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Block branching Miller forcing and covering numbers for prediction

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Abstract

We call a function from $\omega^{<\omega}$ to ω a predictor. A predictor π predicts $f \in \omega^{\omega}$ constantly if there is $n < \omega$ such that for all $i < \omega$ there is $j \in [i, i+n)$ with $f(j) = \pi(f \upharpoonright j)$. θ_{ω} is the smallest size of a set P of predictors such that every $f \in \omega^{\omega}$ is constantly predicted by some predictor in P. θ_{ubd} is the smallest cardinal κ satisfying the following: For every $b \in \omega^{\omega}$ there is a set P of predictors of size κ such that every $f \in \prod_{n<\omega} b(n)$ is constantly predicted by some predictor in P. We prove that θ_{ubd} is consistently smaller than θ_{ω}.

1 Introduction

Blass [2] introduced a combinatorial concept called "predicting and evading". There are some cardinal invariants associated with this notion, and the relations to well-known cardinal invariants, especially those which appear in Cichoń's diagram, were studied by Blass, Brendle, Shelah and others. (See, for example, [2, 3, 4].)

Kamo [5, 6] introduced the notion of "constant prediction" and defined cardinal invariants θ_K for $2 \leq K \leq \omega$. Throughout this paper, we call a function π from $\omega^{<\omega}$ to ω *a predictor*, and $\mathcal{P}$ denotes the set of all predictors.

Definition 1.1. For $\pi \in \mathcal{P}$ and $f \in \omega^{\omega}$, we say π *predicts* f *constantly* if there is $n < \omega$ such that for all $i < \omega$ there is $j \in [i, i+n)$ satisfying $f(j) = \pi(f \upharpoonright j)$.

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Definition 1.2. Let $2 \leq K \leq \omega$. θ_K is the smallest size of $P \subseteq \mathcal{P}$ such that for every function $f \in K^\omega$ there is a predictor $\pi \in P$ predicting f constantly.

It is easily seen that $\theta_2 \leq \theta_3 \leq \cdots \leq \theta_\omega \leq 2^\omega$.

Let us recall the definitions for several cardinal invariants from Cichoń's diagram. $\mathsf{cov}(\mathcal{M})$ (respectively $\mathsf{cov}(\mathcal{N})$) is the smallest size of a set of meager (respectively null) sets of reals whose union covers the real line. $\mathsf{non}(\mathcal{M})$ is the smallest size of a nonmeager set. $\mathsf{cof}(\mathcal{N})$ is the smallest size of a basis for the ideal of null sets. For $f, g \in \omega^\omega$, $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$, and $\mathfrak{d}$ is the smallest size of a cofinal subset of ω^ω with respect to $\leq^*$. It is known that $\omega_1 \leq \mathsf{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathsf{cof}(\mathcal{N}) \leq 2^\omega$ and $\omega_1 \leq \mathsf{non}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N})$. (See [1] for details.)

Kamo [6] pointed out that $\mathsf{cov}(\mathcal{M}) \leq \theta_2$, $\mathsf{cov}(\mathcal{N}) \leq \theta_2$ and $\mathsf{non}(\mathcal{M}) \leq \theta_\omega$. Also, he proved the following consistency results.

Theorem 1.3. *1.* [5, Theorem 2.1] *It is consistent that* $\mathsf{cof}(\mathcal{N}) = \omega_1$ *and* $\theta_2 = \omega_2 = 2^\omega$.

2. [6, Corollary 2.2] *It is consistent that* $\theta_\omega = \omega_1$ *and* $\mathfrak{d} = \omega_2 = 2^\omega$.

3. [5, Theorem 4.2] *It is consistent that* $\theta_K = \omega_1$ *for* $2 \leq K < \omega$ *and* $\theta_\omega = \omega_2 = 2^\omega$.

Here we introduce another cardinal invariant θ_{ubd} by the following.

Definition 1.4. Let $b \in \omega^\omega$. θ_b is the smallest size of $P \subseteq \mathcal{P}$ such that for every function $f \in \prod_{i<\omega} b(i)$ there is a predictor $\pi \in P$ predicting f constantly. Let $\theta_{\text{ubd}} = \sup\{\theta_b : b \in \omega^\omega\}$.

It is easily seen that, $\theta_K \leq \theta_{\text{ubd}} \leq \theta_\omega$ for $2 \leq K < \omega$, and $\theta_\omega \leq \max\{\theta_{\text{ubd}}, \mathfrak{d}\}$.

In the model constructed in the proof of Theorem 1.3(3), $\mathsf{cof}(\mathcal{N}) = \omega_1$ holds [6]. By the relations $\theta_\omega \leq \max\{\theta_{\text{ubd}}, \mathfrak{d}\}$ and $\mathfrak{d} \leq \mathsf{cof}(\mathcal{N})$, θ_{ubd} must be ω_2 in this model. This shows the consistency of "$\theta_K = \omega_1$ for $2 \leq K < \omega$ and $\theta_{\text{ubd}} = \omega_2 = 2^\omega$".

In this paper we will prove the consistency of "$\theta_{\text{ubd}} = \omega_1$ and $\theta_\omega = \omega_2 = 2^\omega$". We introduce a new forcing notion called *block branching Miller forcing*. The required model is obtained by countable support iteration of block branching Miller forcing of length ω_2 over a model of CH.

Our notation is standard and we refer the reader to [1] for undefined notions.

Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, and $\dot{f}$ a $\mathbb{P}$-name for a function in ω^ω. We say $h \in \omega^\omega$ is *an interpretation of* $\dot{f}$ *below* p if there is a decreasing sequence

$\langle p_n : n < \omega\rangle$ of conditions in $\mathbb{P}$ such that $p_0 \leq p$ and $p_n \Vdash_{\mathbb{P}}$ "$\dot{f}\restriction n = h\restriction n$" for each $n < \omega$.

Here we review several notations concerning trees. For a tree T and $s \in T$, $\mathsf{succ}_T(s)$ is the set of all immediate successors of s in T. $s \in T$ is called *a splitting node in T* if $|\mathsf{succ}_T(s)| > 1$. $\mathsf{split}(T)$ is the set of all splitting nodes in T, and $\mathsf{stem}(T)$ is the least node of $\mathsf{split}(T)$.

Definition 1.5. For a tree $H \subseteq \omega^{<\omega}$, let

1. $\mathsf{Max}(H) = \{s \in H : \text{for all } i < \omega,\ s^\frown\langle i\rangle \notin H\}$,
2. $\mathsf{B}(H) = \{|s| : s \in \mathsf{split}(H) \cup \mathsf{Max}(H)\}$, and
3. $\mathsf{Lim}(H) = \{f \in \omega^\omega : \text{for all } i < \omega,\ f\restriction i \in H\}$.

Definition 1.6. We say a tree $H \subseteq \omega^{<\omega}$ is *skip branching* if for all $s \in \mathsf{split}(H)$, $\mathsf{succ}_H(s) \cap (\mathsf{split}(H) \cup \mathsf{Max}(H)) = \emptyset$.

In the following sections we use the following combinatorial lemmata.

For $a \in [\omega]^\omega$, let $\Gamma_a \in \omega^\omega$ be the increasing enumeration of a.

Definition 1.7. For $a \in [\omega]^\omega$ and $g \in \omega^\omega$, we say *a is g-thin* if $g(i) < \Gamma_a(i)$ for all $i < \omega$.

Lemma 1.8. [6, Lemma 2.3] *For any $g \in \omega^\omega$, there is a countable sequence $\langle g_i : i < \omega\rangle$ of functions in ω^ω such that, for a sequence $\langle a_i : i < \omega\rangle$ of infinite subsets of ω, if a_i is g_i-thin for all $i < \omega$, then $\bigcup_{i<\omega} a_i$ is g-thin.*

The following is a slight modification of [6, Lemma 2.4] and proved in the same way.

Lemma 1.9. *Let F be a set of strictly increasing functions in ω^ω such that for every $g \in \omega^\omega$ there is $f \in F$ with $f \not\leq^* g$, and $\langle I_{m,n} : (m,n) \in \omega \times \omega\rangle$ a pairwise disjoint set of intervals in ω. Then there is $f \in F$ such that, for each f-thin set $a \in [\omega]^\omega$ and $m < \omega$ there are infinitely many $n < \omega$ with $I_{m,n} \cap a = \emptyset$.*

2 Block branching Miller forcing

Miller forcing, also called rational perfect set forcing, is the partial order of subtrees of $\omega^{<\omega}$ which have infinitely branching nodes cofinally. The following definition is a modification of Miller forcing.

For each $n < \omega$, let $\mathcal{B}_n = \{W \subseteq \omega^{\leq n} : W \text{ is order-isomorphic to } \omega^{\leq n}\}$. For each $t \in \omega^{<\omega}$ and $W \subseteq \omega^{<\omega}$, let $t * W = \{t^\frown s : s \in W\}$.

Definition 2.1. *Block branching Miller forcing* $\mathbb{BPT}$ is defined as follows: $p \in \mathbb{BPT}$ if $p \subseteq \omega^{<\omega}$ is a tree and for every $s \in p$ and $n < \omega$ there are $t \in p$ and $W \in \mathcal{B}_n$ suth that $s \subseteq t$ and $t * W \subseteq p$. For $p, q \in \mathbb{BPT}$, $p \leq q$ if $p \subseteq q$.

Definition 2.2. For $p \in \mathbb{BPT}$ and $1 \leq n < \omega$, let $S_n(p)$ be the set of nodes s in p such that, $s * W \subseteq p$ for some $W \in \mathcal{B}_n$ and s is minimal with this property. Let $S(p) = \bigcup_{1 \leq n < \omega} S_n(p)$.

Note that, in particular, $S_1(p) = \{\mathsf{stem}(p)\}$.

Definition 2.3. For $p \in \mathbb{BPT}$ and $1 \leq n < \omega$, let $F_n(p) = \{s^\frown t : s \in S_n(p)$ and $t \in \omega^n$ and $s^\frown t \in p\}$

Without loss of generality we can assume that, for any $p \in \mathbb{BPT}$, $1 \leq n < \omega$ and $s \in F_n(p)$ there is a unique $t \in S_{n+1}(p)$ with $s \subseteq t$, because the set of such conditions is dense in $\mathbb{BPT}$.

Now we can introduce the following fusion order in $\mathbb{BPT}$.

Definition 2.4. For $p, q \in \mathbb{BPT}$, $p \leq_0 q$ if $p \leq q$ and $\mathsf{stem}(p) = \mathsf{stem}(q)$, and for $1 \leq n < \omega$, $p \leq_n q$ if $p \leq_0 q$ and $F_n(p) = F_n(q)$.

Proposition 2.5. $\mathbb{BPT}$ *satisfies Axiom A.*

Proof. Easy. □

Proposition 2.6. *Let* $\dot{G}$ *be the canonical name for a generic filter of* $\mathbb{BPT}$, *and* $\dot{g}$ *be the* $\mathbb{BPT}$*-name determined by* $\Vdash_{\mathbb{BPT}} \dot{g} = \bigcup\{\mathsf{stem}(p) : p \in \dot{G}\}$. *Then,*

1. *for any* $f \in \omega^\omega$, $\Vdash_{\mathbb{BPT}} \dot{g} \not\leq^* f$, *and*
2. *for any predictor* $\pi \in \mathcal{P}$, $\Vdash_{\mathbb{BPT}}$ *"π does not predict $\dot{g}$ constantly".*

Proof. Left to the reader. □

Corollary 2.7. *Assume* CH *holds in the ground model* $\mathbf{V}$. *Then* $\mathfrak{d} = \theta_\omega = \omega_2 = 2^\omega$ *holds in the forcing model by the countable support iteration of* $\mathbb{BPT}$ *of length* ω_2 *over* $\mathbf{V}$.

Proposition 2.8. *For* $p \in \mathbb{BPT}$ *and a* $\mathbb{BPT}$*-name* $\dot{h}$ *for a function in* ω^ω, *there are* $q \leq p$ *and* $f \in \omega^\omega$ *such that* $q \Vdash_{\mathbb{BPT}} f \not\leq^* \dot{h}$.

Proof. Almost the same as the case of Miller forcing ([1, Theorem 7.3.46(2)]). □

Let $M_1 = \{\langle\rangle\}$, $M_{n+1} = \prod_{1\le i\le n} \omega^i$ for $n \ge 1$, and $M = \bigcup_{1\le n<\omega} M_n$. Also, let $\tilde{M}_1 = \{\langle\rangle\}$, $\tilde{M}_{n+1} = \{s^\frown\langle t\rangle : s \in M_n \text{ and } t \in \omega^{\le n}\}$ for $n \ge 1$, and $\tilde{M} = \bigcup_{1\le n<\omega} \tilde{M}_n$. For each $p \in \mathbb{BPT}$ we can define a natural order-homomorphism Γ_p from $\tilde{M}$ to $\mathsf{split}(p)$. More precisely, for $p \in \mathbb{BPT}$ we define Γ_p by the following induction: First, let $\Gamma_p(\langle\rangle) = \mathsf{stem}(p)$. Suppose $\Gamma_p(s) \in S_n(p)$ is defined for $s \in M_n$. Fix $W \in \mathcal{B}_n$ satisfying $\Gamma_p(s) * W \subseteq p$ and an order-isomorphism σ from ω^n to W. For each $t \in \omega^n$, let $\Gamma_p(s^\frown\langle t\rangle)$ be the unique node of $S_{n+1}(p)$ extending $\Gamma_p(s)^\frown\sigma(t)$ and for $t \in \omega^{<n}$, let $\Gamma_p(s^\frown\langle t\rangle) = \Gamma_p(s)^\frown\sigma(t)$. Note that $\Gamma_p(s) = \Gamma_p(s^\frown\langle\langle\rangle\rangle)$ for $s \in M$, and so in this sense we may identify $s \in M_n$ to $s^\frown\langle\langle\rangle\rangle \in \tilde{M}_{n+1}$.

For $p \in \mathbb{BPT}$ and $s \in \tilde{M}$, let $p\restriction s = \{t \in p : t \subseteq \Gamma_p(s) \text{ or } \Gamma_p(s) \subseteq t\}$.

For $h \in \omega^\omega$ and $\tau \in \omega^{<\omega}$ with $\tau \not\subseteq h$, let $\Delta(\tau, h) = \min\{i : h(i) \ne \tau(i)\}$.

Definition 2.9. ([6, Definition 2.7]) A function u from a countable set to $\omega^{<\omega}$ is called *a type II function with limit* $h \in \omega^\omega$ if,

1. for all $i \in \mathrm{dom}(u)$, $u(i) \not\subseteq h$ and $\Delta(u(i), h) + 2 \le |u(i)|$, and
2. for all $i, j \in \mathrm{dom}(u)$ with $i \ne j$, $|\Delta(u(i), h) - \Delta(u(j), h)| \ge 2$.

Remark 1. Kamo [6] also defined the notion of type I functions, but now we need only type II functions.

Note that, for a function $b \in \omega^\omega$ and a set $\{f_{n,i} : (n, i) \in \omega \times \omega\}$ of functions in $\prod_{n<\omega} b(n)$, if $f_{n,i} \ne f_{n',i'}$ for any distinct $(n, i), (n', i') \in \omega \times \omega$, then there are $a \in [\omega \times \omega]^\omega$ and a function φ from a to ω such that

1. for all $n < \omega$, $\{i < \omega : (n, i) \in a\}$ is infinite, and
2. $\langle f_{n,i}\restriction\varphi(n, i) : (n, i) \in a\rangle$ is a type II function.

Here we call a subset T of $\omega^{<\omega}$ *a quasi-tree*. For a quasi-tree T and $s \in \omega^\omega$, let $\mathsf{Succ}_T(s) = \{t \in T : s \subsetneq t \text{ and there is no } u \in T \text{ such that } s \subsetneq u \subsetneq t\}$, $\mathsf{pred}_T(s) = t$ if $t \in T$ and $s \in \mathsf{Succ}_T(t)$ (if such t exists; otherwise $\mathsf{pred}_T(s)$ is undefined) and $\mathsf{dcl}(T) = \{t \in \omega^{<\omega} : t \subseteq u \text{ for some } u \in T\}$. By identifying $\langle t_1, \ldots, t_n\rangle \in \tilde{M}$ to $t_1{}^\frown \ldots {}^\frown t_n \in \omega^{<\omega}$, we also regard a subset X of $\tilde{M}$ as a quasi-tree.

For a quasi-tree $T \subseteq \omega^{<\omega}$ without maximal nodes, we define a function Γ_T from $\omega^{<\omega}$ to T by the following induction: First, let $\Gamma_T(\langle\rangle) = \mathsf{stem}(T)$. For $s \in \omega^{<\omega}$, fix an enumeration $\langle t_i : i < \omega\rangle$ of $\mathsf{Succ}_T(s)$, and for each $i < \omega$ let $\Gamma_T(s^\frown\langle i\rangle) = t_i$.

Definition 2.10. $\langle \delta_s : s \in T\rangle$ is *a quasi-tree of type II functions* if:

1. T is a quasi-tree,

2. for all $s \in T$, $\delta_s \in \omega^{<\omega}$,

3. for all $s \in T \setminus \mathsf{Max}(T)$, $\langle \delta_t : t \in \mathsf{Succ}_T(s)\rangle$ is a type II function with some limit $h \in \omega^\omega$ with $\delta_s \subseteq h$.

For a tree $T \subseteq \omega^{<\omega}$ and $s \in T$, we say T *is ω-branching above* s if, for any $t \in T \setminus \mathsf{Max}(T)$, if $s \subseteq t$ then $\mathsf{succ}_T(t)$ is infinite.

Let b be an arbitrary but fixed function in ω^ω.

The following is the main lemma to handle the successor step of the proof of Theorem 4.1.

Lemma 2.11. *Assume that $p \in \mathbb{BPT}$, η is a function from $\tilde{M}$ to ω, and $\dot{f}$ is a $\mathbb{BPT}$-name such that $p \Vdash_{\mathbb{BPT}} \dot{f} \in \prod_{n<\omega} b(n) \setminus \mathbf{V}$. Then there are $q \leq p$, a quasi-tree $X \subseteq \tilde{M}$ and $\langle \delta_s : s \in X\rangle$ such that:*

1. $M \subseteq X$,

2. $\langle \delta_s : s \in X\rangle$ is a quasi-tree of type II functions,

3. for all $s \in X$, $q \restriction s \Vdash_{\mathbb{BPT}} \delta_s \subseteq \dot{f}$, and

4. for all $s \in X$, $|\delta_s| > \eta(s)$.

Proof. By induction on $n < \omega$, we will construct a fusion sequence $\langle p_n : n < \omega\rangle$ of conditions in $\mathbb{BPT}$ starting with $p_0 \leq p$, x_s for $s \in M_n$, and $\delta_{s^\frown\langle t\rangle}$ for $s \in M_n$ and $t \in x_s$.

First, choose $p_0 \leq p$ and $\delta_{\langle\rangle} \in \omega^{<\omega}$ so that $|\delta_{\langle\rangle}| > \eta(\langle\rangle)$ and $p_0 \Vdash_{\mathbb{BPT}} \delta_{\langle\rangle} \subseteq \dot{f}$.

Suppose that

1. $p_{n-1} \in \mathbb{BPT}$,

2. x_s for $s \in M_{n-1}$, satisfying $\langle\rangle \in x_s$ and $\omega^{n-1} \subseteq x_s$, and

3. $\delta_{s^\frown\langle t\rangle}$ for $s \in M_{n-1}$ and $t \in x_s$

have been defined. Fix $s \in M_n$ and let $\delta_{s^\frown\langle\langle\rangle\rangle} = \delta_s$. For $t \in \omega^n$, choose $h^t_s \in \omega^\omega$ so that h^t_s is an interpretation of $\dot{f}$ below $p_{n-1} \restriction (s^\frown\langle t\rangle)$. Note that, for all $t \in \omega^n$, $\delta_s = \delta_{s^\frown\langle\langle\rangle\rangle} \subseteq h^t_s$. Let $y^n_s = \omega^n$. We will construct $y^{n-1}_s, y^{n-2}_s, \dots, y^0_s$ inductively.

Suppose $m < n$, $y^{m+1}_s \subseteq \bigcup\{\omega^k : m+1 \leq k \leq n\}$ and $\{h^u_s : u \in y^{m+1}_s\} \subseteq \omega^\omega$ have been defined. Fix $t \in \omega^m$.

Case 1. $\{h^u_s : u \in \mathsf{Succ}_{y^{m+1}_s}(t)\}$ is infinite. Then there are $X^t_s \in [\mathsf{Succ}_{y^{m+1}_s}(t)]^\omega$ and a function φ^t_s from X^t_s to ω such that

1. for any distinct $u, v \in X_s^t$, $h_s^u \neq h_s^v$,
2. $\langle h_s^u \restriction \varphi_s^t(u) : u \in X_s^t \rangle$ is a type II function with limit $h_s^t \in \omega^\omega$, and
3. $\mathsf{dcl}(X_s^t)$ is ω-branching above t.

By removing a certain finite part from each X_s^u, we can assume that $\mathrm{ran}(\varphi_s^u) \cap (\varphi_s^t(u) + 2) = \emptyset$ for all $u \in \mathsf{Succ}_{y_s^{m+1}}(t) \setminus \omega^n$.

Case 2. Next we assume that $\{h_s^u : u \in \mathsf{Succ}_{y_s^{m+1}}(t)\}$ is finite. Note that in this case $\mathsf{Succ}_{y_s^{m+1}}(t) \cap \omega^n = \emptyset$. We can find $X_s^t \in [\mathsf{Succ}_{y_s^{m+1}}(t)]^\omega$, $h \in \omega^\omega$ and $Y_s^u \in [\mathsf{Succ}_{y_s^{m+1}}(u)]^\omega$ for each $u \in X_s^t$ so that

1. for all $u \in X_s^t$, $h_s^u = h$,
2. $\langle h_s^v \restriction \varphi_s^u(v) : u \in X_s^t \text{ and } v \in Y_s^u \rangle$ is a type II function with limit h, and
3. $\mathsf{dcl}(\bigcup\{Y_s^u : u \in X_s^t\})$ is ω-branching.

Now let y_s^m be the set of following nodes:

1. $t \in \omega^m$ for which Case 1 is applied,
2. $v \in y_s^{m+1}$ such that, there are $t \in \omega^m$ for which Case 1 is applied and $u \in X_s^t$ satisfying $u \subseteq v$, and
3. $w \in y_s^{m+1}$ such that, there are $t \in \omega^m$ for which Case 2 is applied, $u \in X_s^t$ and $v \in Y_s^u$ satisfying $v \subseteq w$.

Finally, let $y_s = \mathsf{dcl}(y_s^0)$ and $x_s = \Gamma_{y_s}^{-1}(y_s^0)$. For each $t \in y_s^0 \cap \omega^n$, choose $p_s^t \leq p_{n-1} \restriction (s^\frown\langle t \rangle)$ so that $p_s^t \Vdash_{\mathbb{BPT}} h_s^t \restriction \varphi_s^u(t) = \dot{f} \restriction \varphi_s^u(t)$, where $u = \mathsf{pred}_{y_s^0}(t)$.

Let $p_n = \bigcup\{p_s^t : s \in M_n \text{ and } t \in y_s^0 \cap \omega^n\}$. Then $p_n \in \mathbb{BPT}$ and $p_n \leq_n p_{n-1}$. For each $s \in M$, $u \in y_s^0 \setminus \omega^n$ and $v \in \mathsf{Succ}_{y_s^0}(u)$ let $\gamma_s^v = h_s^v \restriction \varphi_s^u(v)$, and for each $t \in x_s \setminus \langle\rangle$ let $\delta_{s^\frown\langle t \rangle} = \gamma_{\Gamma_{y_s}(t)}$. Then $q \in \bigcap_{n<\omega} p_n$, $X = \{t^\frown\langle u \rangle : t \in M \text{ and } u \in x_t\}$ and $\langle \delta_s : s \in X \rangle$ satisfy the requirement. □

3 Iteration

In this section we present techniques to handle the iteration, which are due to Kamo [6]. These techniques are developed for the countable support iteration of Miller forcing. But they do not strongly depend on the shape of forcing conditions, and so we can apply them to the iteration of block branching Miller forcing in almost the same fashion.

We are going to prove Theorem 4.1 by induction on the length of iteration. But the proof for a limit step is exactly the same as in the proof of [6, Lemma 5.1]. So we will give only a proof for a successor step.

Throughout this paper, $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle$ denotes the countable support iteration of block branching Miller forcing of length ω_2. For each $\alpha \leq \omega_2$, let $\dot{G}_\alpha$ be the canonical $\mathbb{P}_\alpha$-name for a $\mathbb{P}_\alpha$-generic filter. For $p \in \mathbb{P}_{\omega_2}$, $\mathsf{supp}(p)$ denotes the support of p. For $\xi < \alpha \leq \omega_2$, $\mathbb{P}_{\xi,\alpha}$ denotes the quotient forcing $\mathbb{P}_\alpha / \mathbb{P}_\xi$. $\Vdash_{\mathbb{P}_\alpha}$ is abbreviated as $\Vdash_\alpha$.

We introduce the notion of tentacle trees, which is defined in [6, Section 4].

Definition 3.1. Let $T \subseteq \omega^{<\omega}$ be a tree and $\delta \in \omega^\omega \setminus T$.
$\tilde{\Delta}(T, \delta)$ denotes the maximal node of $T \cap \mathsf{dcl}(\{\delta\})$ and $\Delta(T, \delta) = |\tilde{\Delta}(T, \delta)|$.
δ is *adjoinable on* T if

1. $\Delta(T, \delta) + 2 \leq |\delta|$ and $|\mathsf{stem}(T)| < \Delta(T, \delta)$,
2. $\tilde{\Delta}(T, \delta) \notin \mathsf{split}(T)$,
3. $\mathsf{succ}_T(\tilde{\Delta}(T, \delta)) \cap \mathsf{split}(T) = \emptyset$, and
4. $\delta \restriction (\Delta(T, \delta) - 1) \notin \mathsf{split}(T)$.

T is called *a tentacle tree* if there are a skip branching tree H without maximal nodes and a function u from a countable set to $\omega^{<\omega}$ such that

1. for all $i \in \mathrm{dom}(u)$, $u(i)$ is adjoinable on H,
2. for all $i, j \in \mathrm{dom}(u)$, if $i \neq j$ then $|\Delta(H, u(i)) - \Delta(H, u(j))| \geq 2$, and
3. $T = H \cup \mathsf{dcl}(\mathrm{ran}(u))$.

In this case we say *H and u make up T*, or *T is made up of H and u*.

Note that every tentacle tree is a skip branching tree.

For a tentacle tree T, e_T denotes the enumeration of $\mathsf{Max}(T)$ such that, if $i < j < \omega$ then $|e_T(i)| < |e_T(j)|$.

Definition 3.2. $\mathcal{S}$ denotes the set of all tentacle trees. For each $g \in \omega^\omega$, let $\mathcal{S}(g) = \{H \in \mathcal{S} : \mathsf{B}(H) \text{ is } g\text{-thin}\}$.

Definition 3.3. Let $\mathcal{U}$ be the set of functions $U \in (\omega^\omega)^\omega$ such that

1. for all $i < \omega$, $U(i)$ is increasing, and
2. for all $i, j < \omega$, if $i < j$ then, for all $k < \omega$, $U(i)(k) \leq U(j)(k)$.

Definition 3.4. For $K \in \mathcal{S}$ and $U \in \mathcal{U}$, let $\mathcal{A}(K,U)$ be the set of functions φ from some $a \in [\omega]^\omega$ to $\prod_{i \in a} \mathcal{S}(U(\Gamma_a^{-1}(i))$ such that, there is $c \in [\omega]^\omega$ such that $e_K(\Gamma_c(i)) \subseteq \mathsf{stem}(\varphi(\Gamma_a(i)))$ for all $i < \omega$.

Lemma 3.5. [6, Lemma 4.2] *Let $g \in \omega^\omega$, H a skip branching tree without a maximal node, and $u_n \in (\omega^\omega)^\omega$ for $n < \omega$. Assume that, for all $n < \omega$, H and u_n make up a tentacle tree. Then there is a function v from ω to $\omega^{<\omega}$ such that*

1. *H and v make up a tentacle tree,*
2. *for all $n < \omega$ there are infinitely many $i < \omega$ such that $u_n(i) \in \mathrm{ran}(v)$, and*
3. *$\{|v(j)| : j < \omega\} \cup \{\Delta(H, v(j) : j < \omega\}$ is g-thin.*

Lemma 3.6. [6, Lemma 4.3] *Let $K \in \mathcal{S}$ and $U \in \mathcal{U}$. Then, for any countable subset Ψ of $\mathcal{A}(K,U)$, there is $\psi \in \mathcal{A}(K,U)$ such that, for all $\varphi \in \Psi$ there are infinitely many $i \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$ satisfying $\varphi(i) = \psi(i)$.*

From now on, λ is a "sufficiently large" regular cardinal and $H(\lambda)$ denotes the family of sets hereditarily of cardinality less than λ. N denotes a countable elementary substructure of $H(\lambda)$ unless otherwise defined.

The following is a slightly strengthened version of [6, Lemma 4.4] and proved in almost the same way as the original one.

Lemma 3.7. *Let $\alpha \leq \omega_2$ and $\mathbb{P} = \mathbb{P}_\alpha$.*

1. *Let $H \in N$ be a skip branching tree without a maximal node, and v a function from $\omega \times \omega$ to $\omega^{<\omega}$. Assume that H and v make up a tentacle tree, and for any $u \in (\omega^{<\omega})^{\omega \times \omega} \cap N$,*

   if H and u make up a tentacle tree, then for all $n < \omega$ there are infinitely many $i < \omega$ with $u(n,i) \in \mathrm{ran}(v)$.

   Then for each $p \in \mathbb{P} \cap N$ there is $\tilde{p} \leq p$ such that $\tilde{p}$ is $(N, \mathbb{P})$-generic and forces the following:

   For any $u \in (\omega^{<\omega})^{\omega\times\omega} \cap N[\dot{G}_\alpha]$, if H and u make up a tentacle tree, then for all $n < \omega$ there are infinitely many $i < \omega$ with $u(n,i) \in \mathrm{ran}(v)$.

2. *Let $K_n \in \mathcal{S} \cap N$, $U_n \in \mathcal{U} \cap N$, $\psi_n \in \mathcal{A}(K_n, U_n)$ for $n < \omega$, $\eta \leq \alpha$, $\mathbb{P}^* = \mathbb{P}_{\eta,\alpha}$ and $N^* = N[\dot{G}_\eta]$. Suppose that, in $\mathbf{V}^{\mathbb{P}_\eta}$, for all $n < \omega$,*

> *for all $\varphi \in \mathcal{A}(K_n, U_n) \cap N^*$, if $\mathrm{ran}(\varphi) \subseteq N$ then there are infinitely many $i < \omega$ with $\varphi(i) = \psi_n(i)$.*

Then, in $\mathbf{V}^{\mathbb{P}_\eta}$, for any $p \in \mathbb{P}^* \cap N^*$, there is $\tilde{p} \leq p$ such that $\tilde{p}$ is $(N^*, \mathbb{P}^*)$-generic, $\mathsf{supp}(\tilde{p}) \subseteq N^*$, and for any $n < \omega$, $\tilde{p}$ forces

> *for any $\varphi \in \mathcal{A}(K_n, U_n) \cap N^*[\dot{G}_{\mathbb{P}^*}]$, if $\mathrm{ran}(\varphi) \subseteq N$ then there are infinitely many $i < \omega$ with $\varphi(i) = \psi_n(i)$.*

Corollary 3.8. [6, Corollary 4.5] *Let $\alpha \leq \omega_2$, $\mathbb{P} = \mathbb{P}_\alpha$ and $g \in \omega^\omega$. Then the following hold in $\mathbf{V}^{\mathbb{P}}$: Assume that*

1. *$H \in \mathbf{V}$ is a skip branching tree without a maximal node,*
2. *$\dot{u}$ is a type II function with domain $\omega \times \omega$ and limit $\dot{h} \in \mathsf{Lim}(H)$, and*
3. *H and $\dot{u}$ make up a tentacle tree.*

Then, there is a tentacle tree $T \in \mathbf{V}$ such that:

1. *T is made up of H and some type II function,*
2. *$\{|\delta| : \delta \in \mathsf{Max}(T)\} \cup \{\Delta(H, \delta) : \delta \in \mathsf{Max}(T)\}$ is g-thin, and*
3. *for each $n < \omega$ there are infinitely many $i < \omega$ with $\dot{u}(n, i) \in \mathsf{Max}(T)$.*

4 Proof of the main theorem

Now we are ready to prove the following theorem.

Theorem 4.1. *Let $\alpha \leq \omega_2$, $\mathbb{P} = \mathbb{P}_\alpha$, $g \in \omega^\omega$ and $p \Vdash_\alpha \dot{f} \in \prod_{n<\omega} b(n)$. Then there are $\tilde{p} \leq p$ and $H \subseteq \omega^{<\omega}$ such that*

1. *H is a skip branching tree,*
2. *$\mathsf{B}(H)$ is g-thin, and*
3. *$\tilde{p} \Vdash_\alpha \dot{f} \in \mathsf{Lim}(H)$*

Proof. Induction on $\alpha \leq \omega_2$. As mentioned in the last section, we only give a proof for a successor step and refer the reader to [6] for a limit step.

Suppose that $\alpha = \beta + 1$ and the lemma holds for all $\alpha' \leq \beta$.

Claim 1. *Let $g' \in \omega^\omega$. Then the following holds in $\mathbf{V}^{\mathbb{P}_\beta}$: For any type II function u with domain $\omega \times \omega$, there is a tentacle tree $T \in \mathbf{V}$ such that*

1. $\mathsf{B}(T)$ *is g'-thin, and*

2. for all $m < \omega$ there are infinitely many $i < \omega$ with $u(m, i) \in \mathsf{Max}(T)$.

Proof. Work in $\mathbf{V}^{\mathbb{P}_\beta}$. Suppose that a function u from $\omega \times \omega$ to $\omega^{<\omega}$ is a type II function with limit $h \in \omega^\omega$. Take a pairwise disjoint set $\{I_{m,n} : (m, n) \in \omega \times \omega\}$ of intervals in ω so that for all $(m, n) \in \omega \times \omega$ there is $i < \omega$ with $[\Delta(h, u(m, i) - 1), |u(m, i)| + 2) \subseteq I_{m,n}$. Using Lemmata 1.8, 1.9 and Proposition 2.8, choose $g_1 \in \omega^\omega \cap \mathbf{V}$ so that, for any g_1-thin sets $a, c \in [\omega]^\omega$,

1. for any $m < \omega$ there are infinitely many $n < \omega$ with $a \cap I_{m,n} = \emptyset$, and

2. $a \cup c$ is g'-thin.

By the induction hypothesis, we find a skip branching tree $H \in \mathbf{V}$ without a maximal node such that $h \in \mathsf{Lim}(H)$ and $\mathsf{B}(H)$ is g_1-thin. By the choice of g_1, for all $m < \omega$ there is $a_m \in [\omega]^\omega$ such that, for all $i \in a_m$, $[\Delta(h, u(m, i), |u(m, i)| + 2) \cap \mathsf{B}(H) = \emptyset$.

Now we define a function v from $\omega \times \omega$ to $\omega^{<\omega}$ by letting $v(m, n) = u(m, \Gamma_{a_m}(n))$ for each $(m, n) \in \omega \times \omega$. Then H and v make up a tentacle tree.

By Corollary 3.8, there is a tentacle tree $T \in \mathbf{V}$ such that:

1. T is made up of H and some type II function,

2. $\{|\delta| : \delta \in \mathsf{Max}(T)\} \cup \{\Delta(h, \delta) : \delta \in \mathsf{Max}(T)\}$ is g_1-thin, and

3. for all $m < \omega$ there are infinitely many $i < \omega$ such that $v(m, i) \in \mathsf{Max}(T)$.

Then T is as required. □

Using Lemma 1.8, take a set $\{g_s : s \in \omega^{<\omega}\}$ of increasing functions in ω^ω so that

1. for $\{a_s : s \in \omega^{<\omega}\} \subseteq [\omega]^\omega$, if a_s is g_s-thin for all $s \in \omega^{<\omega}$, then $\bigcup\{a_s : s \in \omega^{<\omega}\}$ is g-thin,

2. for $n < \omega$ and $s, t \in \omega^n$, if $s(i) \leq t(i)$ for all $i < n$, then $g_s(i) \leq g_s(i)$ for all $i < \omega$, and

3. for $s, t \in \omega^{<\omega}$, if $s \subseteq t$, then $g_s(0) < g_t(0)$.

Without loss of generality we may assume $p \Vdash_\alpha \dot{f} \notin \mathbf{V}^{\mathbb{P}_\beta}$.

We work in $\mathbf{V}^{\mathbb{P}_\beta}$. Using Lemma 2.11, take $\dot{q} \leq p(\beta)$, $\dot{X} \subseteq \tilde{M}$ and $\left\langle \dot{\delta}_s : s \in \dot{X} \right\rangle$ so that,

1. $M \subseteq \dot{X}$,

2. $\left\langle \dot{\delta}_s : s \in \dot{X} \right\rangle$ is a quasi-tree of type II functions,

3. for all $s \in \dot{X}$, $q \restriction s \Vdash \dot{\delta}_s \subseteq \dot{f}$, and

4. for all $s \in \dot{X}$, $|\dot{\delta}_s| > g_{\Gamma^{-1}_{\dot{X}}(s)}(0)$.

Using Claim 1, for all $s \in \omega^{<\omega}$ we can take a tentacle tree $\dot{T}_s \in \mathbf{V}$ so that for all $s \in \omega^{<\omega}$,

1. $\mathsf{B}(\dot{T}_s)$ is g_s-thin, and

2. there is $a_s \in [\omega]^\omega$ such that

   (a) for all $i \in a_s$, $\dot{\delta}_{s^\frown\langle i\rangle} \in \mathsf{Max}(\dot{T}_s)$, and

   (b) $\mathsf{dcl}(\{\Gamma_{\dot{X}}(s^\frown\langle i\rangle) : i \in a_s\})$ is ω-branching above $\Gamma_{\dot{X}}(s)$.

Fix $s \in \omega^{<\omega}$. Let $\{\dot{t}_j : j \in \omega\}$ be an enumeration of $\{\mathsf{pred}_{\mathsf{dcl}(\dot{X})}(t) : t \in \mathsf{Succ}_{\dot{X}}(s)\}$ if this set is infinite; otherwise $\dot{t}_j = s$ for all $j < \omega$. Let $a^j_s = \{i \in a_s : \dot{t}_j \subseteq \Gamma_{\dot{X}}(s^\frown\langle i\rangle)\}$. Note that a^j_s is infinite for every $j < \omega$. For $j < \omega$, $\dot{\varphi}^j_s = \left\langle \dot{T}_{s^\frown\langle i\rangle} : i \in a^j_s \right\rangle$. Then $\dot{\varphi}^j_s \in \mathcal{A}(\dot{T}_s, U_s)$ and $\mathrm{ran}(\dot{\varphi}^j_s) \subseteq \mathbf{V}$.

Return to $\mathbf{V}$. Take a countable elementary substrutcure N of $H(\lambda)$ so that the above arguments were done in N. Using Lemma 3.6, for each $K \in \mathcal{S} \cap N$ and $U \in \mathcal{U} \cap N$ take $\psi_{K,U} \in \mathcal{A}(K, U)$ so that

1. for all $\varphi \in \mathcal{A}(K, U)$ there are infinitely many $i < \omega$ with $\varphi(i) = \psi_{K,U}(i)$, and

2. $\mathrm{ran}(\psi_{K,U}) \subseteq N$.

By Lemma 3.7, there is $\tilde{p} \leq p \restriction \beta$ such that, for all $K \in \mathcal{S} \cap N$ and $U \in \mathcal{U} \cap N$, $\tilde{p}$ forces

> for all $\varphi \in \dot{\mathcal{A}}(K, U) \cap N[\dot{G}_\beta]$, if $\mathrm{ran}(\varphi) \subseteq N$, then there are infinitely many $i < \omega$ with $\varphi(i) = \psi_{K,U}(i)$.

In particular, $\tilde{p}$ forces

> (*) for all $s \in \omega^{<\omega}$ and $j < \omega$, there are infinitely many $i < \omega$ with $\dot{\varphi}^j_s(i) = \psi_{K,U}(i)$.

Without loss of generality we can assume that $\tilde{p} \Vdash_\beta \dot{T}_{\langle\rangle} = T$ for some $T \in N$.

By induction, define $C_n \subseteq \omega^n$ and $K_t \in N$ for $t \in C_n$ by

1. $C_0 = \{\langle\rangle\}$,

2. $K_{\langle\rangle} = T$,

3. $C_{n+1} = \{s^\frown\langle i\rangle : s \in C_n \text{ and } i \in \operatorname{dom}(\psi_{K_s,U_s})\}$,

4. $K_{s^\frown\langle i\rangle} = \psi_{K_s,U_s}(i)$ for all $s^\frown\langle i\rangle \in C_{n+1}$.

Let $C = \bigcup_{n<\omega} C_n$ and $K = \bigcup\{K_s : s \in C\}$. It is easy to see that K is a skip branching tree.

Claim 2. *For all $s \in \omega^{<\omega}$, $\mathsf{B}(K_{\Gamma_C(s)})$ is g_s-thin.*

Proof. Induction on the length of s. The case $s = \langle\rangle$ is clear. Assume $s = t^\frown\langle i\rangle$. Let $s' = \Gamma_C(s)$, $t' = \Gamma_C(t)$, $a = \operatorname{dom}(\psi_{K_{t'},U_{t'}})$, $i' = \Gamma_a(i)$ and $u = t'^\frown\langle i\rangle$. Note that $s' = t'^\frown\langle i'\rangle$ and so $K_{\Gamma_C(s)} = K_{s'} = K_{t'^\frown\langle i'\rangle} = \psi_{K_{t'},U_{t'}}(i')$. Since $\psi_{K_{t'},U_{t'}}(i') \in \mathcal{S}(U_{t'}(\Gamma_a^{-1}(i'))) = \mathcal{S}(g_u)$, $\mathsf{B}(K_{\Gamma_C(s)})$ is g_u-thin. On the other hand, $s(j) \le u(j)$ for all $j < |u|$ and so $g_s(k) \le g_u(k)$ for all $k < \omega$. Hence, $\mathsf{B}(K_{\Gamma_C(s)})$ is g_s-thin. □

Since $\mathsf{B}(K) \subseteq \bigcup\{\mathsf{B}(K_s) : s \in C\}$, $\mathsf{B}(K)$ is g-thin.
Work in $\mathbf{V}^{\mathbb{P}_\beta}$ below $\tilde{p}$. By induction on $n < \omega$, define $\dot{D}_n \subseteq C_n$ by

1. $\dot{D}_0 = \{\langle\rangle\}$,

2. for $s \in \dot{D}_n$ and $j < \omega$, $\dot{D}_s^j = \{s^\frown\langle i\rangle \in C_{n+1} : i \in \operatorname{dom}(\dot{\varphi}_s) \text{ and } \dot{\varphi}_s^j(i) = \psi_{K_s,U_s}(i)\}$, and $\dot{D}_{n+1} = \bigcup\{\dot{D}_s^j : s \in \dot{D}_n \text{ and } j < \omega\}$.

Claim 3. *For all $n < \omega$, $\tilde{p} \Vdash_\beta$ "for all $s \in \dot{D}_n$, $K_s = \dot{T}_s$".*

Proof. Easy. □

By the above claim and the property (*), $\tilde{p}$ forces

> for all $n < \omega$ and $s \in \dot{D}_n$ and $j < \omega$, there are infinitely many $i < \omega$ with $s^\frown\langle i\rangle \in \dot{D}_s^j$.

Define a $\mathbb{P}_\beta$-name $\dot{r}$ by $\Vdash_\beta \dot{r} = \bigcap_{n<\omega}\bigcup\{\dot{q}\restriction\Gamma_{\dot{X}}(s) : s \in \dot{D}_n\}$. By the construction of $\dot{D}_n$, for all $n < \omega$ and $s \in \dot{D}_n$, $\mathsf{dcl}(\{\Gamma_{\dot{X}}(t) : t \in \mathsf{Succ}_{\dot{D}_{n+1}}(s)\})$ is ω-branching above $\Gamma_{\dot{X}}(s)$, and hence $\tilde{p} \Vdash_\beta \dot{r} \in \mathbb{BPT}$. Note that $\tilde{p}$ forces

> $\dot{r} \le \dot{q}$ and $\{\dot{q}\restriction s : s \in \dot{D}_n\}$ is predense below $\dot{r}$.

Since $\tilde{p}$ forces

> $\{\dot{\delta}_s : s \in \bigcup_{n<\omega} \dot{D}_n\} \subseteq K$ and $\dot{q}\restriction s \Vdash \dot{\delta}_s \subseteq \dot{f}$ for all $s \in \omega^{<\omega}$,

we have $\tilde{p} \Vdash_\beta$ "$\dot{r} \Vdash \dot{f} \in \mathsf{Lim}(K)$" and hence $\tilde{p}^\frown\langle\dot{r}\rangle \Vdash_\alpha \dot{f} \in \mathsf{Lim}(K)$. $\square$

Corollary 4.2. *Assume that* CH *holds in* $\mathbf{V}$. *Then in* $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, *for every* $b \in \omega^\omega$ *there is a set* $P \subseteq \mathcal{P}$ *of predictors of size* ω_1 *such that, for any* $f \in \prod_{n<\omega} b(n)$ *there is a predictor* $\pi \in P$ *predicting* f *constantly.*

Proof. Choose $\beta < \omega_2$ so that $b \in \mathbf{V}^{\mathbb{P}_\beta}$. Work in $\mathbf{V}^{\mathbb{P}_\beta}$. Let $p \in \mathbb{P}_{\beta,\omega_2}$ and $\dot{f}$ be a $\mathbb{P}_{\beta,\omega_2}$-name of a function in $\prod_{n<\omega} b(n)$. Apply Theorem 4.1 to get $q \leq p$ and a skip branching tree H so that $q \Vdash_{\beta,\omega_2} \dot{f} \in \mathsf{Lim}(H)$. Now it is easy to find a predictor π such that

$$q \Vdash_{\beta,\omega_2} \text{"for all } n < \omega,\ \dot{f}(n) = \pi(\dot{f}\restriction n) \text{ or } \dot{f}(n+1) = \pi(\dot{f}\restriction(n+1))\text{"}.$$

Since CH holds in $\mathbf{V}^{\mathbb{P}_\beta}$, $P = \mathcal{P} \cap \mathbf{V}^{\mathbb{P}_\beta}$ satisfies the requirement. $\square$

By Corollary 4.2, $\theta_{\mathrm{ubd}} = \omega_1$ holds in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. On the other hand, by Proposition 2.6, $\theta_\omega = \omega_2 = 2^\omega$ holds in the same model.

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