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GENERALIZED WHITTAKER MODELS AND $\mathfrak{n}$-HOMOLOGY FOR SOME SMALL IRREDUCIBLE REPRESENTATIONS OF SIMPLE LIE GROUPS

HIROSHI YAMASHITA (山下 博)

1. INTRODUCTION

Let G be a connected simple linear Lie group, and let K be a maximal compact subgroup of G. We denote by $G_{\mathbb{C}}, K_{\mathbb{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) the complexifications of G, K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$, and let θ denote the corresponding Cartan involution of $\mathfrak{g}$. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$).

We assume the Harish-Chandra rank condition $\operatorname{rank} G = \operatorname{rank} K$, which is necessary and sufficient for G to have the irreducible unitary representations of discrete series. Then, the Borel-de Siebenthal Theorem says that

Theorem 1.1 (cf. Rubenthaler [21, Th.3.1], Knapp [15, Th.6.96]). *The Lie algebra $\mathfrak{g}$ admits a θ-stable gradation*

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2) \tag{1.1}$$

with the following properties (a)–(c).

(a) $\mathfrak{k} = \oplus_{j:even}\, \mathfrak{g}(j)$ *and* $\mathfrak{p} = \oplus_{j:odd}\, \mathfrak{g}(j)$,

(b) $\mathfrak{q} := \oplus_{j\geq 0}\, \mathfrak{g}(j)$ *is a maximal parabolic subalgebra of* $\mathfrak{g}$, *and one has* $\overline{\mathfrak{g}(j)} = \mathfrak{g}(-j)$, *where* $\bar{\cdot}$ *denotes the complex conjugation of* $\mathfrak{g}$ *with respect to the real form* $\mathfrak{g}_0$.

(c) *The subspaces $\mathfrak{g}(\pm 2)$ vanish if and only if the Lie algebra $\mathfrak{k}$ is not semisimple but reductive. This occurs exactly when the symmetric space $K\backslash G$ is Hermitian. In this case, the triangular decomposition $\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$ with $\mathfrak{p}_\pm := \mathfrak{g}(\pm 1)$ and $\mathfrak{k} = \mathfrak{g}(0)$ comes from the unique (up to sign) G-invariant complex structure on $K\backslash G$ in the canonical way.*

The purpose of this paper is to describe the generalized Whittaker models and the 0th $\mathfrak{n}$-homology spaces for Harish-Chandra modules of some small irreducible G-representations which are closely related to the above gradation (1.1) of $\mathfrak{g}$. To be more specific, we are concerned with the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ, when G is of Hermitian type (Case H). Such an $L(\tau)$ is, by construction, the unique simple quotient of a generalized Verma module induced from $\mathfrak{q} = \mathfrak{k} + \mathfrak{p}_+$. Also, when G is of quaternionic type (Case Q), the Borel-de Siebenthal discrete series $(\mathfrak{g}, K)$-modules $\boldsymbol{X}_\Lambda$ are studied. Here the Harish-Chandra parameter Λ of $\boldsymbol{X}_\Lambda$ lies in the open Weyl chamber defined by a Borel subalgebra contained in $\mathfrak{q}$.

Now let us explain the results of this paper in more detail.

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Case H. Assume that G is of Hermitian type. Let $\{\mathcal{O}_m \mid m = 0, 1, \ldots, r\}$ be the totality of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_+ = \mathfrak{g}(1)$ arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$. Following the recipe by Kawanaka [14] (see also [30, II]), we can construct a generalized Gelfand-Graev representation $\Gamma_m = \mathrm{Ind}_{\mathfrak{n}(m)}^{G}(\eta_m)$ (GGGR for short; see Definition 5.3) attached to the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponding to each $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ through the Kostant-Sekiguchi bijection.

Our aim in Case H is to study the generalized Whittaker models, i.e., the $(\mathfrak{g}, K)$-embeddings of highest weight modules $L(\tau)$ into these GGGRs Γ_m. This is a continuation of our earlier work [31] on Whittaker models for holomorphic discrete series.

If G is one of the classical groups $Sp(2n, \mathbb{R})$, $U(p,q)$ and $O^*(2p)$, the theory of reductive dual pair gives explicit realizations of unitarizable highest weight modules $L(\tau)$ ([13], [6]). It is not difficult to describe the generalized Whittaker models for such $L(\tau)$'s by using the Segel-Shale-Weil representation. For this, see [24] and [34].

Our emphasis in this article is placed on an intrinsic understanding of the embeddings $L(\tau) \hookrightarrow \Gamma_m$ for arbitrary $L(\tau)$. To specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 3.3). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke ([2],[3],[4]), and its K-finite kernel realizes the dual lowest weight module $L(\tau)^*$. By virtue of the kernel theorem given as Corollary 2.6, we find that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions F of differential equation $\mathcal{D}_{\tau^*}F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question (see (5.16)), where η_m is the character of nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$ that defines our GGGR Γ_m.

The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by using the unbounded realization of $K\backslash G$ via a Cayley transform on $G_{\mathbb{C}}$, and also by using some remarkable results of Enright and Joseph [5], Jakobsen [17] and Vogan [26]. As a result, we get the following conclusions **(A)** and **(B)** (see Theorem 5.6–5.8).

(A) *$L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. In this case, the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ consists only of elementary functions on the unbounded domain $\mathcal{S}$ $(\subset \mathfrak{p}_-)$ which realizes $K\backslash G$.*

(B) *If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau)$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K(X(m(\tau)))$ of $K_{\mathbb{C}}$ at a point $X(m(\tau)) \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$, that is, the multiplicity of embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$, coincides with the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$.*

The last statement in **(B)** clarifies the relationship between the generalized Whittaker models and the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of unitarizable $L(\tau)$. For the classical groups, the latter $\mathcal{AC}(L(\tau))$ and the Bernstein degree have been described by Nishiyama, Ochiai and Taniguchi [20] through detailed study of K-types of $L(\tau)$.

Case Q. Next, let G a connected simple linear Lie group of quaternionic type, which is not of type type AIII (purely from technical reason). Assume for simplicity that G admits the simply connected complexification $G_{\mathbb{C}}$. Let $G = KA_{\mathfrak{p}}N$ be an Iwasawa decomposition of G, and let $P_0 = M_0A_{\mathfrak{p}}N$ be a Langlands decomposition of the identity component P_0 of a minimal parabolic subgroup of G. We write $\mathfrak{n}$ for the complexified Lie algebra of N.

We describe the 0th $\mathfrak{n}$-homology space $H_0(\mathfrak{n}, \Lambda) = \boldsymbol{X}_\Lambda/\mathfrak{n}\boldsymbol{X}_\Lambda$, or equivalently the embeddings into the principal series, of the Borel-de Siebenthal discrete series $\boldsymbol{X}_\Lambda$, by using the Schmid differential operator whose kernel realizes the maximal globalization of dual $(\mathfrak{g}, K)$-module $\boldsymbol{X}_\Lambda^*$ (see also the related works [32] and [35]).

With the Zuckerman translation principle in mind, we can concentrate on the quaternionic discrete series $\boldsymbol{X}_{c\delta+\rho}$ (Definition 6.1) with lowest K-type arising from an irreducible representation of a simple factor of K of type A_1. Then, $M_0A_\mathfrak{p}$-module structure of $H_0(\mathfrak{n}, c\delta+\rho)$ is explicitly determined in Theorem 6.3. We find in particular that the space $H_0(\mathfrak{n}, c\delta+\rho)$ has exactly two exponents if the real rank of G is at least two.

The organization of this paper is as follows.

Section 2 gives general theory on the embeddings of irreducible $(\mathfrak{g}, K)$-modules into induced G-representations. The kernel theorem (Corollary 2.6) is our main tool for studying generalized Whittaker models and $\mathfrak{n}$-homology spaces.

Sections 3–5 deal with the groups G of Hermitian type. We introduce in Section 3 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [4]. In addition, the solutions F of $\mathcal{D}_{\tau^*}F = 0$ of exponential type are specified in Proposition 3.7. Section 4 is devoted to to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 4.9). In Section 5 we give our main results in Case H (Theorems 5.6–5.8) that describe the generalized Whittaker models for highest weight modules $L(\tau)$.

Last in Section 6, we specify the 0th $\mathfrak{n}$-homology spaces of the Borel-de Siebenthal discrete series $(\mathfrak{g}, K)$-modules $\boldsymbol{X}_\Lambda$, when G is of quaternionic type (Case Q).

The detail of this article with complete proofs will appear elsewhere.

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2. Embeddings of Harish-Chandra modules

This section prepares some generalities about the embeddings of irreducible Harish-Chandra modules into C^∞-induced representations of a semisimple Lie group, by developing our earlier observation [32, I, §2] for the discrete series in full generality. The results stated in this section seem to be more or less folklore for the experts, or they are consequences of some known facts on the maximal globalization of Harish-Chandra modules (cf. [23], [12]). We will use the kernel theorem (Corollary 2.6) in the succeeding sections to specify the generalized Whittaker models and $\mathfrak{n}$-homology spaces.

2.1. **A duality of Peter-Weyl type.** Throughout this section, let G be any connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We employ the notation at the beginning of Introduction.

A $U(\mathfrak{g})$-module $\boldsymbol{X}$ is called a *$(\mathfrak{g}, K)$-module* if the subalgebra $U(\mathfrak{k})$ acts on $\boldsymbol{X}$ locally finitely, and if the $\mathfrak{k}_0$-action gives rise to a representation of K on $\boldsymbol{X}$ through exponential map. By a *Harish-Chandra module* is meant a $(\mathfrak{g}, K)$-module of finite length as a $U(\mathfrak{g})$-module. By basic results of Harish-Chandra (see e.g., [28, Chap.3]), any admissible (i.e., K-multiplicity finite) representation of G on a Hilbert space $\boldsymbol{H}$ yields, through differentiation, a $(\mathfrak{g}, K)$-module structure on the subspace $\boldsymbol{H}_K$ of all K-finite vectors in $\boldsymbol{H}$. The continuous G-module $\boldsymbol{H}$ is irreducible if and only if the corresponding $\boldsymbol{H}_K$ is irreducible as a $(\mathfrak{g}, K)$-module. Each irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ can be extended to an irreducible Hilbert G-module $\boldsymbol{H}$ with K-finite part $\boldsymbol{H}_K = \boldsymbol{X}$. Notice that the $(\mathfrak{g}, K)$-module corresponding to the irreducible G-module $\boldsymbol{H}^*$ contragredient to $\boldsymbol{H}$ is isomorphic to the K-finite part of the full dual space $\boldsymbol{X}' = \mathrm{Hom}_{\mathbb{C}}(\boldsymbol{X}, \mathbb{C})$. We denote this irreducible $(\mathfrak{g}, K)$-module by $\boldsymbol{X}^*$, and call it the dual Harish-Chandra module of $\boldsymbol{X}$.

We study in this paper the embeddings of irreducible $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ into certain smoothly induced Fréchet G-modules $\boldsymbol{F}$. Such an $\boldsymbol{F}$ has a compatible $\mathfrak{g}$ and K module structure through differentiation, and its K-finite part $\boldsymbol{F}_K$ is a $(\mathfrak{g}, K)$-module. We note that the image of $\boldsymbol{X}$ by any $\mathfrak{g}$ and K homomorphism into $\boldsymbol{F}$ is necessarily contained in $\boldsymbol{F}_K$, i.e., $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}) = \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}_K)$.

The group G acts on the space $C^\infty(G)$ of all smooth functions on G by left translation and by right translation as follows:

$$g^L f(x) := f(g^{-1}x), \quad g^R f(x) := f(xg) \quad (g \in G, x \in G, f \in C^\infty(G)). \tag{2.1}$$

These two actions L and R commute with each other. Through differentiation one gets two $U(\mathfrak{g})$-representations on $C^\infty(G)$ denoted again by L and R respectively. Let $C_K^\infty(G)$ be the space of functions $f \in C^\infty(G)$ which are left K-finite and also right K-finite. Then $C_K^\infty(G)$ becomes a $(\mathfrak{g}, K)$-module through L or R.

The following lemma is well-known. It gives a duality of Peter-Weyl type for irreducible Harish-Chandra modules of noncompact semisimple Lie groups.

Lemma 2.1. *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let f be in $C_K^\infty(G)$. Then the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})^L f$ generated by f through L is isomorphic to $\boldsymbol{X}$ if and only if the corresponding $U(\mathfrak{g})^R f$ through R is isomorphic to $\boldsymbol{X}^*$.*

We give a proof below, introducing some important notion used in this paper.

Proof of Lemma 2.1. Let us prove the *if* part only since the converse can be proved in the same way. So, assume that $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$ as $(\mathfrak{g}, K)$-modules.

Take a finite-dimensional K-module (τ, V_τ) which is isomorphic to $U(\mathfrak{k})^L f$. Let $i : V_\tau \xrightarrow{\sim} U(\mathfrak{k})^L f$ be a K-isomorphism. We define a V_τ^*-valued smooth function F on G by

$$\langle F(g), v \rangle = i(v)(g) \quad (v \in V_\tau,\ g \in G), \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural dual pairing on $V_\tau^* \times V_\tau$. Then it is immediate to verify that F lies in the following space:

$$C_{\tau^*}^\infty(G) := \{\Phi : G \xrightarrow{C^\infty} V_\tau^* \mid \Phi(kg) = \tau^*(k)\Phi(g) \ (g \in G,\ k \in K)\}. \tag{2.3}$$

Here (τ^*, V_τ^*) denotes the representation of K contragredient to τ. The space $C_{\tau^*}^\infty(G)$ has G- and $U(\mathfrak{g})$-module structures through right translation R. The function F is in the K-finite part, say $C_{\tau^*}^\infty(G)_K$, of $C_{\tau^*}^\infty(G)$ since $U(\mathfrak{k})^L f \subset C_K^\infty(G)$. By definition we see

$$f(g) = \langle F(g), i^{-1}(f) \rangle. \tag{2.4}$$

Now the assignment $D^R F \mapsto D^R f = \langle D^R F(\cdot), i^{-1}(f) \rangle$ $(D \in U(\mathfrak{g}))$ gives a $(\mathfrak{g}, K)$-homomorphism from $U(\mathfrak{g})^R F$ onto $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$. We see that this homomorphism is injective. Thus we have found a $(\mathfrak{g}, K)$-module embedding, say A_0, from $\boldsymbol{X}^*$ into $C_{\tau^*}^\infty(G)_K$ whose image equals $U(\mathfrak{g})^R F$.

Let $(\pi, \boldsymbol{H})$ be an irreducible admissible G-representation with Harish-Chandra module $\boldsymbol{X}$, and $(\pi^*, \boldsymbol{H}^*)$ be the representation of G contragredient to π. We have $\boldsymbol{H}_K^* = \boldsymbol{X}^*$ as remarked before. By virtue of the Frobenius reciprocity for smoothly induced representation $\mathrm{Ind}_K^G(\tau^*)$ of G acting on $C_{\tau^*}^\infty(G)$, one obtains a linear isomorphism

$$\mathrm{Hom}_K(\boldsymbol{X}^*,\ V_\tau^*) \simeq \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}^*,\ C_{\tau^*}^\infty(G)_K), \tag{2.5}$$

which is given as follows. Take a K-homomorphism $T : \boldsymbol{X}^* \to V_\tau^*$. Then we can define $A(\varphi) \in C_{\tau^*}^\infty(G)$ for every $\varphi \in \boldsymbol{X}^*$ by

$$A(\varphi)(g) = \tilde{T}(\pi^*(g)\varphi) \quad (g \in G). \tag{2.6}$$

Here $\tilde{T}$ denotes the unique continuous extension of $T : \boldsymbol{X}^* \to V_\tau^*$ to $\boldsymbol{H}^*$. Then, the assignment $T \mapsto A$ gives (2.5).

We now consider our specified embedding $A_0 : \boldsymbol{X}^* \simeq U(\mathfrak{g})^R F \hookrightarrow C_{\tau^*}^\infty(G)_K$. Let T_0 denote the element of $\mathrm{Hom}_K(\boldsymbol{X}^*, V_\tau^*)$ corresponding to A_0 by (2.6). Set $\varphi_0 := A_0^{-1}(F) \in \boldsymbol{X}^*$ and $\psi_0 := i^{-1}(f) \circ \tilde{T}_0 \in \boldsymbol{X} = ((\boldsymbol{H}^*)^*)_K$, where

$$\psi_0 : \boldsymbol{H}^* \xrightarrow{\tilde{T}_0} V_\tau^* \xrightarrow{i^{-1}(f)} \mathbb{C} \tag{2.7}$$

with $i^{-1}(f) \in V_\tau = \mathrm{Hom}_{\mathbb{C}}(V_\tau^*, \mathbb{C})$. In view of (2.4) and (2.6) we find

$$f(g) = \langle \pi^*(g)\varphi_0, \psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} = \langle \varphi_0, \pi(g)^{-1}\psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} \quad (g \in G) \tag{2.8}$$

Finally, (2.8) implies that the map

$$\boldsymbol{X} \ni D\psi_0 \mapsto D^L f = \langle \varphi_0, \pi(g)^{-1} D\psi_0 \rangle \in U(\mathfrak{g})^L f \quad (D \in U(\mathfrak{g})) \tag{2.9}$$

gives a $(\mathfrak{g}, K)$-isomorphism, i.e., $\boldsymbol{X} \simeq U(\mathfrak{g})^L f$ as desired. □

2.2. **Maximal globarization.** Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module. We fix once and for all an irreducible finite-dimensional representation (τ, V_τ) of K which occurs in $\boldsymbol{X}$, and fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules. Then the adjoint operator i_τ^* of i_τ gives a surjective K-homomorphism from $\boldsymbol{X}^*$ to V_τ^*. We denote by A_{τ^*} the $(\mathfrak{g}, K)$-embedding from $\boldsymbol{X}^*$ into $C_{\tau^*}^\infty(G)$ (see (2.3)) corresponding to i_τ^* through (2.5) and (2.6).

Equip $C_{\tau^*}^\infty(G)$ with a Fréchet space topology of compact uniform convergence of functions on G and each of their derivatives. The following proposition characterizes the closure $A_{\tau^*}(\boldsymbol{X}^*)^-$ of $A_{\tau^*}(\boldsymbol{X}^*)$ in $C_{\tau^*}^\infty(G)$.

Theorem 2.2 (cf. [23], [12]). *Under the above notation, $A_{\tau^*}(\boldsymbol{X}^*)^-$ is a G-submodule of $C_{\tau^*}^\infty(G)$, and one gets an isomorphism of G-modules*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \ni W \overset{\sim}{\longmapsto} F \in A_{\tau^*}(\boldsymbol{X}^*)^- \tag{2.10}$$

through

$$\langle F(g), v \rangle = ((W \circ i_\tau)(v))(g) \quad (g \in G,\ v \in V_\tau). \tag{2.11}$$

Here $C^\infty(G)$ is viewed as a smooth G-module by left translation L, and the right action R on $C^\infty(G)$ naturally gives a G-module structure on $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$.

We can prove this theorem by using Lemma 2.1.

It follows essentially from [23, page 316] that the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ gives a *maximal globalization* of the Harish-Chandra module $\boldsymbol{X}^*$. Namely, if a complete, locally convex Hausdorff topological vector space $\boldsymbol{F}$ admits a continuous G-action with underlying Harish-Chandra module $\boldsymbol{X}^*$, then the identity map on $\boldsymbol{X}^*$ extends uniquely to a continuous embedding $\boldsymbol{F} \hookrightarrow A_{\tau^*}(\boldsymbol{X}^*)^-$ as G-modules.

2.3. **Kernel theorem.** To study the embeddings of $\boldsymbol{X}$ into various induced G-modules, it is useful to characterize the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ as the full kernel space of a continuous G-homomorphism $\mathcal{D}$ defined on $C^\infty_{\tau^*}(G)$ in the following way.

Theorem 2.3. *Keep the notation in 2.2. If $\mathcal{D}$ is any continuous G-homomorphism from the $C^\infty_{\tau^*}(G)$ to a smooth Fréchet G-module $\boldsymbol{M}$ such that*

$$A_{\tau^*}(\boldsymbol{X}^*) = \{F \in C^\infty_{\tau^*}(G) \mid F \text{ is right } K\text{-finite and } \mathcal{D}F = 0\}, \tag{2.12}$$

then the full kernel space $\operatorname{Ker}\mathcal{D}$ *of $\mathcal{D}$ in $C^\infty_{\tau^*}(G)$ coincides with the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$, the closure of $A_{\tau^*}(\boldsymbol{X}^*)$ in $C^\infty_{\tau^*}(G)$. Hence one finds from Theorem 2.2*

$$\operatorname{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \simeq \operatorname{Ker}\mathcal{D} = A_{\tau^*}(\boldsymbol{X}^*)^- \quad \textit{as } G\textit{-modules.} \tag{2.13}$$

Example 2.4. We mention that an operator $\mathcal{D}$ satisfying the requirement in Theorem 2.3 has been constructed when $\boldsymbol{X}^*$ is the $(\mathfrak{g}, K)$-modules associated with: (a) discrete series ([22], [11]) more generally Zuckerman cohomologically induced module ([29], [1]), with parameter "far from the walls", or (b) highest weight module ([2], [4]; see also Definition 3.3). In each of these cases, $\mathcal{D}$ is given as a *G-invariant differential operator of gradient type* on $C^\infty_{\tau^*}(G)$, where τ^* is the unique extreme K-type of $\boldsymbol{X}^*$.

We conclude this section by giving an application of Theorem 2.3. For this we need

Definition 2.5. Let $\mathfrak{n}$ be a complex Lie subalgebra of $\mathfrak{g}$, and $(\eta, \boldsymbol{E})$ be a representation of $\mathfrak{n}$ on a Fréchet space $\boldsymbol{E}$ such that the linear endomorphism $\eta(Z)$ is continuous on $\boldsymbol{E}$ for every $Z \in \mathfrak{n}$. Then the space

$$C^\infty(G;\eta) := \{f : G \xrightarrow{C^\infty} \boldsymbol{E} \mid Z^R f = -\eta(Z)f \quad (Z \in \mathfrak{n})\}, \tag{2.14}$$

endowed with the natural Fréchet space topology, has a structure of smooth G-module by L. We write Γ_η for the resulting G-representation on $C^\infty(G;\eta)$, and call it the *representation of G induced from η in C^∞-context.*

Let the notation and assumption be as in Theorem 2.3 and in Definition 2.5. We write $C^\infty_{\tau^*}(G;\eta)$ for the space of C^∞-functions on G with values in $V_\tau^* \otimes \boldsymbol{E}$ such that

$$\begin{aligned} Z^R F &= -(\mathrm{id}_{V_\tau^*} \otimes \eta(Z))F \quad (Z \in \mathfrak{n}) \quad \text{and} \\ k^L F &= (\tau^*(k^{-1}) \otimes \mathrm{id}_{\boldsymbol{E}})F \quad (k \in K), \end{aligned} \tag{2.15}$$

where id_V denotes the identity map on a set V. We define a linear map

$$\mathcal{D}_\eta : C^\infty_{\tau^*}(G;\eta) \longrightarrow \operatorname{Hom}_{\mathbb{C}}(\boldsymbol{E}', \boldsymbol{M}) \tag{2.16}$$

through $\mathcal{D}$ by

$$(\mathcal{D}_\eta F)(\zeta) = \mathcal{D}(\langle F(\cdot), \zeta\rangle) \quad (F \in C^\infty_{\tau^*}(G;\eta),\ \zeta \in \boldsymbol{E}'). \tag{2.17}$$

Here $\boldsymbol{E}'$ denotes the space of continuous linear functionals on $\boldsymbol{E}$ equipped with dual $U(\mathfrak{n})$-action, and $\langle\,\cdot\,,\,\cdot\,\rangle$ the canonical dual pairing on $(V_\tau^* \otimes \boldsymbol{E}) \times \boldsymbol{E}'$ with values in V_τ^*. If η is a one-dimensional $\mathfrak{n}$-representation, the above $\mathcal{D}_\eta$ is naturally identified with the restriction of $\mathcal{D}$ to the subspace $C^\infty_{\tau^*}(G;\eta)$ of $C^\infty_{\tau^*}(G)$.

By using (2.13), we can deduce the following

Corollary 2.6 (Kernel Theorem). *Under the above notation, assume that the representation $(\eta, \boldsymbol{E})$ of $\mathfrak{n}$ is weakly cyclic in the following sense: there exists a $\zeta_0 \in \boldsymbol{E}'$ such that $U(\mathfrak{n})\zeta_0$ is dense in $\boldsymbol{E}'$ with respect to the weak $*$-topology. Then the embeddings of irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ into induced module $C^\infty(G;\eta)$ are characterized as*

$$\operatorname{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, C^\infty(G;\eta)) \simeq \operatorname{Ker}\mathcal{D}_\eta \quad \textit{as vector spaces.} \tag{2.18}$$

Here the isomorphism is given as in (2.11).

Remark 2.7. The above kernel thoerem has been proved in our earlier work [32, I, Th.2.4] in case that $\boldsymbol{X}$ is the $(\mathfrak{g}, K)$-module of discrete series and that $\mathcal{D}$ is a differential operator of gradient type (Schmid operator).

3. Differential operators, and lowest or highest weight modules

Until the end of Section 5, let G be a connected, simple linear Lie group such that $K\backslash G$ is a Hermitian symmetric space. We consider the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ. In this section we describe, following [4], the differential operators $\mathcal{D}_{\tau^*}$ of gradient type on $K\backslash G$ whose K-finite kernels realize the dual lowest weight $(\mathfrak{g}, K)$-modules $L(\tau)^*$ (Theorem 3.5). This combined with Theorem 2.3 enables us to specify the maximal globalization of $L(\tau)^*$ as the full (not necessarily K-finite) kernel space of $\mathcal{D}_{\tau^*}$ (Proposition 3.6).

3.1. **Simple Lie group of Hermitian type.** We begin with summarizing some basic facts on fine structure for simple Lie groups of Hermitian type, following the notation in [31, Part I, §5] and [9, 3.3]. Fix a complexification $G_{\mathbb{C}}$ of G, and the analytic subgroup $K_{\mathbb{C}}$ of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Then there exists a unique (up to sign) central element Z_0 of $\mathfrak{k}_0$ such that $\operatorname{ad} Z_0$ restricted to $\mathfrak{p}_0$ gives an $\operatorname{Ad}(K)$-invariant complex structure on $\mathfrak{p}_0$. One gets a triangular decomposition (cf. Theorem 1.1) of $\mathfrak{g}$ as follows:

$$\begin{aligned} &\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{such that} \\ &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \end{aligned} \tag{3.1}$$

where $\mathfrak{p}_\pm$ denotes the eigenspace of $\operatorname{ad} Z_0$ on $\mathfrak{g}$ with eigenvalue $\pm\sqrt{-1}$ respectively.

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and for each $\gamma \in \Delta$ the corresponding root subspace of $\mathfrak{g}$ will be denoted by $\mathfrak{g}(\mathfrak{t}; \gamma)$. We can choose root vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ $(\gamma \in \Delta)$ such that

$$X_\gamma - X_{-\gamma},\ \sqrt{-1}(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma, \tag{3.2}$$

where H_γ is the element of $\sqrt{-1}\mathfrak{t}_0$ corresponding the coroot $\gamma^\vee := 2\gamma/(\gamma,\gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B of $\mathfrak{g}$. Let Δ_c (resp. Δ_n) denote the subset of all compact (resp. noncompact) roots in Δ.

Take a positive system Δ^+ of Δ compatible with the decomposition (3.1), and fix a lexicographic order on $\sqrt{-1}\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \ldots, \gamma_r$. Then r equals the real rank of G.

Now, put $\mathfrak{t}^- := \sum_{k=1}^r \mathbb{C}H_{\gamma_k} \subset \mathfrak{t}$, and denote by $\gamma^- \in (\mathfrak{t}^-)^*$ the restriction to $\mathfrak{t}^-$ of a linear form $\gamma \in \mathfrak{t}^*$. For integers k, l with $1 \leq l < k \leq r$, we define subsets P_{kl}, P_k, P_0 of

Δ_n^+ and subsets C_{kl}, C_k, C_0 of Δ_c^+ respectively by

$$(3.3) \qquad P_{kl} := \left\{ \gamma \in \Delta_n^+ \middle| \ \gamma^- = \left(\frac{\gamma_k + \gamma_l}{2} \right)^- \right\},$$

$$(3.4) \qquad C_{kl} := \left\{ \gamma \in \Delta_c^+ \middle| \ \gamma^- = \left(\frac{\gamma_k - \gamma_l}{2} \right)^- \right\},$$

$$(3.5) \qquad P_k := \left\{ \gamma \in \Delta_n^+ \middle| \ \gamma^- = \left(\tfrac{\gamma_k}{2} \right)^- \right\}, \quad C_k := \left\{ \gamma \in \Delta_c^+ \middle| \ \gamma^- = \left(\tfrac{\gamma_k}{2} \right)^- \right\},$$

$$(3.6) \qquad P_0 := \{\gamma_1, \gamma_2, \ldots, \gamma_r\}, \quad C_0 := \{\gamma \in \Delta_c^+ \mid \gamma^- = 0\}.$$

By Harish-Chandra the subsets Δ_n^+ and Δ_c^+ are decomposed as

$$(3.7) \qquad \begin{aligned} \Delta_n^+ &= (\bigcup_{1 \le k \le r} P_k) \bigcup P_0 \bigcup (\bigcup_{1 \le l < k \le r} P_{kl}), \\ \Delta_c^+ &= C_0 \bigcup (\bigcup_{1 \le k \le r} C_k) \bigcup (\bigcup_{1 \le l < k \le r} C_{kl}), \end{aligned}$$

where the unions are disjoint. We denote by $\mathbf{c} = \mathrm{Ad}(c)$ a *Cayley transform* on $\mathfrak{g}$ defined by the element:

$$(3.8) \qquad c = \exp\left(\frac{\pi}{4} \cdot \sum_{k=1}^{r} (X_{\gamma_k} - X_{-\gamma_k}) \right) \in G_{\mathbb{C}}.$$

3.2. **Generalized Verma module and its maximal submodule.** Let (τ, V_τ) be any irreducible finite-dimensional representation of K with Δ_c^+-highest weight $\lambda = \lambda(\tau)$. We consider the *generalized Verma* $U(\mathfrak{g})$*-module* induced from τ:

$$(3.9) \qquad M(\tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau.$$

Then $M(\tau)$ has a structure of $(\mathfrak{g}, K)$-module. Let $N(\tau)$ be the unique maximal proper $(\mathfrak{g}, K)$-submodule of $M(\tau)$. Then the quotient $L(\tau) := M(\tau)/N(\tau)$ gives an irreducible $(\mathfrak{g}, K)$-module with Δ^+-highest weight λ.

We now summarize for later use some basic facts on the structure of $N(\tau)$.

One finds from the decomposition (3.1) that $M(\tau) = U(\mathfrak{p}_-)V_\tau$ is canonically isomorphic to the tensor product $S(\mathfrak{p}_-) \otimes V_\tau = S(\mathfrak{p}_-) \otimes_{\mathbb{C}} V_\tau$ as a K-module, where $S(\mathfrak{p}_-)$ $(\simeq U(\mathfrak{p}_-))$ denotes the symmetric algebra of $\mathfrak{p}_-$ looked upon as a K-module by the adjoint action. This isomorphism yields a natural gradation of the K-module $M(\tau)$:

$$(3.10) \qquad M(\tau) = \bigoplus_{j=0}^{\infty} M_j(\tau) \quad \text{with} \quad M_j(\tau) := S^j(\mathfrak{p}_-)V_\tau \simeq S^j(\mathfrak{p}_-) \otimes V_\tau.$$

Here we write $S^j(\mathfrak{p}_-)$ for the K-submodule of $S(\mathfrak{p}_-)$ consisting of all homogeneous elements of $S(\mathfrak{p}_-)$ of degree j. Note that the submodule $N(\tau)$ is graded:

$$(3.11) \qquad N(\tau) = \bigoplus_{j=0}^{\infty} N_j(\tau) \quad \text{with} \quad N_j(\tau) := N(\tau) \cap M_j(\tau).$$

Since $M(\tau) = S(\mathfrak{p}_-)V_\tau$ is finitely generated over the Noetherian ring $S(\mathfrak{p}_-)$, so is the submodule $N(\tau)$, too. This implies that, if $N(\tau) \neq \{0\}$, there exist finitely many irreducible K-submodules $W_1, \dots, W_q$ of $N(\tau)$ such that

$$N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{3.12}$$

for some positive integers i_u $(u = 1, \dots, q)$ arranged as

$$i(\tau) := i_1 = \min\{j | \; N_j(\tau) \neq \{0\}\}. \tag{3.13}$$

We call $i(\tau)$ *the level of reduction* of $M(\tau)$.

An irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ is called *unitarizable* if $\boldsymbol{X}$ is isomorphic to the Harish-Chandra module $\boldsymbol{H}_K$ of an irreducible unitary representation of G on a Hilbert space $\boldsymbol{H}$. For unitarizable $L(\tau)$'s, Enright and Joseph [5] gives a simple description of the maximal submodule $N(\tau)$ as follows. Assume that $L(\tau)$ is unitarizable and that $N(\tau) \neq \{0\}$. Then the level $i(\tau)$ of reduction of $M(\tau)$ is an integer such that $1 \leq i(\tau) \leq r$. Let $Q_{i(\tau)}$ be the irreducible K-submodule of $S^{i(\tau)}(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \dots - \gamma_{r-i(\tau)+1}$. Then the tensor product $Q_{i(\tau)} \otimes V_\tau$ has a unique irreducible K-submodule W_1, called the PRV-component, with extreme weight $\lambda - \gamma_r - \dots - \gamma_{r-i(\tau)+1}$. Noting that

$$Q_{i(\tau)} \otimes V_\tau \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau \simeq M_{i(\tau)}(\tau), \tag{3.14}$$

we regard W_1 as a K-submodule of $M_{i(\tau)}(\tau)$.

Theorem 3.1 ([5, 5.2, 6.5 and 8.3], see also [3, 3.1]). *The maximal submodule $N(\tau)$ of $M(\tau)$ is a highest weight $(\mathfrak{g}, K)$-module generated over $S(\mathfrak{p}_-)$ by the PRV-component W_1.*

3.3. **A realization of lowest weight module $L(\tau)^*$.** For each irreducible representation (τ, V_τ) of K, let $L(\tau)^*$ be the irreducible lowest weight $(\mathfrak{g}, K)$-module which is dual to $L(\tau)$. This subsection gives a realization of $L(\tau)^*$ as the K-finite kernel of a certain G-invariant differential operator of gradient type defined on the symmetric space $K\backslash G$.

Now, let $\overline{O}_{\tau^*}(G)$ denote the space of functions F in $C^\infty_{\tau^*}(G)$ (see (2.3)) satisfying

$$X^L F = 0 \quad \text{for all } X \in \mathfrak{p}_+. \tag{3.15}$$

Then we see that $\overline{O}_{\tau^*}(G)$ is a closed G-submodule of $C^\infty_{\tau^*}(G)$ through right translation R, and that it is canonically isomorphic to the space of anti-holomorphic sections of the G-homogeneous vector bundle on $K\backslash G$ associated to the K-module V_τ^*.

It is useful to employ another realization of the G-module $\overline{O}_{\tau^*}(G)$ as a space of holomorphic V_τ^*-valued functions on a bounded domain $\mathcal{B}$ of $\mathfrak{p}_-$. To be more precise, let $P_\pm := \exp \mathfrak{p}_\pm$ be the connected Lie subgroups of $G_\mathbb{C}$ with Lie algebras $\mathfrak{p}_\pm$, respectively. Note that the exponential map gives holomorphic diffeomorphisms from $\mathfrak{p}_\pm$ onto $P_\pm$. Consider an open dense subset $P_+K_\mathbb{C}P_-$ of $G_\mathbb{C}$, which is holomorphically diffeomorphic to the direct product $P_+ \times K_\mathbb{C} \times P_-$ through multiplication. For each $x \in P_+K_\mathbb{C}P_-$, let $p_+(x)$, $k_\mathbb{C}(x)$, and $p_-(x)$ denote respectively the elements of P_+, $K_\mathbb{C}$, and P_- such that $x = p_+(x)k_\mathbb{C}(x)p_-(x)$. Set $\xi(x) := \log p_-(x) \in \mathfrak{p}_-$. It then follows that $G \subset P_+K_\mathbb{C}P_-$ and that the assignment $x \mapsto \xi(x)$ $(x \in G)$ naturally induces an anti-holomorphic diffeomorphism, say $\tilde{\xi}$, from the symmetric space $K\backslash G$ onto a bounded domain

$$\mathcal{B} := \{\xi(x) \in \mathfrak{p}_- \mid x \in G\} \tag{3.16}$$

of $\mathfrak{p}_-$, where $\tilde{\xi}(Kx) := \xi(x)$. (See for example [15, 7.129].)

Let $O(\mathcal{B}, V_\tau^*)$ be the space of all V_τ^*-valued holomorphic functions on $\mathcal{B}$. We see easily that the above $\tilde{\xi}$ gives a linear isomorphism Θ from $\overline{O}_{\tau^*}(G)$ onto $O(\mathcal{B}, V_\tau^*)$ by

$$(\Theta F)(\tilde{\xi}(Kx)) := \tau^*(k_\mathbb{C}(x))^{-1}F(x) \quad (x \in G) \tag{3.17}$$

for $F \in \overline{O}_{\tau^*}(G)$. Then $O(\mathcal{B}, V_\tau^*)$ has a G-module structure inherited from $(R, \overline{O}_{\tau^*}(G))$ through Θ:

$$(3.18) \qquad (g \cdot f)(\xi(x)) = \tau^*(k_{\mathbb{C}}(\exp \xi(x)\, g)) f(\xi(xg)) \quad (x \in G)$$

for $g \in G$ and $f \in O(\mathcal{B}, V_\tau^*)$. By differentiating the G-action (3.18) one obtains a $\mathfrak{g}$-module $O(\mathcal{B}, V_\tau^*)$. Note that $f \in O(\mathcal{B}, V_\tau^*)$ is K-finite if and only if f is a polynomial. Hence the K-finite part $\overline{O}_{\tau^*}(G)_K$ of $\overline{O}_{\tau^*}(G)$ is isomorphic, through Θ, to the space $\mathcal{P}(\mathfrak{p}_-, V_\tau^*) = S(\mathfrak{p}_+) \otimes V_\tau^*$ of V_τ^*-valued polynomial functions on $\mathfrak{p}_-$. Here we identify the symmetric algebra $S(\mathfrak{p}_+)$ of $\mathfrak{p}_+$ with the ring of polynomial functions on $\mathfrak{p}_-$ through $B|_{\mathfrak{p}_+ \times \mathfrak{p}_-}$.

We now define a bilinear form $\langle \cdot, \cdot \rangle_\tau$ on $\overline{O}_{\tau^*}(G) \times (U(\mathfrak{g}) \otimes_{\mathbb{C}} V_\tau)$ by

$$(3.19) \qquad \langle F, D \otimes v \rangle_\tau := \langle D^L F(e), v \rangle = \langle ({}^T D)^R F(e), v \rangle$$

for $F \in \overline{O}_{\tau^*}(G), D \in U(\mathfrak{g})$, and $v \in V_\tau$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $V_\tau^* \times V_\tau$, and $D \mapsto {}^T D$ the principal anti-automorphism of $U(\mathfrak{g})$, respectively. Then it is a routine task to verify that $\langle \cdot, \cdot \rangle_\tau$ naturally gives rise to a $(\mathfrak{g}, K)$-invariant bilinear form on $\overline{O}_{\tau^*}(G) \times M(\tau)$, which we denote again by $\langle \cdot, \cdot \rangle_\tau$. Note that

$$(3.20) \qquad \langle F, D \otimes v \rangle_\tau = \langle ({}^T D \cdot f)(0), v \rangle \quad \text{with } f := \Theta F \in O(\mathcal{B}, V_\tau^*),$$

where $D \in U(\mathfrak{p}_-) = S(\mathfrak{p}_-)$, $v \in V_\tau$, and ${}^T D \cdot f$ is defined through the directional derivative action. This implies the following

Lemma 3.2 (cf. [3, §2]). (1) *The $(\mathfrak{g}, K)$-invariant pairing $\langle \cdot, \cdot \rangle_\tau$ is nondegenerate on $\overline{O}_{\tau^*}(G)_K \times M(\tau)$.*

(2) *Let $R(\tau^*)$ be the orthogonal of the maximal submodule $N(\tau)$ in $\overline{O}_{\tau^*}(G)_K \simeq \mathcal{P}(\mathfrak{p}_-, V_\tau^*)$ with respect to $\langle \cdot, \cdot \rangle_\tau$. Then $R(\tau^*)$ is the unique, nonzero irreducible $(\mathfrak{g}, K)$-submodule of $\overline{O}_{\tau^*}(G)_K$, and it is isomorphic to the lowest weight module $L(\tau)^*$ dual to $L(\tau) = M(\tau)/N(\tau)$. The $(\mathfrak{g}, K)$-isomorphism A_{τ^*} from $L(\tau)^*$ onto $R(\tau^*)$ is given by*

$$(3.21) \qquad \langle A_{\tau^*}(\varphi), w \rangle_\tau = \langle \varphi, \ w + N(\tau) \rangle_{L(\tau)^* \times L(\tau)} \quad (w \in M(\tau))$$

for $\varphi \in L(\tau)^*$.

We are going to introduce a differential operator of gradient type whose K-finite kernel equals the $(\mathfrak{g}, K)$-module $R(\tau^*) = A_{\tau^*}(L(\tau)^*)$. For this, we take a basis $X_1, \ldots, X_s$ of the $\mathbb{C}$-vector space $\mathfrak{p}_+$ such that $B(X_j, \overline{X}_k) = \delta_{jk}$ (Kronecker's δ), where $\overline{X}_i \in \mathfrak{p}_-$ denotes the complex conjugate of an $X_i \in \mathfrak{p}_+$ with respect to $\mathfrak{g}_0$. Set

$$(3.22) \qquad X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s} \in U(\mathfrak{p}_+) \quad \text{and} \quad \overline{X}^\alpha := \overline{X}_1^{\alpha_1} \cdots \overline{X}_s^{\alpha_s} \in U(\mathfrak{p}_-)$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_s)$ of nonnegative integers $\alpha_1, \ldots, \alpha_s$. We denote by $|\alpha| := \alpha_1 + \cdots + \alpha_s$ the length of α. For each positive integer n we define the gradients ∇^n and $\overline{\nabla}^n$ of order n on $C^\infty_{\tau^*}(G)$ as follows.

$$(3.23) \qquad \nabla^n F(x) := \sum_{|\alpha|=n} \overline{X}^\alpha \otimes (X^\alpha)^L F(x),$$

$$(3.24) \qquad \overline{\nabla}^n F(x) := \sum_{|\alpha|=n} X^\alpha \otimes (\overline{X}^\alpha)^L F(x),$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. It is then easy to see that $\nabla^n F$ and $\overline{\nabla}^n F$ are independent of the choice of a basis $X_1, \ldots, X_s$, and that the operators ∇^n and $\overline{\nabla}^n$ give continuous G-homomorphisms

$$(3.25) \qquad \nabla^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(-n)}(G), \quad \overline{\nabla}^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(+n)}(G).$$

Here $\tau^*(\pm n)$ denotes the K-representation on the tensor product $S^n(\mathfrak{p}_\pm)\otimes V_\tau^*$ respectively.

Let W_u $(u=1,\dots,q)$ be, as in (3.12), the irreducible K-submodules of $S^{i_u}(\mathfrak{p}_-)V_\tau \subset N(\tau)$ which generate $N(\tau)$ over $S(\mathfrak{p}_-)$ when $N(\tau)\neq\{0\}$. For each u, the adjoint operator P_u of the embedding

$$W_u \hookrightarrow S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-)\otimes V_\tau \tag{3.26}$$

gives a surjective K-homomorphism:

$$P_u : S^{i_u}(\mathfrak{p}_+)\otimes V_\tau^* \simeq (S^{i_u}(\mathfrak{p}_-)\otimes V_\tau)^* \longrightarrow W_u^*. \tag{3.27}$$

Definition 3.3. Under the above notation, let $\mathcal{D}_{\tau^*}$ be a continuous G-homomorphism from $C^\infty_{\tau^*}(G)$ to $C^\infty_\varrho(G)$ defined by

$$\mathcal{D}_{\tau^*}F(x) := \nabla^1 F(x) \oplus (\oplus_{u=1}^q P_u(\overline{\nabla}^{i_u}F(x))) \tag{3.28}$$

for $x\in G$ and $F\in C^\infty_{\tau^*}(G)$. Here we write for $\varrho=\varrho(\tau^*)$ the representation of K on

$$(\mathfrak{p}_-\otimes V_\tau^*)\oplus(\oplus_{u=1}^q W_u^*), \tag{3.29}$$

and $\mathcal{D}_{\tau^*}$ should be understood as $\mathcal{D}_{\tau^*}=\nabla^1$ if $N(\tau)=\{0\}$, or if $M(\tau)=L(\tau)$. We call $\mathcal{D}_{\tau^*}$ the *differential operator of gradient type* associated to τ^*.

Remark 3.4. A function $F\in C^\infty_{\tau^*}(G)$ lies in the G-submodule $\overline{O}_{\tau^*}(G)$ defined by (3.15) if and only if $\nabla^1 F=0$. Hence we have $\operatorname{Ker}\mathcal{D}_{\tau^*}\subset\overline{O}_{\tau^*}(G)$ for every τ^*, and the equality holds if and only if $N(\tau)=\{0\}$.

The following theorem is equivalent to [4, Prop.7.6] due to Davidson and Stanke.

Theorem 3.5. *The image $R(\tau^*)$ of the $(\mathfrak{g},K)$-embedding A_{τ^*} from $L(\tau)^*$ into $\overline{O}_{\tau^*}(G)_K$ defined in Lemma 3.2 coincides with the K-finite kernel of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type:*

$$R(\tau^*) = \{F\in C^\infty_{\tau^*}(G) \mid F \textit{ is right } K\textit{-finite and } \mathcal{D}_{\tau^*}F=0\}. \tag{3.30}$$

3.4. **Maximal globalization of $L(\tau)^*$.** The above theorem together with Theorem 2.3 implies that the full kernel space $\operatorname{Ker}\mathcal{D}_{\tau^*}$ gives a maximal globarization of $L(\tau)^*$.

Proposition 3.6. (1) *The closure $R(\tau^*)^-$ of $R(\tau^*)=A_{\tau^*}(L(\tau)^*)$ in $C^\infty_{\tau^*}(G)$ coincides with* $\operatorname{Ker}\mathcal{D}_{\tau^*}$. *It coincides also with the orthogonal, say $R'(\tau^*)$, of $N(\tau)$ in the whole (not necessarily K-finite) space $\overline{O}_{\tau^*}(G)$ with respect to the paring $\langle\,\cdot,\cdot\,\rangle_\tau$ in* (3.19).

(2) *One has an isomorphism of G-modules*

$$\operatorname{Hom}_{\mathfrak{g},K}(L(\tau),\ C^\infty(G)) \simeq \operatorname{Ker}\mathcal{D}_{\tau^*}(=R(\tau^*)^-=R'(\tau^*)) \tag{3.31}$$

by the correspondence given in Theorem 2.2.

We end this section by specifying for later use the solutions $F\in\overline{O}_{\tau^*}(G)$ of exponential type of the differential equation $\mathcal{D}_{\tau^*}F=0$.

For each $X\in\mathfrak{p}_+$ and each $v^*\in V_\tau^*$, let $f_{X,v^*}=\exp X\otimes v^*$ denote the V_τ^*-valued holomorphic function on $\mathfrak{p}_-$ defined by

$$f_{X,v^*}(z) := \exp B(X,z)\cdot v^* \quad (z\in\mathfrak{p}_-). \tag{3.32}$$

We set $F_{X,v^*} := \Theta^{-1}f_{X,v^*}\in\overline{O}_{\tau^*}(G)$. Then the function F_{X,v^*} is described as

$$F_{X,v^*}(x) = \exp B(X,\xi(x))\cdot\tau^*(k_{\mathbb{C}}(x))v^* \quad (x\in G) \tag{3.33}$$

by the definition of Θ (see (3.17)).

Proposition 3.7. *The function F_{X,v^*} satisfies the differential equation $\mathcal{D}_{\tau^*}F = 0$ if and only if*

$$(3.34) \qquad P_u(X^{i_u} \otimes v^*) = 0 \quad \textit{for } u = 1, \dots, q.$$

Here P_u is a K-homomorphism defined in (3.26) *and in* (3.27).

4. Associated variety and multiplicity of highest weight modules

The purpose of this section is to understand the associated variety and multiplicity for each $L(\tau)$ by means of the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type. The harvest of our discussion is summarized as Theorem 4.9.

4.1. **Associated variety $\mathcal{V}(L(\tau))$.** We keep the notation in 3.1. For every integer m such that $0 \le m \le r = \mathbb{R}\text{-rank}\, G$, we set

$$(4.1) \qquad \mathcal{O}_m := \mathrm{Ad}(K_{\mathbb{C}})X(m) \quad \text{with} \quad X(m) := \sum_{k=r-m+1}^{r} X_{\gamma_k} \text{ (see (3.2))} .$$

Here $X(0)$ should be understood as 0. The following proposition is well-known.

Proposition 4.1. *The subspace $\mathfrak{p}_+$ splits into a disjoint union of $r+1$ number of $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \le m \le r)$: $\mathfrak{p}_+ = \coprod_{0 \le m \le r} \mathcal{O}_m$, and the closure $\overline{\mathcal{O}_m}$ of orbit $\mathcal{O}_m$ is equal to $\cup_{k \le m} \mathcal{O}_k$ for every m.*

Let $L(\tau) = M(\tau)/N(\tau)$ be, as in 3.2, the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type (τ, V_τ). Consider the annihilator ideal

$$(4.2) \qquad \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau) := \{D \in S(\mathfrak{p}_-) \mid Dw = 0 \quad \text{for all } w \in L(\tau)\}.$$

of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$.

Definition 4.2. The algebraic variety

$$(4.3) \qquad \mathcal{V}(L(\tau)) := \{X \in \mathfrak{p}_+ \mid D(X) = 0 \quad \text{for all } D \in \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)\} \subset \mathfrak{p}_+$$

defined by the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is called the *associated variety* of the $(\mathfrak{g}, K)$-module $L(\tau)$. Here $S(\mathfrak{p}_-)$ is identified with the ring of polynomial functions on $\mathfrak{p}_+$.

Since the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is stable under $\mathrm{Ad}(K_{\mathbb{C}})$, so is the variety $\mathcal{V}(L(\tau))$. In view of Proposition 4.1, there exists a unique integer $m = m(\tau)$ $(0 \le m \le r)$ such as

$$(4.4) \qquad \mathcal{V}(L(\tau)) = \overline{\mathcal{O}_m} \quad \text{with} \quad \mathcal{O}_m = \mathrm{Ad}(K_{\mathbb{C}})X(m).$$

In particular, the variety $\mathcal{V}(L(\tau))$ is irreducible.

Now let I_m be the prime ideal of $S(\mathfrak{p}_-)$ which defines the irreducible variety $\overline{\mathcal{O}_m}$ $(m = 0, \dots, r)$. It holds that $I_r = \{0\}$ since $\overline{\mathcal{O}_r} = \mathfrak{p}_+$. If $m < r$, one knows that

$$(4.5) \qquad I_m = S(\mathfrak{p}_-) Q_{m+1}$$

by [5, 8.1] and [18, Prop.2.3], where Q_{m+1} denotes as in (3.14) the irreducible K-submodule of $S^{m+1}(\mathfrak{p}_-) \subset S(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \dots - \gamma_{r-m}$.

By Hilbert's Nullstellensatz , $I_{m(\tau)}$ coincides with the radical of the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ for every τ. This allows us to deduce the following

Lemma 4.3. *The annihilator in $S(\mathfrak{p}_-)$ of the quotient $(S(\mathfrak{p}_-), K)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ is equal to $I_{m(\tau)}$.*

For each $X \in \mathfrak{p}_+$, let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ which defines the variety $\{X\}$ of one element X:

$$\mathfrak{m}(X) := \sum_{Y \in \mathfrak{p}_-} (Y - B(X,Y))S(\mathfrak{p}_-). \tag{4.6}$$

The isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at X acts naturally on the quotient space

$$\mathcal{W}(X,\tau) := L(\tau)/\mathfrak{m}(X)L(\tau). \tag{4.7}$$

We note that $\dim \mathcal{W}(X,\tau) < \infty$.

By applying a result of Vogan [26, Cor.2.10 and Def.2.12] in view of Lemma 4.3, we immediately deduce

Proposition 4.4. *Assume that $X \in \mathcal{O}_{m(\tau)}$. Then the dimension of $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ coincides with the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau))$ of the $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the unique minimal associated prime $I_{m(\tau)}$. So in particular, one has $\mathcal{W}(X,\tau) \neq \{0\}$.*

See [26, §2] for the definition of the multiplicities of finitely generated modules over a commutative Noetherian ring (in connection with Harish-Chandra modules).

As for the unitarizable highest weight modules, the following remarkable result of Joseph gives a clearer understanding of the above proposition.

Theorem 4.5 ([18, Lem.2.4 and Th.5.16]). *Suppose that $L(\tau)$ is unitarizable. Then, the annihilator $\mathrm{Ann}_{S(\mathfrak{p}_-)}w$ in $S(\mathfrak{p}_-)$ of any nonzero vector $w \in L(\tau)$ coincides with the prime ideal $I_{m(\tau)}$. Especially, one has $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) = I_{m(\tau)}$.*

Corollary 4.6 (to Prop.4.4 and Th.4.5). *One has $\mathrm{mult}_{I_{m(\tau)}}(L(\tau)) = \dim \mathcal{W}(X,\tau)$ $(X \in \mathcal{O}_{m(\tau)})$ for every irreducible unitarizable highest weight module $L(\tau)$.*

Remark 4.7. For classical groups $Sp(2n,\mathbb{R})$, $U(p,q)$ and $O^*(2p)$, Nishiyama, Ochiai and Taniguchi [20, Th.7.18 and Th.9.1] have described the associated cycle $\mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \cdot [\overline{\mathcal{O}_{m(\tau)}}]$ and the Bernstein degree of unitarizable highest weight module $L(\tau)$ by using the theory of reductive dual pairs (G,G') with compact G'. They treat the case where the dual pair (G,G') is in the stable range, and the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$ is specified as the dimension of corresponding irreducible G'-module, through detailed study of K-types of $L(\tau)$. On the other hand, the above corollary allows us to give another simple proof of this description of the multiplicity by investigating the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ (see [34] and also [24]), where the dual pairs (G,G') need not be in the stable range .

4.2. **Principal symbol σ and variety $\mathcal{V}(L(\tau))$.** Let $\mathcal{D}_{\tau^*} = \nabla^1 \oplus (\oplus_{u=1}^q P_u \circ \overline{\nabla}^{i_u})$ be, as in Definition 3.3, the differential operator of gradient type whose kernel realizes the maximal globarization of dual lowest weight module $L(\tau)^*$ (see Proposition 3.6). We put

$$\sigma(X,v^*) := \sum_{u=1}^{q} P_u(X^{i_u} \otimes v^*) \in W^* := \oplus_{u=1}^q W_u^* \tag{4.8}$$

for $X \in \mathfrak{p}_+$ and $v^* \in V_\tau^*$, where $P_u : S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^* \longrightarrow W_u^*$ is the K-homomorphism in (3.27). We call σ the *principal symbol* of $\mathcal{D}_{\tau^*}$ at the origin. Here σ should be understood as $\sigma(X,v^*) = 0$ for all $X \in \mathfrak{p}_+$ and $v^* \in V_\tau^*$, when $\mathcal{D}_{\tau^*} = \nabla^1$, i.e., $N(\tau) = \{0\}$.

We want to describe the associated variety $\mathcal{V}(L(\tau))$ by means of σ. To do this, fix any $X \in \mathfrak{p}_+$ for a while. Then the map $v^* \mapsto \sigma(X,v^*)$ gives a $K_{\mathbb{C}}(X)$-homomorphism $\sigma(X,\cdot)$

from V_τ^* to W^*. Hence $\operatorname{Ker}\sigma(X,\,\cdot\,)$ is a $K_{\mathbb{C}}(X)$-submodule of V_τ^*. By Proposition 3.7 we can describe $\operatorname{Ker}\sigma(X,\,\cdot\,)$ as

$$\operatorname{Ker}\sigma(X,\,\cdot\,) = \{v^* \in V_\tau^* \mid \mathcal{D}_{\tau^*}F_{X,v^*} = 0\}, \tag{4.9}$$

where $F_{X,v^*} \in C^\infty_{\tau^*}(G)$ is the function of exponential type defined by (3.33).

The following lemma relates the above kernel with the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ in (4.7).

Lemma 4.8. *For each $X \in \mathfrak{p}_+$, the natural map*

$$V_\tau \hookrightarrow M(\tau) \to L(\tau) = M(\tau)/N(\tau) \to \mathcal{W}(X,\tau) = L(\tau)/\mathfrak{m}(X)L(\tau) \tag{4.10}$$

from V_τ onto $\mathcal{W}(X,\tau)$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}(X,\tau)^* \simeq \operatorname{Ker}\sigma(X,\,\cdot\,) \subset V_\tau^* \tag{4.11}$$

through the contravariant functor $\operatorname{Hom}_{\mathbb{C}}(\,\cdot\,,\mathbb{C})$.

Now, we can give the following characterization of the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$ and the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau))$ in terms of the symbol σ.

Theorem 4.9. *Let $L(\tau)$ be any irreducible highest weight $(\mathfrak{g},K)$-module with extreme K-type τ, and let $\sigma : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type associated to τ^*. Then it holds that*

$$\mathcal{V}(L(\tau)) = \{X \in \mathfrak{p}_+ \mid \operatorname{Ker}\sigma(X,\,\cdot\,) \neq \{0\}\}. \tag{4.12}$$

Moreover, if X is an element of the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ of $\mathcal{V}(L(\tau))$, the dimension of vector space $\operatorname{Ker}\sigma(X,\,\cdot\,)$ *coincides with the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$ of $S(\mathfrak{p}_-)$.*

Remark 4.10. We can give the same kind of characterization of the associated variety and the multiplicity also for irreducible $(\mathfrak{g},K)$-modules of discrete series, by using some results in [33].

5. Generalized Whittaker models for highest weight modules

In this section we describe the generalized Whittaker models for irreducible highest weight modules $L(\tau)$. The main results are summarized as Theorems 5.6–5.8.

5.1. **Generalized Gelfand-Graev representations.** We begin with introducing in this subsection the generalized Gelfand-Graev representations of G attached to the Cayley transforms of nilpotent $K_{\mathbb{C}}$-orbits $\mathcal{O}_m = \operatorname{Ad}(K_{\mathbb{C}})X(m)$ in $\mathfrak{p}_+$.

For this, we consider the $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$X(m) = \sum_{k=r-m+1}^{r} X_{\gamma_k}, \quad H(m) := \sum_{k=r-m+1}^{r} H_{\gamma_k}, \quad Y(m) := \sum_{k=r-m+1}^{r} X_{-\gamma_k}, \tag{5.1}$$

with commutation relation

$$[H(m),X(m)] = 2X(m), \quad [H(m),Y(m)] = -2Y(m), \quad [X(m),Y(m)] = H(m). \tag{5.2}$$

Let $\boldsymbol{c} = \operatorname{Ad}(c)$ (cf. (3.8)) be the Cayley transform on $\mathfrak{g}$. We put

$$\begin{aligned} X'(m) :&= -\sqrt{-1}\boldsymbol{c}^{-1}(X(m)) = \frac{\sqrt{-1}}{2}(H(m) - X(m) + Y(m)), \\ H'(m) :&= \boldsymbol{c}^{-1}(H(m)) = X(m) + Y(m), \\ Y'(m) :&= \sqrt{-1}\boldsymbol{c}^{-1}(Y(m)) = -\frac{\sqrt{-1}}{2}(H(m) + X(m) - Y(m)). \end{aligned} \tag{5.3}$$

Then $(X'(m), H'(m), Y'(m))$ forms an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_0$. Set $\mathcal{O}'_m := \mathrm{Ad}(G)X'(m)$. Note that the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponds to the $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+ \subset \mathfrak{p}$ through the Kostant-Sekiguchi correspondence (cf. [9, Th.3.1]).

Lemma 5.1 ([9, Lemma 3.2]). (1) *The Lie algebra $\mathfrak{g}$ decomposes into a direct sum of the j-eigensubspaces $\mathfrak{g}_j(m)$ for $\mathrm{ad}\, H'(m)$ as*

$$\mathfrak{g} = \mathfrak{g}_{-2}(m) \oplus \mathfrak{g}_{-1}(m) \oplus \mathfrak{g}_0(m) \oplus \mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m). \tag{5.4}$$

(2) *Let $\Delta(m,j)$ ($j = 0, \pm 1, \pm 2$) be the subsets of the root system Δ of $(\mathfrak{g}, \mathfrak{t})$ defined by*

$$\Delta(m,2) := \{\gamma_{r-m+1}, \dots, \gamma_r\} \bigcup \Big(\bigcup_{r-m<l<k} P_{kl} \Big), \tag{5.5}$$

$$\Delta(m,1) := \Big(\bigcup_{l \le r-m < k} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{r-m<k} (P_k \cup C_k) \Big), \tag{5.6}$$

$$\begin{aligned} \Delta^+(m,0) := C_0 \bigcup \{\gamma_1, \dots, \gamma_{r-m}\} \bigcup \Big(\bigcup_{r-m<l<k} C_{kl} \Big) \\ \bigcup \Big(\bigcup_{l<k\le r-m} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{k \le r-m} (P_k \cup C_k) \Big), \end{aligned} \tag{5.7}$$

$$\Delta(m,0) := \Delta^+(m,0) \cup (-\Delta^+(m,0)), \quad \Delta(m,-j) := -\Delta(m,j) \quad (j = 1,2). \tag{5.8}$$

Then each subspace $\boldsymbol{c}(\mathfrak{g}_j(m)) = \mathrm{Ad}(c)\mathfrak{g}_j(m)$ is described in terms of root subspaces as

$$\boldsymbol{c}(\mathfrak{g}_j(m)) = \begin{cases} \oplus_{\gamma \in \Delta(m,j)} \mathfrak{g}(\mathfrak{t}; \gamma) & \text{if } j \neq 0, \\ \mathfrak{t} \oplus \big(\oplus_{\gamma \in \Delta(m,0)} \mathfrak{g}(\mathfrak{t}; \gamma) \big) & \text{if } j = 0. \end{cases} \tag{5.9}$$

Now we set

$$\Delta^-(m) := (\Delta(m,-2) \cup \Delta(m,-1)) \cap \Delta_n, \tag{5.10}$$

and let $\mathfrak{p}_-(m)$ and $\mathfrak{n}(m)$ be nilpotent, abelian Lie subalgebras of $\mathfrak{g}$ defined respectively by

$$\mathfrak{p}_-(m) := \bigoplus_{\gamma \in \Delta^-(m)} \mathfrak{g}(\mathfrak{t}; \gamma) \quad \text{and} \quad \mathfrak{n}(m) := \boldsymbol{c}(\mathfrak{p}_-(m)). \tag{5.11}$$

If $K \backslash G$ is of tube type, $\mathfrak{n}(m)$ is the complexification of a real Lie subalgebra $\mathfrak{n}(m)_0$ of $\mathfrak{g}_0$.

Lemma 5.2. (1) *One has the equality $\mathfrak{p}_-(m) = [\mathfrak{k}, Y(m)]$. Namely, $\mathfrak{p}_-(m)$ is canonically isomorphic to the tangent space of the $K_{\mathbb{C}}$-orbit $\mathcal{O}^*_m := \mathrm{Ad}(K_{\mathbb{C}})Y(m)$ at the point $Y(m)$.*

(2) *Let $\mathfrak{v}(m)$ be the subspace of $\mathfrak{g}_1(m)$ such that*

$$\mathfrak{v}(m) := \boldsymbol{c}^{-1}(\oplus_{\gamma \in \Xi(m)} \mathfrak{g}(\mathfrak{t}; \gamma)) \quad \textit{with} \quad \Xi(m) := \Big(\bigcup_{l \le r-m<k} P_{kl} \Big) \bigcup \Big(\bigcup_{k > r-m} C_k \Big). \tag{5.12}$$

Then it holds that

$$\mathfrak{n}(m) = \mathfrak{v}(m) \oplus \mathfrak{g}_2(m) \quad \textit{and} \quad \dim \mathfrak{v}(m) = \frac{1}{2} \dim \mathfrak{g}_1(m). \tag{5.13}$$

Let η_m be the one-dimensional representation (i.e., character) of abelian Lie subalgebra $\mathfrak{n}(m) = \mathfrak{v}(m) \oplus \mathfrak{g}_2(m)$ defined by

$$\eta_m(U) := \sqrt{-1} B(U, \theta X'(m)) = -\sqrt{-1} B(U, Y'(m)) \quad \text{for} \quad U \in \mathfrak{n}(m). \tag{5.14}$$

Here θ denotes the complexified Cartan involution of $\mathfrak{g}$. Just as in Definition 2.5, we get a C^∞-induced representation $\Gamma_m := \Gamma_{\eta_m}$ of G acting on $C^\infty(G; \eta_m)$ by left translation L.

Definition 5.3. We call $(\Gamma_m, C^\infty(G;\eta_m))$ the *generalized Gelfand-Graev representation* (GGGR for short) attached to the nilpotent G-orbit $\mathcal{O}'_m = \mathrm{Ad}(G)X'(m)$ in $\mathfrak{g}_0$.

Remark 5.4. The GGGRs attached to nilpotent orbits have been constructed in full generality by Kawanaka [14] for reductive algebraic groups. See also [30] and [31].

5.2. **Generalized Whittaker models.** For any irreducible finite-dimensional K-module (τ, V_τ), let $L(\tau) = M(\tau)/N(\tau)$ (see 3.2) be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Consider the GGGRs $(\Gamma_m, C^\infty(G;\eta_m))$ $(m = 0, \ldots, r)$ induced from the characters $\eta_m : \mathfrak{n}(m) \to \mathbb{C}$. We say that $L(\tau)$ has a *generalized Whittaker model* of type η_m if $L(\tau)$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $C^\infty(G;\eta_m)$.

We are going to describe the generalized Whittaker models for $L(\tau)$ by specifying the vector space $\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m))$. Let $\mathcal{D}_{\tau^*} : C^\infty_{\tau^*}(G) \to C^\infty_{\varrho}(G)$ be, as in Definition 3.3, the G-invariant differential operator of gradient type associated to τ^*. Set

$$(5.15)\qquad \mathcal{Y}(\tau, m) := \{F \in C^\infty_{\tau^*}(G) \mid \mathcal{D}_{\tau^*}F = 0 \text{ and } U^R F = -\eta_m(U)F \ (U \in \mathfrak{n}(m))\}.$$

Then the kernel theorem (Corollary 2.6) gives a linear isomorphism

$$(5.16)\qquad \mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau, m).$$

Now our aim is to describe the space $\mathcal{Y}(\tau, m)$ for each τ and m. For this purpose, we use the following unbounded realization of Hermitian symmetric space $K\backslash G$.

Proposition 5.5 (cf. [15, page 455], [10]). *Retain the notation in the beginning of* 3.3, *and let $P_+K_{\mathbb{C}}P_-$ be the open dense subset of $G_{\mathbb{C}}$ with $P_\pm = \exp \mathfrak{p}_\pm$. Then,*

(1) *one has $Gc \subset P_+K_{\mathbb{C}}P_-$, where c is the Cayley element of $G_{\mathbb{C}}$.*

(2) *Set $\xi'(x) := \log p_-(xc) \in \mathfrak{p}_-$ $(x \in G)$, where $xc = p_+(xc)k_{\mathbb{C}}(xc)p_+(xc)$ with $k_{\mathbb{C}}(xc) \in K_{\mathbb{C}}$ and $p_\pm(xc) \in P_\pm$. The map $x \mapsto \xi'(x)$ $(x \in G)$ sets up an anti-holomorphic diffeomorphism from $K\backslash G$ onto an unbounded domain $\mathcal{S} := \{\xi'(x) \mid x \in G\}$ of $\mathfrak{p}_-$.*

Now we state the principal results of this section. Let $\mathcal{O}_{m(\tau)}$ be, as in (4.4), the unique open $K_{\mathbb{C}}$-orbit in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. Among the generalized Whittaker models for $L(\tau)$, those of type $\eta_{m(\tau)}$ are most important, and we can specify the corresponding linear space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ as follows.

Theorem 5.6. (1) *$\mathcal{Y}(\tau)$ is a nonzero, finite-dimensional vector space.*

(2) *For any $F \in \mathcal{Y}(\tau)$, there exists a unique polynomial function φ on $\mathfrak{p}_-$ with values in V^*_τ such that*

$$(5.17)\qquad F(x) = \exp B(X(m(\tau)), \xi'(x))\tau^*(k_{\mathbb{C}}(xc))\varphi(\xi'(x)) \quad (x \in G).$$

(3) *Let $\sigma : \mathfrak{p}_+ \times V^*_\tau \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type, defined by* (4.8). *Consider the functions $F_{X(m(\tau)),v^*} \in C^\infty_{\tau^*}(G)$ of exponential type in Proposition 3.7. Then the assignment*

$$(5.18)\qquad v^* \longmapsto c^R F_{X(m(\tau)),v^*} = F_{X(m(\tau)),v^*}(\cdot\, c) \quad (v^* \in \mathrm{Ker}\,\sigma(X(m(\tau)),\ \cdot\ ))$$

yields an injective linear map

$$(5.19)\qquad \chi_\tau : \mathrm{Ker}\,\sigma(X(m(\tau)),\ \cdot\ ) \hookrightarrow \mathcal{Y}(\tau).$$

Second, we can show the surjectivity of χ_τ for relevant $L(\tau)$'s.

Theorem 5.7. *Assume that $L(\tau)$ is unitarizable. Then the linear embedding χ_τ in* (5.19) *is surjective. Hence one gets*

$$(5.20)\qquad \mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau) \simeq \mathrm{Ker}\,\sigma(X(m(\tau)),\ \cdot\ )$$

as vector spaces. Moreover, the dimension of these spaces equals the multiplicity

$$\text{(5.21)} \qquad \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) = \dim(L(\tau)/\mathfrak{m}(X(m(\tau)))L(\tau)) \quad (\textit{see Corollary 4.6})$$

of the $S(\mathfrak{p}_-)$*-module* $L(\tau)$ *at the unique associated prime* $I_{m(\tau)} \subset S(\mathfrak{p}_-)$.

Third, Theorem 5.6 for $m = m(\tau)$ allows us to deduce the following

Theorem 5.8. *The linear space* $\mathcal{Y}(\tau, m)$ *vanishes (resp. is infinite-dimensional) if* $m > m(\tau)$ *(resp.* $m < m(\tau)$*).*

Remark 5.9. Theorem 5.7 recovers our earlier result [31, Part II] on the generalized Whittaker models for holomorphic discrete series $L(\tau) = M(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau$:

$$\text{(5.22)} \qquad \mathrm{Hom}_{\mathfrak{g},K}(M(\tau), C^\infty(G;\eta_r)) \simeq V_\tau^*.$$

Remark 5.10. The vanishing of $\mathcal{Y}(\tau, m)$ $(m > m(\tau))$ in Theorem 5.8 follows also from a general result of Matumoto [19, Th.1].

6. $\mathfrak{n}$-HOMOLOGY OF BOREL-DE SIEBENTHAL DISCRETE SERIES

This section describes the $\mathfrak{n}$-homology spaces for the Borel-de Siebenthal discrete series representations of simple Lie groups of quaternionic type (Theorem 6.3).

6.1. **Simple Lie groups of quaternionic type.** First, let us identify the groups of quaternionic type which we concern in this section. Let $G, K, G_{\mathbb{C}}, K_{\mathbb{C}}$ and $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{g}, \mathfrak{k}$ be the Lie groups and the corresponding Lie algebras as in Introduction, respectively. We assume that

$$\text{(6.1)} \qquad \mathrm{rank}\, G = \mathrm{rank}\, K \quad \text{and} \quad \mathfrak{k} \text{ is semisimple.}$$

Take a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ as in 3.1 for the root system of $(\mathfrak{g}, \mathfrak{t})$. Then the Borel-de Siebenthal theorem (Theorem 1.1) implies that there exist a simple system Π of Δ and a noncompact root $\alpha_1 \in \Pi$ which yield the gradation (1.1) by putting

$$\text{(6.2)} \qquad \mathfrak{g}(j) := \bigoplus_{\substack{\gamma \in \Delta \cup \{0\} \\ m_{\alpha_1}(\gamma) = j}} \mathfrak{g}(\mathfrak{t};\gamma).$$

Here, we set $\mathfrak{g}(\mathfrak{t};0) := \mathfrak{t}$, and $m_{\alpha_1}(\gamma)$ denotes the coefficient of α_1 in the expression $\gamma = \sum_{\alpha\in\Pi} m_\alpha(\gamma)\alpha$ of γ as a linear combination of simple roots. Note that the Dynkin diagram of $\mathfrak{k}$ is obtained from the extended Dynkin diagram of $\mathfrak{g}$, by excluding the vertex corresponding to α_1.

Let Δ^+ be the positive system of Δ defined by Π, and let $\delta \in \Delta^+$ be the highest root. We assume further that

$$\text{(6.3)} \qquad \delta \text{ is not orthogonal to } \alpha_1, \text{ i.e., } (\delta, \alpha_1) \neq 0.$$

Then the Dynkin diagram of $\mathfrak{k}$ splits into two components $\Pi \backslash \{\alpha_1\}$ and $\{-\delta\}$. Accordingly, the Lie algebra $\mathfrak{k}$ decomposes into a direct sum of two ideals as

$$\text{(6.4)} \qquad \mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \quad \text{with} \quad \mathfrak{k}_1 = [\mathfrak{g}(0), \mathfrak{g}(0)] \quad \text{and} \quad \mathfrak{k}_2 \simeq \mathfrak{sl}_2(\mathbb{C}),$$

where $\mathfrak{k}_2$ is generated by the highest and lowest root spaces $\mathfrak{g}(\mathfrak{t};\delta)$ and $\mathfrak{g}(\mathfrak{t};-\delta)$.

In what follows, we deal with the groups G satisfying the above assumptions (6.1) and (6.3). Up to isomorphism, the corresponding Lie algebras $\mathfrak{g}_0$ are enumerated as

$$\text{(6.5)} \qquad \mathrm{BDI}(p \geq 3, q = 4),\ \mathrm{CII}(p \geq 1, q = 1),\ \mathrm{EII},\ \mathrm{EVI},\ \mathrm{EIX},\ \mathrm{FI},\ \mathrm{G},$$

under Cartan's notation. Here we write BDI(p,q) and CII(p,q) for $\mathfrak{so}(p,q)$ and $\mathfrak{sp}(p,q)$ respectively. The list (6.5) exhausts all the real simple Lie algebras of quaternionic type, except the Lie algebras $\mathfrak{su}(p,2)$ of Hermitian type (see [8, Table 4.7]).

6.2. **Borel-de Siebenthal discrete series.** We assume that G has the simply connected complexification $G_{\mathbb{C}}$. For any regular integral linear form Λ on $\mathfrak{t}$, let $\boldsymbol{X}_\Lambda$ be the irreducible $(\mathfrak{g},K)$-module of discrete series with Harish-Chandra parameter Λ.

Definition 6.1. (1) The discrete series $\boldsymbol{X}_\Lambda$ is called *of Borel-de Siebenthal* if the parameter Λ is Δ^+-dominant. (2) $\boldsymbol{X}_\Lambda$ is called *quaternionic* if $\Lambda = c\delta+\rho$ for some nonnegative integer c, where ρ denotes half the sum of all positive roots.

Note that the quaternionic discrete series forms a subfamily of the Borel-de Siebenthal discrete series. The following fact on the quaternionic discrete series is fundamental.

Proposition 6.2 ([8, Prop.5.7]). *Let $\boldsymbol{X}_{c\delta+\rho}$ be a quaternionic discrete series.*

(1) *The Blattner parameter λ of $\boldsymbol{X}_{c\delta+\rho}$ is equal to $(d/2)\delta$, where $d := 2c+\dim\mathfrak{g}(1)$ is a positive even integer. Therefore, the corresponding lowest K-representation τ_λ of $\boldsymbol{X}_{c\delta+\rho}$ is equivalent to the exterior tensor product of the trivial $\mathfrak{k}_1$-module with an irreducible representation τ_d of $\mathfrak{k}_2 \simeq \mathfrak{sl}_2(\mathbb{C})$ of dimension $d+1$, i.e., $\tau_\lambda \simeq id_{\mathfrak{k}_1}\hat{\otimes}\tau_d$.*

(2) *The $(\mathfrak{g},K)$-module $\boldsymbol{X}_{c\delta+\rho}$ is self-contragredient.*

(3) *The Gelfand-Kirillov dimension and the Bernstein degree of $\boldsymbol{X}_{c\delta+\rho}$ are given respectively by* $\dim\mathfrak{g}(1)+1$ *and by* $\dim\mathfrak{g}(1)$.

6.3. **$\mathfrak{n}$-homology.** Take a maximal family of strongly orthogonal noncompact positive roots $\gamma_1,\dots,\gamma_r$ arranged as $\gamma_1 = \alpha_1 < \dots < \gamma_r$. One finds from the list (6.5) that $r = \mathbb{R}\text{-rank}\, G$ is at most 4. We choose root vectors $X_{\gamma_k} \in \mathfrak{g}(\mathfrak{t};\gamma_k)$ $(1 \le k \le r)$ as in (3.2). Set

$$\mathfrak{a}_{\mathfrak{p},0} := \sum_{k=1}^{r} \mathbb{R}H_k \quad \text{with} \quad H_k := X_{\gamma_k} + X_{-\gamma_k}. \tag{6.6}$$

Then we see that $\mathfrak{a}_{\mathfrak{p},0}$ is a maximal abelian subspace of $\mathfrak{p}_0$ with orthogonal basis $H_1,\dots,H_r$. Let Ψ be the restricted root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_{\mathfrak{p},0}$. We introduce a positive system Ψ^+ of Ψ through the lexicographic order defined by the basis $H_1,\dots,H_r$ of $\mathfrak{a}_{\mathfrak{p},0}$. Let $\mathfrak{n}_0$ denote the maximal nilpotent Lie subalgebra of $\mathfrak{g}_0$ which is the sum of root subspaces for all positive restricted roots, and N the corresponding analytic subgroup of G. Then one gets an Iwasawa decomposition $G = KA_{\mathfrak{p}}N$ of G with $A_{\mathfrak{p}} := \exp\mathfrak{a}_{\mathfrak{p},0}$. Set $M := Z_K(\mathfrak{a}_{\mathfrak{p},0})$. We write M_0 for the identity component of M.

Now let

$$H_0(\mathfrak{n},\Lambda) := \boldsymbol{X}_\Lambda/\mathfrak{n}\boldsymbol{X}_\Lambda \tag{6.7}$$

be the 0th $\mathfrak{n}$-homology space of Borel-de Siebenthal discrete series $(\mathfrak{g},K)$-module $\boldsymbol{X}_\Lambda$. Then the group $MA_{\mathfrak{p}}$ acts on $H_0(\mathfrak{n},\Lambda)$ naturally. We want to clarify the $M_0A_{\mathfrak{p}}$-module structure of $H_0(\mathfrak{n},\Lambda)$. For this, it is enough to describe the $\mathfrak{n}$-homology of the quaternionic discrete series $\boldsymbol{X}_{c\delta+\rho}$ by virtue of the Zuckerman translation principle. We can achieve this as follows.

Theorem 6.3. *Let $\boldsymbol{X}_{c\delta+\rho}$ be the quaternionic discrete series $(\mathfrak{g},K)$-module with Blattner parameter $\lambda = (d/2)\delta$. Define two linear forms μ, μ' on $\mathfrak{a}_{\mathfrak{p},0}$ by*

$$\mu(H_1) = 2+\frac{d}{2}, \qquad \mu(H_i) = (\delta,\alpha_i^\vee)\frac{d}{2} \qquad (i \ge 2), \tag{6.8}$$

$$\mu'(H_1) = d+2, \qquad \mu'(H_i) = 0 \qquad (i \ge 2), \tag{6.9}$$

where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$ *denotes the coroot of* α_i. *Then the 0th* $\mathfrak{n}$*-homology space of* $\boldsymbol{X}_{c\delta+\rho}$ *is described as*

$$(6.10) \qquad H_0(\mathfrak{n}, c\delta+\rho) \simeq \begin{cases} \sigma\hat{\otimes}\exp\mu' & (\,\mathfrak{g}_0 \simeq \mathfrak{sp}(p,1)) \\ id_{M_0}\hat{\otimes}\{[2]\cdot\exp\mu \oplus \exp\mu'\} & (\textit{otherwise}) \end{cases}$$

as $M_0A_\mathfrak{p}$*-modules. Here,* $[2]\cdot\exp\mu := \exp\mu \oplus \exp\mu$ *(two copies),* id_{M_0} *is the trivial representation of* M_0*, and* $\sigma := \tau_\lambda|M_0$*, the restriction of* τ_λ *to* M_0*, is an irreducible representation of* M_0 *for* $\mathfrak{g}_0 \simeq \mathfrak{sp}(p,1)$.

To prove this theorem, we use the differential operator $\mathcal{D}$ of gradient type (Schmid operator; see [32] for the definition) on $C^\infty_{\tau_\lambda}(G)$ (cf. (2.3)) whose kernel realizes the maximal globalization of $\boldsymbol{X}^*_{c\delta+\rho} = \boldsymbol{X}_{c\delta+\rho}$ for sufficiently large c. To be a little more precise, [32, I, Prop.3.2] together with Corollary 2.6 gives the isomorphisms

$$(6.11) \qquad H_0(\mathfrak{n}, c\delta+\rho)^* \simeq \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}_{c\delta+\rho}, C^\infty(G; id_N)) \simeq \mathrm{Ker}\,\mathcal{D}_{id_N}$$

of $MA_\mathfrak{p}$-modules. Then the space $\mathrm{Ker}\,\mathcal{D}_{id_N}$, the totality of right N-fixed solutions F of the differential equation $\mathcal{D}F = 0$, can be specified by explicit but rather long calculation.

Remark 6.4. For simple Lie groups of real rank one, the 0th $\mathfrak{n}$-homology spaces for arbitrary discrete series have been described by Silva and also by Collingwood. Also for higher rank groups, the work [16] of Knapp and Wallach specifies a certain number of irreducible $MA_\mathfrak{p}$-submodules of the $\mathfrak{n}$-homology spaces of discrete series. For example, one can find out the submodule $\mathrm{id}_{M_0} \otimes [2]\cdot\exp\mu$ of $H_0(\mathfrak{n}, c\delta+\rho)$ from their result.

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DIVISION OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO, 060-0810 JAPAN (〒 060-0810 札幌市北区北 10 条西 8 丁目 北海道大学大学院理学研究科数学専攻)
E-mail address: `yamasita@math.sci.hokudai.ac.jp`