $j \geq j_0$, $\max|\pi_2(L_j)| = r_0$. It can be shown similarly.

Define $L \subset \{0\} \times \overline{\mathbb{D}}_{r_0}$ as follows.

$$z \in L \iff \liminf_{j\to\infty} d(z, L_j) = 0 \iff z \in \bigcap_{n=0}^{\infty} \overline{\bigcup_{j=n}^{\infty} L_j}.$$

It is clear that $\Phi(L) \subset K^+$. The following holds.

Proposition 4.1. *Assume a point $z_0 \in W^s(a)$ is accessible from* $\operatorname{int} K^+$. *Then L is a connected subset of $\{0\} \times \overline{\mathbb{D}}_{r_0}$. Hence $\widetilde{K}^+ = H^{-1}(K^+)$ is bridged. Therefore Yoccoz inequality holds there.*

Proof. Assume L is disconnected. Because L is a compact set contained in y-axis, the components can be separated by a closed curve Γ contained in y-axis. Take $z_1 \in L$ such that z_1 and 0 are in opposite sides of Γ each other. By definition we can choose a subsequence $\{L_{j_k}\}$ so that

$$d(z_1, L_{j_k}) \leq \frac{1}{k} d(z_1, \Gamma) \quad (k \in \mathbb{N}).$$

By the way, because $\{L_{j_k}\}$ is connected, we have

$$\left(\overline{\mathbb{D}}_{r_0} \times \pi_2(\Gamma)\right) \cap L_{j_k} \neq \emptyset$$

for any $k \in \mathbb{N}$. Therefore it can be concluded that $\Gamma \cap L \neq \emptyset$ because L_j approaches y-axis uniformly. It contradicts.

Because $\Phi(L) \subset K^+$ and $\Phi(L) \subset W^u(a)$ we have

$$H^{-1} \circ \Phi(L) \subset H^{-1}(K^+) = \widetilde{K}^+.$$

$H^{-1} \circ \Phi(L)$ is a connected set which contains 0 and is not a point. Therefore by Proposition 3.10, $\widetilde{K}^+$ is bridged, i.e. some component is not compact. □

図 2: Unstable slice for $H(x,y) = (y, y^2 - 1.37 - 0.36x)$.

4.2 Example

Example 4.2. In an example of Hénon mapping which Buff and Hubbard have given in [BxH], we show that there exists a stable manifold $W^s(a)$ such that any points on it are not accessible from $\operatorname{int} K^+$ though $W^s(a)$ is dense in $\partial \operatorname{int} K^+$. Especially a is not accessible from $\operatorname{int} K^+$. It contrasts sharply with Corollary 3.7.

The Hénon mapping is:

$$F(x,y) = (y, y^2 - 1.37 - 0.36x).$$

It has two fixed points, $x = y \doteqdot -0.674$ and $x = y \doteqdot 2.034$.

When $x = y \doteqdot -0.674$, eigenvalues λ, λ' of DF are $\lambda \doteqdot -0.980$ and $\lambda' \doteqdot -0.367$. Therefore the point is a sink. Let U be the basin of the sink. In [BS2] Bedford and Smillie have shown that $J^+ = \partial U$. Therefore $W^s(a) \subset J^+ = \partial U = \partial \operatorname{int} K^+$.

When $x = y \doteqdot 2.034$, $\lambda \doteqdot 3.977$, $\lambda' \doteqdot 0.091$. Therefore the point is of saddle type. On the other hand, the *order* is:

$$\rho = \frac{\log d}{\log|\lambda|} \doteqdot 0.502 > \frac{1}{2}.$$

Yoccoz inequality doesn't decide whether $\widetilde{K}^+ = H^{-1}(K^+)$ is bridged or not.

But Buff and Hubbard say that the $\widetilde{K}^+$ is not connected according to computer graphics. As far as the author watches the picture, it seems not bridged. See figure 2. The right vertex is the origin.

Therefore by Proposition 4.1, every point on $W^s(a)$ is not accessible from $\operatorname{int} K^+$. Moreover in [BS2] it has been shown that $\overline{W^s(a)} = J^+$.

参考文献

[B] Boas R. P., Jr., Entire Functions, Academic Press Inc., Publishers (1954).

[BxH] Buff X., Hubbard J. H., Yoccoz inequality for Hénon mappings., Cornell University.

[BS1] Bedford E., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity, Invent. math. 103 (1991), 69–99.

[BS2] Bedford E., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$. II: Stable manifolds and recurrence, Journal of the American Mathematical Society, 4 (1991), 657–679.

[EL] Erëmenko A. È, Levin G. M., PERIODIC POINTS OF POLYNOMIALS, Ukrainian Math. J. 41 (1989), 1258–1262.

[FM] Friedland S., Milnor J., Dynamical properties of plane polynomial automorphisms, Ergod. Th. & Dynam. Sys. 9 (1989), 67–99.

[H] Hubbard J. H., The Hénon mapping in the complex domain, Chaotic Dynamics and Fractals, Academic Press (1986), 101–111.

[Ka] Katok A., Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publ. Math. IHES. 51 (1980), 137–174.

[Km] Klimek M., Pluripotential Theory, Clarendon Press (1991).

[MNTU] Morosawa S., Nishimura Y., Taniguchi M., Ueda T., Holomorphic dynamics, preprint.

[Nt] Nishino T., Tahensuu Kansuuron (Theory of Function of Several Complex Variables), University of Tokyo Press (1996) (in Japanese).

[Ny] Nishimura Y., $\mathbb{C}^2$ no takoushiki jiko doukei no tsukuru rikigakukei: Potential ron teki houhou. (Dynamics of Polynomial Automorphisms in $\mathbb{C}^2$: The way of Potential Theory), Tahensuu no fukuso rikigakukei (Holomorphic Dynamics of Several Variables), Topics in Complex Analysys (1993), 23–69 (in Japanese).

[O] Ozawa M., Kindai Kansuuron I: Atai bunpu riron (The Modern Function Theory I: Value Distribution Theory), Morikita (1976) (in Japanese).

[P] Petersen C. L., On the Pommerenke-Levin-Yoccoz inequality, Ergod. Th. & Dynam. Sys. (1993), 13, 785–806.

[T] Tsuji M., Potential Theory, Maruzen (1959).