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On the cohomology of finite Chevalley groups

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Introduction

Let $G(\mathbf{F}_q)$ be a finite Chevalley group defined over the finite field $\mathbf{F}_q$ with q elements and l a prime number with $(\mathrm{ch}(\mathbf{F}_q), l) = 1$. In this note, we consider the cohomology $H^*(G(\mathbf{F}_q), \mathbf{Z}/l)$ by the étale method inaugurated by a mile-stone paper of Quillen [Q1, Q2]. Friedlander has developed and published a book [F1].

Let G be a Chevalley $\mathbf{Z}$-scheme and G_k a scalar extension by an algebraically closed field k with $\mathrm{ch}(k) = p$. Let X be a k-scheme equipped with an G_k-action and $B(X, G_k)_\bullet$ a classifying simplicial scheme. Then the Deligne spectral sequence [D] is of the form

$$E_2 = \mathrm{Cotor}_{H^*(G_k,\mathbf{Z}/l)}(H^*_{et}(X), \mathbf{Z}/l)) \Rightarrow H^*(B(X, G)_\bullet, \mathbf{Z}/l))$$

using the Lang isogeny [L]: $G(\mathbf{F}_q)\backslash G_k \simeq G_k$, the right G_k-action of G is given by the F-conjuguation where F is the q-th Frobenius map. Then the above spectral sequence takes the form

$$E_2 = \mathrm{Cotor}_{H^*(G_k,\mathbf{Z}/l)}(H^*_{et}(G_k, \mathbf{Z}/l), \mathbf{Z}/l) \Rightarrow H^*(B(G(\mathbf{F}_q)\backslash G_k, G)_\bullet, \mathbf{Z}/l)).$$

We prove it in §1. The Lang map L induces a Galois covering $G_k \to G(\mathbf{F}_q)\backslash G_k$ and $B(G_k, G_k)_\bullet \to B(G(\mathbf{F}_q)\backslash G_k, G_k)_\bullet$ is considered as a Galois covering between the simplicial schemes. Generally, let $p : Y_\bullet \to X_\bullet$ be a Galois covering with its Galois group. Then we construct the Hochschild-Serre spectral sequence. In the above case, there is a spectral sequence such that

$$E_2 = H^p(G(\mathbf{F}_q), H^q(B(G_k, G_k)_\bullet, \mathbf{Z}/l)) \Rightarrow H^{p+q}(G(\mathbf{F}_q), \mathbf{Z}/l).$$

Applying the Deligne spectral sequence, it is easily shown that $H^*(B(G_k, G_k)_\bullet, \mathbf{Z}/l)$ is acyclic. Hence the above spectral sequence collapes at the E_2-term. After all, we get a spectral sequence which converges to the cohomology of a finite Chevalley group. We call the spectral sequence Deligne-Eilenberg-Moore spectral sequence.

1. Deligne-Eilenberg-Moore spectral sequence

In this section, we introduce a spectral sequence of Eilenberg-Moore type converging to $H^*(G(\mathbb{F}_q);\mathbb{Z}/l)$. For general arguments, we refer to Friedlander [F1].

First let us recall the simplicial scheme $B(X,G)$ from [F1, 1. Example 2]. Let S be a scheme, G a group scheme over a scheme S and X a scheme over S equipped with a right G-action $X \times_S G \to X$. Then the simplicial scheme $B(X,G)$ is defined by

$$B(X,G)_n = X \times_S \overbrace{G \times_S \cdots \times_S G}^{n}. \tag{1.1}$$

We define the face operators $d_i : B(X,G)_n \to B(X,G)_{n-1}$ for $0 \leq i \leq n$ by

$$\begin{aligned} d_0(x, g_1, \cdots, g_n) &= (xg_1, g_2, \cdots, g_n) && (i=0), \\ d_i(x, g_1, \cdots, g_n) &= (x, g_1, \cdots, g_i g_{i+1}, \cdots, g_n) && (1 \leq i \leq n-1), \\ d_n(x, g_1, \cdots, g_n) &= (x, g_1, \cdots, g_{n-1}) && (i=n), \end{aligned} \tag{1.2}$$

for $x \in X(T)$, $g_i \in G(T)$, where T is a scheme over S and $X(T)$ and $G(T)$ are T-valued points defined by $X(T) = \mathrm{Hom}_S(X,T)$ and $G(T) = \mathrm{Hom}_S(G,T)$.

Let X, T be schemes over S. Then we denote $T \times_S X$ by X_T, which is considered as a scheme over T.

Now we recall here the definition of the Lang map (see [L]). Let $G_{\mathbb{F}_q}$ be a linear algebraic group over $\mathbb{F}_q$ and ϕ the Frobenius automorphism defined by $\phi(x) = x^q$ for $x \in k$, where k is an algebraically closed field of $\mathbb{F}_q$. Then ϕ can be considered as a morphism of $G_{\mathbb{F}_q}$ and G_k which is a linear algebraic group over k. We consider the Lang map $\mathcal{L} : G_k \to G_k$ which can be defined by $\mathcal{L}(x) = \phi^{-1}(x)x$ for $x \in G(k)$.

Lemma 1.1 (Lang [L]). There holds $\mathcal{L}(x) = \mathcal{L}(y)$ for $x, y \in G(k)$ if and only if $y = ax$ for some $a \in G(\mathbb{F}_q)$.

Lemma 1.2 (Lang [L]). (1) The map $\mathcal{L} : G(k) \to G(k)$ is surjective. Hence G_k is a principal left $G(\mathbb{F}_q)$-space over G_k and $\mathcal{L}$ induces an isomorphism $G(\mathbb{F}_q)\backslash G_k \cong G_k$.
(2) If we define a right $G_{\mathbb{F}_q}$-action on G_k by

$$z \cdot x = \phi(x)^{-1} z x \quad \text{for } x, z \in G(k), \tag{1.3}$$

then there holds

$$z \cdot (xy) = (z \cdot x) \cdot y, \quad z \cdot 1 = z.$$

Moreover, the induced isomorphism

$$\mathcal{L} : G(\mathbb{F}_q)\backslash G_k \cong G_k$$

is a right equivariant $G_{\mathbb{F}_q}$-map, that is, there holds

$$\mathcal{L}([z]x) = z \cdot x$$

for $[z] \in (G(\mathbb{F}_q)\backslash G_k)(k)$ and $x \in G(k)$.

Corollary 1.3. The isomorphism in the above lemma induces an isomorphism

$$B(G(\mathbb{F}_q)\backslash G_k, G_k) \cong B(G_k, G_k)$$

as a simplicial scheme, where the right G_k-action on G_k is given by $z \cdot x = \phi(x)^{-1}zx$ for $x, y \in G_k$.

Theorem 1.4. Let $G_{\mathbb{Z}}$ be a Chevalley group scheme of Lie type over $\mathbb{Z}$. Then we have a spectral sequence $\{E_r\}$ of Eilenberg-Moore type such that

$$E_2 = \mathrm{Cotor}_{H^*_{et}(G_k;\mathbb{Z}/l)}(H^*_{et}(G_k;\mathbb{Z}/l), \mathbb{Z}/l),$$
$$E_\infty = \mathrm{gr}H^*(G(\mathbb{F}_q);\mathbb{Z}/l),$$

where l is a prime such that $(l,q) = 1$ and k is an algebraically closed field of $\mathbb{F}_q$. The comodule structure of $H^*_{et}(G_k;\mathbb{Z}/l)$ is induced from (1.3) in Lemma 1.2.

The Eilenberg-Moore spectral sequence of a simplicial scheme for complex algebraic groups is given by Deligne [D]. However, as his proof seems not to be appropriate in our context, we give here a proof following Friedlander [F1], and so we use his notations.

We recall a constant sheaf $\mathbb{Z}/l$ on the étale site $\mathrm{Et}(B(Y,G))$. If we denote by $(\mathbb{Z}/l)_n$ a constant sheaf $\mathbb{Z}/l$ on $\mathrm{Et}(B(Y,G))$, then a constant sheaf $\mathbb{Z}/l$ is a collection of $(\mathbb{Z}/l)_n$ for $n \geq 0$ satisfying the following property; if $\alpha^* : X_m \to X_n$ is a map induced from a simplicial map $\alpha : \Delta(n) \to \Delta(m)$, then it induces the identification $\alpha^*(\mathbb{Z}/l)_m = (\mathbb{Z}/l)_n$.

Proposition 1.5 ([F1] Proposition 2.2). Let $X_\bullet$ be a simplicial scheme and F an abelian sheaf on $\mathrm{Et}(X_\bullet)$. Then $F \to \prod_{n=0}^{\infty} R_n(I_n^\bullet)$ is an injective resolution in $\mathrm{Absh}(X_\bullet)$, where the function

$$R_n(\ ) : \mathrm{Absh}(X_n) \to \mathrm{Absh}(X_\bullet)$$

is defined by

$$(R_n(G))_m = \prod_{\Delta[n]_m} \alpha^* G,$$

such that each restriction $F_n \to I_n^\bullet$ is an injective resolution on $\mathrm{Absh}(X_n)$. Moreover we have

$$\mathrm{Hom}_{X_\bullet}(R_n(G), F) \cong \mathrm{Hom}_{X_n}(G, F_n).$$

Proof of Theorem. Let us recall the complex defined in [F1, Proposition 2.4]; let $L^n(\ ) : \mathrm{Absh}(X_n) \to \mathrm{Absh}(X_\bullet)$ be defined by

$$(L^n(G))_m = \bigoplus_{\alpha \in \Delta[m]_n} \alpha^* G, \ n \geq 0$$

for $G \in \mathrm{Absh}(X_n)$. From the definition of a sheaf on a simplicial scheme, we see that

$$\mathrm{Hom}_{X_\bullet}(L^n(G), E) \cong \mathrm{Hom}_{X_n}(G, F_n)$$

for $F \in \mathrm{Absh}(X_\bullet)$. We set

$$L^m(\mathbb{Z}|_{X_m}) = \mathbb{Z}\langle m\rangle$$

for $\mathbb{Z}|_{X_m} \in \mathrm{Absh}(X_m)$. From the definition of L^m, we have

$$(\mathbb{Z}\langle m\rangle)_n = \bigoplus_{\Delta[n]_m} \mathbb{Z}$$

on X_n. We define the augmented complex of sheaves

$$\{C(\cdot) = \bigoplus_{m=0} \mathbb{Z}\langle m\rangle, \partial\langle m\rangle : \mathbb{Z}\langle m\rangle \to \mathbb{Z}\langle m-1\rangle\}$$

in the following manner. Restricting to X_n, an augmentation and a boundary operator

$$(\varepsilon)_n : (\mathbb{Z}\langle 0\rangle)_n \to (\mathbb{Z})_n,$$
$$\partial\langle n\rangle_n : (\mathbb{Z}\langle m\rangle)_n \to (\mathbb{Z}\langle m-1\rangle)_n$$

are given by the summation

$$\bigoplus_{\Delta[n]_0} Z(U) \to \mathbb{Z}(U),$$
$$\sum_{i=0}^{m}(-1)^i\partial_i : \bigoplus_{\Delta[n]_m} \mathbb{Z}(U) \to \bigoplus_{\Delta[n]_{m-1}} \mathbb{Z}(U)$$

for $U \to X_n$ in $\mathrm{Et}(X_\bullet)$ respectively. □

When we restrict the complex to X_n, we see that

$$C\langle\cdot\rangle_n \simeq C_\bullet(\Delta[n]),$$

where $C_\bullet(\Delta[n])$ is the augmented chain complex of a simplex $\Delta[n]$. Since the restriction functor $(\ )_n$ (see [F1]) is exact and since $C_\bullet(\Delta[n])$ is acyclic, the complex $C\langle\cdot\rangle$ is acyclic in $\mathrm{Absh}(X_\bullet)$. We denote for simplicity $C\langle m\rangle$ and $\partial\langle m\rangle$ by C^{-m} and ∂^{-m} respectively.

Let $F \to I^\bullet$ be an injective resolution of F in $\mathrm{Absh}(X_\bullet)$ and $\delta^i : I^{i+1} \to I^{i+1}$ a difference. We denote

$$\prod_{q\geq 0} \mathrm{Hom}^n_{\mathrm{Absh}(X_\bullet)}(C^{-q}, I^{-q+n})$$

simply by $\mathrm{Hom}^n_{\mathrm{Absh}(X_\bullet)}(C^\bullet, I^\bullet)$. We define that

$$\mathrm{Hom}^\bullet(C^\bullet, I^\bullet) = \bigoplus_{n\geq 0} \mathrm{Hom}^n_{\mathrm{Absh}(X_\bullet)}(C^\bullet, I^\bullet)$$

and that

$$(\delta^n f)^{-q} = \delta^{-q+n} f^{-q} + (-1)^{n+1} f^{-q+1}\partial^{-q}$$

for $f = (f^{-q}) \in \mathrm{Hom}^n(C^{-q}, C^{-q+n}) = \mathrm{Hom}^n(C^\bullet, I^\bullet)$.

We consider a spectral sequence associated with the double complex defined as follows. We define the first filtration by

$$F^{\mathrm{I}} = \mathrm{Hom}^\bullet(C^\bullet, \bigoplus_{n\le p} I^n),$$

where we define two kinds of differentials δ_{I} and δ_{II} respectively by

$$(\delta_{\mathrm{I}} f)^{-q} = \delta^{-q+n} f^{-q},$$
$$(\delta_{\mathrm{II}} f)^{-q} = (-1)^{n+1} f^{-q+1}\partial^{-q}$$

and define

$$\delta = \delta_{\mathrm{I}} + \delta_{\mathrm{II}}.$$

Since $C^\bullet$ is acyclic and since $I^\bullet$ is injective, we see that

$$E_1^{\mathrm{I}p} = H(F_p^{\mathrm{I}}/F_{p-1}^{\mathrm{I}}, \delta_2) = \begin{cases} 0 & (p \ge 0) \\ \mathrm{Hom}(\mathbb{Z}, I^\bullet) & (p = 0). \end{cases}$$

From the definition of the cohomology, we have

$$E_2^{\mathrm{I}p} = H^p(\mathrm{Hom}(\mathbb{Z}, I), \delta_{\mathrm{I}}) = H^p(X_\bullet, F).$$

We see immediately that $E_2' = E_\infty'$, which implies that

$$H^n(\mathrm{Hom}^\bullet(C^\bullet, I^\bullet)\delta) = H^n(X_\bullet, F).$$

We define the second filtration by

$$F_p^{\mathrm{II}} = \mathrm{Hom}^\bullet(\bigoplus_{m\le p} C^{-m}, I^\bullet).$$

Then we see that (where $q = \deg f - p$):

$$\begin{aligned} E_1^{p,q} &= H^q(\mathrm{Hom}^\bullet(C^{-p}, I^\bullet), \delta_{\mathrm{I}}) = H^q(\mathrm{Hom}^\bullet(L^p(\mathbb{Z}_{X_p}), I^\bullet), \delta_{\mathrm{I}}) \\ &= H^q(\mathrm{Hom}^\bullet_{X_p}(\mathbb{Z}, I^\bullet[p]), \delta_{\mathrm{II}}|_{X_p}) = H^{p+q}(X_p, F_p). \end{aligned}$$

Proposition 1.6 ([F1], Proposition 2.4). We have a spectral sequence $\{E_r^{p,q}\}$ such that

$$E_1^{p,q} = H^{p+q}(X_p; F_p),$$
$$E_\infty^{p,q} = \mathrm{gr} H^{p+q}(X_\bullet; F).$$

We apply this spectral sequence to

$$X_\bullet = B(G_k, G_k) \cong B(G(\mathbb{F}_q)\backslash G_k, G_k)$$

with $F = \mathbb{Z}/l$.

Lemma 1.7. We have a spectral sequence $\{E_r\}$ such that

$$E_1^{p,*} = H_{et}^*(G_k; \mathbb{Z}/l) \otimes \bigotimes^p (H_{et}^*(G_k; \mathbb{Z}/l)[1]),$$
$$E_\infty = \mathrm{gr} H^*(B(G(\mathbb{F}_q)\backslash G_k, G_k); \mathbb{Z}/l).$$

Proposition 1.8. We have

$$E_2^{p,*} = \mathrm{Cotor}^p_{H_{et}^*(G_k;\mathbb{Z}/l)}(H_{et}^*(X; \mathbb{Z}/l), \mathbb{Z}/l).$$

Proof. The non-decreasing function $\delta_i^* : [p] \to [p+1]$ such that $i \notin \operatorname{Im} d_1$ induces a simplicial map $\partial_i : \Delta[p+1] \to \Delta[p]$ defined by

$$\partial_i[0,1,\dots,p+1] = [0,1,\dots,i,\dots,p]$$

and the morphism $d_i : X_{p+1} \to X_p$ defined by (1.2). So, the morphism

$$\partial_i^* : \operatorname{Hom}(C_{1X_p}^{-p}, I_{1X_p}^{\bullet}) \to \operatorname{Hom}(C_{1X_{p+1}}^{-p-1}, I_{1X_{p+1}}^{\bullet})$$

defined by $f\partial_i$ is induced from the inverse image of a sheaf $\mathbb{Z}/l$ by $d_i : X_{p+1} \to X_p$. Hence we have

$$E_2 = H(E_1, \delta_{\mathrm{I}})$$

and

$$\delta_{\mathrm{I}} = (^1)^{p+1} \sum_{i=0}^{p} (-1)^i \partial_i^* = (-1)^{p+1} \sum_{i=0}^{p} (-1)^i d_i^* : E_1^{p,*} \to E_1^{p,*}.$$

In this case, we can give an explicit representation of d_i^* as follows; let

$$\Delta_X : H_{et}^*(X;\mathbb{Z}/l) \to H_{et}^*(X;\mathbb{Z}/l) \otimes H_{et}^*(G_k;\mathbb{Z}/l)$$

and

$$\Delta : H^*(G_k;\mathbb{Z}/l) \to H_{et}^*(G_k;\mathbb{Z}/l) \otimes H^*(G_k;\mathbb{Z}/l)$$

be the comodule map and the coalgebra map respectively induced from a right G-action $X \times G_k \to X$ and a multiplication $G_k \times G_k \to G_k$. Then we obtain

$$d_i^*(m \otimes x_1 \otimes \cdots \otimes x_p) = \begin{cases} \Delta_X(m) \otimes x_1 \otimes \cdots \otimes x_p & (i = 0) \\ m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes \Delta(x_i) \\ \quad \otimes x_{i+1} \otimes \cdots \otimes x_p & (1 \le i \le p-1) \\ m \otimes x_1 \otimes \cdots \otimes x_p \otimes 1 & (i = p). \end{cases}$$

Therefore we have shown that

$$E_2^{p,*} = \operatorname{Cotor}^p_{H_{et}^*(G_k;\mathbb{Z}/l)}(H_{et}^*(X;\mathbb{Z}/l), \mathbb{Z}/l).$$

□

2. Hochschild-Serre spectral sequence

In this section, we construct the Hochschild-Serre spectral sequence for simplicial schemes in a little more direct way than in Milne [Mi].

Let $X_\bullet$ and $Y_\bullet$ be simplicial schemes over a field k. Then we call $\pi_\bullet : Y_\bullet \to X_\bullet$ a *finite Galois cover* with Galois group G if $\pi_n : Y_n \to X_n$ is a finite Galois cover with Galois group G for all n and if $\pi_\bullet$ is compatible with the face and degeneracy operators.

Theorem 2.1. Let $\pi_\bullet : Y_\bullet \to X_\bullet$ be a finite Galois cover with Galois group G for simplicial schemes. Let F be an abelian sheaf on $\mathrm{Et}(X_\bullet)$. Then we have a Hochschild-Serre spectral sequence $\{E_r^{p,q}\}$ such that

$$E_2^{p,q} = H^p(G, H^q(X_\bullet; F)),$$
$$E_\infty = \mathrm{gr} H^{p+q}(Y_\bullet; F).$$

To prove the theorem, we prepare some notations.

Let $(B_\bullet(G,G), \partial_\bullet, \sigma_\bullet)$ and $(Y_\bullet, d_\bullet, s_\bullet)$ be simplicial schemes defined in the section 1. Then we define a double simplicial scheme $B(G,G)_\bullet \boxtimes Y_\bullet$ as follows; as schemes, we set

$$B(G,G)_p \boxtimes Y_q = \coprod_{g_I \in G^{p+1}} Y_{q,g_I},$$

where Y_{q,g_I} is indexed by $g_I \in G^{p+1} = B(G,G)_p$ and we have $Y_{q,g_I} \cong Y_q$ as schemes.

We denote Y_{t,g_I} by $g_I \boxtimes Y_t$. Then we define two kinds of face operators

$$\partial_p^i \boxtimes 1_q : B(G,G)_p \boxtimes Y_q \to B(G,G)_{p-1} \boxtimes Y_q,$$
$$1_p \boxtimes d_q^i : B(G,G)_p \boxtimes Y_q \to B(G,G)_p \boxtimes Y_{q-1}$$

by

$$\partial_p^i \boxtimes 1_q((g_0, g_1, \dots, g_p, y)) = \begin{cases} (g_0, \dots, g_i g_{i+1}, \dots, g_p, y) & (0 \le i \le p-1) \\ (g_0, g_1, \dots, g_{p-1}, y) & (i = p), \end{cases}$$
$$1_p \boxtimes d_q^j((g_0, g_1, \dots, g_p, y)) = (g_0, g_1, \dots, g_p, d_q^j(y))$$

respectively, where we identify $B(G,G)_p \boxtimes Y_q(S)$ with $G^{\times(p+1)} \times Y_q(S)$ for a k-scheme S. Similarly we define two kinds of degeneracy operators

$$\sigma_p^i \boxtimes 1_q : B(G,G)_p \boxtimes Y_q \to B(G,G)_{p+1} \boxtimes Y_q,$$
$$1_p \boxtimes s_q^i : B(G,G)_p \boxtimes Y_q \to B(G,G)_p \boxtimes Y_{q+1}$$

by

$$\sigma_p^i \boxtimes 1_q((g_0, g_1, \dots, g_p, y)) = (g_0, \dots, g_i, e, g_{i+1}, \dots, g_p, y)$$
$$1_p \boxtimes s_q^i((g_0, g_1, \dots, g_p, y)) = (g_0, g_1, \dots, g_p, s_q^i(y)).$$

By abuse of notation we put

$$\partial_p^i \boxtimes 1_q = \partial_p^i, \ 1_p \boxtimes d_q^j = d_q^j, \ \sigma_p^i \boxtimes 1_q = \sigma_p^i, \ 1_p \boxtimes s_q^i = s_q^i.$$

We define a G-action on $B(G,G)_\bullet \boxtimes Y_\bullet$ by

$$g((g_0, g_1, \dots, g_p, y)) = (gg_0, g_1, \dots, g_p, y), \; g \in G.$$

Clearly we see that the G-action is compatible with all the face and degeneracy operators, and we have the identities

$$\partial_p^i d_q^j = d_q^j \partial_p^i, \quad \sigma_p^i s_q^j = s_q^j \sigma_p^i.$$

Remark 2.2. Let F be an abelian sheaf on $Y_{\bullet et}$. Then we can consider F as a sheaf on $X_{\bullet et}$, because $Y_\bullet$ is a Galois cover over $X_\bullet$.

The Galois group G acts on F from the right hand side and on $Y_\bullet$ from the left one. Moreover $\prod_{g_I \in G^{p+1}} F_{g_I}$ is a sheaf on

$$B(G,G)_p \boxtimes (Y_{\bullet et}) = \coprod_{g_I \in G^{p+1}} (Y_{\bullet et})_{g_I},$$

where F_{g_I} is a sheaf F indexed by $g_I \in G^{p+1}$.

In the similar manner to before, we can associate the sheaf on

$$B(G,G)_p \boxtimes (Y_{\bullet et})$$

to a sheaf F on $Y_{\bullet et}$ and denote its sheaf by

$$B(G,G)_p \boxtimes F. \tag{2.1}$$

Now we describe the face operators ∂_p^{i*} induced on the sheaf F explicitly. Let $U \to Y_q$ be an étale map. Then we have the étale map induced by it:

$$B(G,G)_p \boxtimes U = \coprod_{g_I \in G^{p+1}} U_{g_I} \to \coprod_{g_I \in G^{p+1}} Y_{g_I} = B(G,G)_p \boxtimes Y_q.$$

We also have

$$F(B(G,G)_p \boxtimes U) = \prod_{g_I \in G^{p+1}} F(U_{g_I}),$$

and denote its section by $s = (s_{g_I})$. The face operator

$$\partial_p^{i*} : F(B(G,G)_p \boxtimes U) \to F(B(G,G)_{p+1} \boxtimes U)$$

is given by

$$(\partial_p^{i*} s)_{(g_0, \dots, g_p)} = \begin{cases} s(g_0, \dots, g_i g_{i+1}, \dots, g_p) & (0 \le i \le p-1) \\ s(g_0, g_1, \dots, g_{p-1}) & (i = p). \end{cases} \tag{2.2}$$

We consider an injective resolution

$$0 \to F \xrightarrow{d_F^\bullet} I^\bullet$$

of F on $X_{\bullet et}$ and define the sheaf complex

$$(C^\bullet, d^\bullet) = (\bigoplus_{n \ge 0} \bigoplus_{p+q=r} C^{p,q}, \bigoplus_{n \ge 0} \bigoplus_{p,q} d^{p,q})$$

by

$$C^{p,q} = B(G,G) \boxtimes I^q,$$
$$d^{p,q} = (-1)^i \partial_p^{i*} + (-1)^p d_F^q.$$

Then we have

Lemma 2.3. The G-free complex $C^\bullet$ gives rise to also an injective resolution on $X_{\bullet et}$:

$$0 \to F \xrightarrow{d^\bullet} C^\bullet.$$

Proof. Since I^q is injective and since $B(G,G)_p \boxtimes I^q$ is a direct product of I^q, we see that $B(G,G)_p \boxtimes I^q$ is injective and C^n is injective on $X_{\bullet et}$. Hence we will show that $0 \to F \to C^\bullet$ is acyclic. For a fixed geometric point $\bar{x}$, it is enough to show that $0 \to F_{\bar{x}}^\bullet \to C_{\bar{x}}^\bullet$ is acyclic. We calculate the homology of the double complex $(C_{\bar{x}}^\bullet, d^\bullet)$ by using a spectral sequence. We introduce filtration $F^n C_{\bar{x}}^\bullet$ by $\bigoplus_{p \geq n} C_{\bar{x}}^{p,\bullet}$ for n and consider the associated spectral sequence. From the injective resolution of F, we see that

$$\begin{aligned} E_1^{p,q} &= H^q(B(G,G)_p \boxtimes I_{\bar{x}}^q, (-1)^p d_F^q) \\ &= \begin{cases} 0 & (q \geq 0) \\ B(G,G)_p \boxtimes F_{\bar{x}} & (q = 0). \end{cases} \end{aligned}$$

The differential d_1 is given by $\sum_{i=0}^{p} (-1)^i \partial_p^{i*}$ from (2.2) in Remark 2.2. Forgetting the G-action on $F_{\bar{x}}$, we obtain

$$E_1^{p,0} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes B^p(G), \mathbb{Z}) \otimes_{\mathbb{Z}} F_{\bar{x}},$$

where $\mathbb{Z}[G] \otimes B^\bullet(G)$ is the standard bar complex of G over $\mathbb{Z}$ and $\mathbb{Z}[G]$ is a group ring over $\mathbb{Z}$ [Mc]. Hence we obtain that

$$E_2^{p,0} = \begin{cases} 0 & p > 0 \\ F_{\bar{x}} & p = 0. \end{cases}$$

It is easy to prove that $H^\bullet(C, d^\bullet) = F_{\bar{x}}$. □

Lemma 2.4. Let $\Gamma_{X_\bullet}$ and $\Gamma_{Y_\bullet}$ be the section functors of $X_\bullet = \{X_n\}$ and $Y_\bullet = \{Y_n\}$ respectively. Then for a sheaf F on $Y_{\bullet et}$, we have

$$\Gamma_{X_\bullet}(F) = \Gamma_{Y_\bullet}(F)^G.$$

Proof. From the definition of the section functor [F1, D] we recall that

$$\Gamma_{Y_\bullet}(F) = \mathrm{Ker}(F(Y_0) \underset{d_0^1}{\overset{d_0^0}{\rightrightarrows}} F(Y_1)).$$

Since Y_i/X_i is a Galois cover with the same Galois group G, we have

$$F(X_i) = F(Y_i/G) = F(Y_i)^G.$$

Observing that the face operators are compatible with the G-action, we have

$$\Gamma_{X_\bullet}(F) = \Gamma_{Y_\bullet}(F) \cap \Gamma(X_0, F) = \Gamma_{Y_\bullet}(F) \cap \Gamma(Y_\bullet, F)^G = \Gamma_{Y_\bullet}(F)^G.$$

From Lemmas 1.1 and 1.2, we summarize that

$$H^n(X_\bullet; F) = H^n(\Gamma_{Y_\bullet}(C^\bullet)^G).$$

□

Lemma 2.5. We have

$$\Gamma_{Y_\bullet}(C^{p,q}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes B^p(G), \Gamma^q_{Y_\bullet}(I^\bullet)),$$
$$\Gamma_{Y_\bullet}(C^{p,q})^G \cong \mathrm{Hom}_{\mathbb{Z}}(B^p(G), \Gamma^q_{Y_\bullet}(I^q)),$$

where $\mathbb{Z}[G] \otimes B^p(G)$ is the standard bar complex of G over $\mathbb{Z}$.

Proof. From the construction of $C^{p,q}$, we have

$$\Gamma_{Y_\bullet}(C^{p,q}) = \Gamma_{Y_\bullet}\Big(\prod_{g_I \in G^{p+1}} I^q_{g_I}\Big) = \prod_{g_I \in G^{p+1}} \Gamma_{Y_\bullet}(I^q_{g_I}),$$

where $I^q_{g_I} \cong I^q$. Noting Remark 2.2, we see that $I^\bullet$ is a right G-module and so the left G-action is given by

$$g^{-1}s = sg$$

for $g \in G(k) = G_{\mathbb{Z}}(k)$ and $s \in I^\bullet(U)$, where $U \to X_n$ is any étale map.
Hence the left action of G on $\prod_{g_I \in G^{p+1}} \Gamma_{Y_\bullet}(I^q_{g_I})$ is given by

$$g * s_{(g_0, g_1, \dots, g_p)} = g^{-1}(s_{(gg_0, g_1, \dots, g_p)}) \tag{2.3}$$

for $(s_{g_I}) \in \prod_{g_I \in G^{p+1}} \Gamma_{Y_\bullet}(I^q_{g_I})$, $g_I = (g_0, g_1, \dots, g_p)$ and $g, g_i \in G$.

When we identify $\prod_{g_I \in G^{p+1}} \Gamma_{Y_\bullet}(I^q_{g_I})$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes B^p(G), \Gamma_{Y_\bullet}(I^q))$ by

$$f(g_0, g_1, \dots, g_p) = s_{(g_0, g_1, \dots, g_p)}$$

for $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes B^p(G), \Gamma_{Y_\bullet}(I^q))$, the G-module structure is given by

$$(gf)(g_0, g_1, \dots, g_p) = g^{-1} f(gg_0, g_1, \dots, g_p).$$

Therefore we see that

$$\Gamma_{Y_\bullet}(C^{p,q})^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes B^p(G), \Gamma_{Y_\bullet}(I^q))$$
$$\cong \mathrm{Hom}_{\mathbb{Z}}(B^p(G), \Gamma_{Y_\bullet}(I^q)).$$

□

Under these preparations, the construction of the spectral sequence is a routine argument from the double complex. We define the filtration of the complex $\Gamma_{Y_\bullet}(C^\bullet)^G$ by $F^n = \bigoplus_{p\geq n} \Gamma_{Y_\bullet}(C^{p,\bullet})^G$. From Lemma 2.5, it follows that

$$E_1^{p,q} \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\otimes B^p(G), H^q(\Gamma_{Y_\bullet}(I^\bullet), d_F)) \\ = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]\otimes B^p(G), H^q(Y_\bullet; F)).$$

As shown in the proof of Lemma 2.3, the differential d_1 is given by

$$d_1 = \sum_{i=0}^{p}(-1)^i \partial_r^{i*}.$$

Hence we obtain that

$$E_2^{p,q} = H^p(G, H^q(Y_\bullet; F)).$$

Thus we have the spectral sequence which converges to

$$E_\infty^{*,*} = \mathrm{gr}H^*(X_\bullet; F).$$

Now we apply the Hochschild-Serre spectral sequence in the following form. Let G_k be an algebraic group defined over a prime field $\mathbb{F}_p$ and $G(\mathbb{F}_q)$ the finite group consisting of its $\mathbb{F}_q$-rational points with $q = p^n$. Then according to Lang [L], the left coset $G(\mathbb{F}_q)\backslash G_k$ is a k-affine scheme [Se, III, 12] and $G(\mathbb{F}_q)$ is a finite Galois cover over $G(\mathbb{F}_q)\backslash G_k$ with Galois group $G(\mathbb{F}_q)$. So we can take $B_\bullet(G_k, G_k)$ and $B_\bullet(G(\mathbb{F}_q)\backslash G_k, G_k)$ as $Y_\bullet$ and $X_\bullet$ in the above argument. Under the present context, the spectral sequence $\{E_r^{p,q}\}$ takes the form

$$E_2^{p,q} = H^p(G(\mathbb{F}_q), H^q(B_\bullet(G_k, G_k); F)) \\ \Rightarrow H^{p+q}(B_\bullet(G(\mathbb{F}_q)\backslash G_k, G_k); F). \tag{2.4}$$

Lemma 2.6 (Friedlander [F2]). For a reductive algebraic group G_k defined and split over $\mathbb{F}_p$, we have

$$H^n(B_\bullet(G_k, G_k); \mathbb{Z}/l) = 0 \quad \text{for } n > 0.$$

Proof. We consider the Deligne-Eilenberg-Moore spectral sequence

$$E_1^{n,*} = H^*(B_n(G_k, G_k); \mathbb{Z}/l) \Rightarrow H^*(B_\bullet(G_k, G_k); \mathbb{Z}/l).$$

From Friedlander-Parshall [FP], we can apply the Künneth formula to $H^*(B_n(G_k, G_k); \mathbb{Z}/l)$. We have

$$H^*(B_n(G_k, G_k); \mathbb{Z}/l) \cong H^*_{et}(G_k; \mathbb{Z}/l)^{\otimes n},$$

which implies that the E_1-term is the cobar complex of $H^*_{et}(G_k; \mathbb{Z}/l)$ over $\mathbb{Z}/l$. Hence we have

$$E_2^{p,q} = 0 \quad \text{except } p = q = 0.$$

□

Theorem 2.7. For a reductive algebraic group G_k defined and split over $\mathbb{F}_q$, we have

$$H^*(G(\mathbb{F}_q); \mathbb{Z}/l) \cong H^*(B(G(\mathbb{F}_q)\backslash G_k, G_k); \mathbb{Z}/l).$$

Proof. That the spectral sequence (2.4) collapses follows from Lemma 2.6. Then the rest of the assertion can be proved straightforwardly. □

Together with the Deligne-Eilenberg-Moore spectral sequence, we can now state the main theorem.

Theorem 2.8. For a reductive algebraic group G_k defined and split over $\mathbb{F}_q$, we obtain the spectral sequence $\{E_r\}$ such that

$$E_2 = \mathrm{Cotor}_{H^\bullet_{et}(G;\mathbb{Z}/l)}(H^\bullet_{et}(G;\mathbb{Z}/l), \mathbb{Z}/l),$$
$$E_\infty = \mathrm{gr}H^\bullet(G(\mathbb{F}_q), \mathbb{Z}/l).$$

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