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HÖLDER TYPE OPERATOR INEQUALITIES

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ABSTRACT. For commutative operators, we give the estimation of Hölder's type operator inequality which represents the complementary inequality on the difference derived from Hölder's operator inequality, and show that its estimation is the best. As special cases, we give some well-known difference and ratio inequalities by considering the 2-th power mean or only one operator. Finally by using the geometric mean in the Kubo-Ando theory we shall generalize Hölder's type operator inequality for noncommutative operators.

1. INTRODUCTION

This paper is in continucation to our preceding paper [13]. In this note, an operator means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

The Hölder inequality is one of the most important inequalities: If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are n-tuples of real numbers, then for any real number $p > 1$, $q > 1$ satisfying $1/p + 1/q = 1$,

$$(\sum a_k^p)^{\frac{1}{p}} (\sum b_k^q)^{\frac{1}{q}} \geq \sum a_k b_k. \tag{1}$$

In [13], we introduced out that the complementary inequality derived from (1), Hölder's type inequality which represented the estimation for the difference of the p-th power mean and positive scalar multiple of the usual arithmetic mean. We consider the operator version of Hölder's type inequalities.

Now we consider the following difference on both sides of Hölder's operator inequality: Let A, B be two commutative positive operators. Then for a unit vector $x \in H$

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} - \langle ABx, x\rangle.$$

Recently one of the upper bound of this difference was obtained in [10] and it was shown that it's estimation was the best bound in [11].

On the other hand, in [4, Theorem 3] the following interested ratio inequality which is the complementary inequality of the Hölder-McCarthy inequality [3]

Theorem A. *Let A be a selfadjoint operator on H with $0 < m \leq A \leq M$. Then for a unit vector $x \in H$.*

$$\langle A^p x, x\rangle^{\frac{1}{p}} \leq \lambda \langle Ax, x\rangle, \tag{2}$$

where $\lambda = \frac{1-\alpha^p}{p^{1/p} q^{1/q} (1-\alpha)^{1/p} (\alpha - \alpha^p)^{1/q}}$.

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In this note, we attempt to unify these difference and ratio operator inequalities. To do it, we define $S(X)$ as follows: For positive operators A, B and any positive real number λ,

$$S(\lambda) := \langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} - \lambda\langle ABx, x\rangle,$$

and denote by $F(\lambda)$ the upper bound of $S(\lambda)$. Here we call the following inequality Hölder's type operator inequality

$$S(\lambda) \leq F(\lambda).$$

We note that Hölder's type operator inequality implies the estimation of the above difference for $\lambda = 1$, and ratio inequality (1) for $\lambda = \lambda_0$ with $F(\lambda_0) = 0$. Moreover by putting $p = 2$ or $B \to I$ (I is the identity operator) we induce some difference and ratio inequalities, and point out that some of them are operator version of well-known inequalities. The final result is a noncommutative operator version of Hölder's type operator inequality $S(\lambda) \leq F(\lambda)$. For this, we use the s-geometric mean $\sharp_s$ by viture of the Kubo-Ando theory [14] which is defined by

$$A\sharp_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}} \quad (0 < s \leq 1)$$

for all positive operators A and B. We note that $B^q\sharp_{1/p}A^p = AB$ if A and B commute.

Throughout this paper, we assume that p and q are real numbers such that $p > 1$, $q > 1$ and $1/p + 1/q = 1$.

2. An operator version of Hölder's type inequality

The purpose in this section is to present an extension to commutative operators of the complemantary inequality derived from Hölder's inequality. For the sake of convenience, we give some notations needed later. We denote by K_α, K_β, K, $\tilde{K}_\alpha$, $\tilde{K}_\beta$ and $\tilde{K}$ respectively:

$$\frac{1-\alpha^p}{1-\alpha}, \quad \frac{1-\beta^q}{1-\beta}, \quad \frac{{K_\alpha}^{1/p}{K_\beta}^{1/q}}{p^{1/p}q^{1/q}}, \quad \frac{K_\alpha}{\alpha^{p/q}}, \quad \frac{K_\beta}{\beta^{q/p}} \quad \text{and} \quad \frac{K}{\alpha^{1/q}\beta^{1/p}}, \tag{3}$$

where α and β are real numbers with $0 < \alpha < 1$ and $0 < \beta < 1$.

In our preceding paper [13, Lemma 2.3], we pointed out that for any positive real number λ, the equation

$$(1-\alpha)(\lambda - K\tau^{\frac{1}{q}}) = (1-\beta)(\lambda - K\tau^{-\frac{1}{p}}) \tag{4}$$

has a (unique) positive solution (which we denote by $\tau = \tau_\lambda$). We define a constant c_λ as follows:

$$c_\lambda = (1-\alpha)(\lambda - K\tau_\lambda^{\frac{1}{q}}) \left(= (1-\beta)(\lambda - K\tau_\lambda^{-\frac{1}{p}})\right). \tag{5}$$

Furthermore in [13, Theorem 3.3], we showed the following theorem which gave the upper bound of the difference of the p-th power mean and positive scalar multiple of the usual arithmetic mean.

Theorem B. *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be n-tuples satisfying $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$ $(k = 1, 2, \dots, n)$, $m_1 < M_1$ and $m_2 < M_2$. Suppose that $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then for any $\lambda > 0$,*

$$S(\lambda) := (\sum a_k^p)^{\frac{1}{p}}(\sum b_k^q)^{\frac{1}{q}} - \lambda \sum a_k b_k \leq nM_1M_2F_0(\lambda), \tag{6}$$

where $F_0(\lambda)$ is a constant defined as follows:

$$F_0(\lambda) = \begin{cases} \alpha\beta(1-\lambda) & \text{if } \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\} < \lambda \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p + \frac{\beta^q}{K_\beta} - \beta\}\alpha\lambda & \text{if } \tilde{K} < \lambda \leq \frac{\tilde{K}_\beta}{q} (= \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\}) \\ \{\frac{\alpha^p}{K_\alpha} + \frac{1}{K_\beta}(\frac{K}{\lambda})^q - \alpha\}\beta\lambda & \text{if } \tilde{K} < \lambda \leq \frac{\tilde{K}_\alpha}{p} (= \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\}) \\ (\frac{1}{K_\alpha} + \frac{1}{K_\beta} - 1)\lambda - c_\lambda(\frac{1}{1-\alpha^p} + \frac{1}{1-\beta^q} - 1) & \text{if } K \leq \lambda \leq \tilde{K} \\ \{\frac{1}{K_\alpha} + \frac{1}{K_\beta}(\frac{K}{\lambda})^q - 1\}\lambda & \text{if } \frac{K_\alpha}{p} (= \min\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\}) \leq \lambda < K \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p + \frac{1}{K_\beta} - 1\}\lambda & \text{if } \frac{K_\beta}{q} (= \min\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\}) \leq \lambda < K \\ 1 - \lambda & \text{if } 0 < \lambda < \min\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\}. \end{cases}$$

Furthermore for each λ, the constant $nM_1M_2F_0(\lambda)$ in (6) is the best bound of $S(\lambda)$.

We call the inequality (6) Hölder's type inequality.

Moreover we pointed out the following facts; If $\lambda = 1$, then by $K \leq 1 \leq \tilde{K}$,

$$F_0(1) = \frac{1}{K_\alpha} + \frac{1}{K_\beta} - 1 - c_1\left(\frac{1}{1-\alpha^p} + \frac{1}{1-\beta^q} - 1\right), \tag{7}$$

and so the following Izumino inequality [10, Theorem 2.2] is obtained

$$(\sum a_k^p)^{\frac{1}{p}}(\sum b_k^q)^{\frac{1}{q}} - \sum a_k b_k \leq nM_1M_2F_0(1), \tag{8}$$

and if $F_0(\lambda) = 0$, then by [13, Theorem 3.4 and Lemma 5.1] there exists a unique solution

$$\lambda = \lambda_0 = \frac{1 - \alpha^p\beta^q}{p^{1/p}q^{1/q}(\beta - \alpha\beta^q)^{1/p}(\alpha - \alpha^p\beta)^{1/q}} \; (\in [K, \tilde{K}]), \tag{9}$$

and so the following Gheorghiu inequality ([7] and [17, p.685]) is obtained

$$(\sum a_k^p)^{\frac{1}{p}}(\sum b_k^q)^{\frac{1}{q}} \leq \lambda_0 \sum a_k b_k. \tag{10}$$

The constant (9) was intoroduced in [6].

Now we shall give Hölder's type operator inequality which is an operator version of (6). Moreover we consider cases of $\lambda = 1$ and $F_0(\lambda) = 0$ in it. To complete this, we use the same method as [10, Theorems 4.1-4.3] or [4].

Theorem 1. *Let A and B be two commuting positive operators on H satisfying $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$, $m_1 < M_1$ and $m_2 < M_2$, and let $F_0(\lambda)$ be a constant defined by Theorem B. Suppose that $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then for any $\lambda > 0$ and any unit vector $x \in H$,*

$$S_0(\lambda) := \langle A^p x, x\rangle^{\frac{1}{p}}\langle B^q x, x\rangle^{\frac{1}{q}} - \lambda\langle ABx, x\rangle \leq M_1M_2F_0(\lambda). \tag{11}$$

Furthermore for each λ, *the constant* $M_1M_2F_0(\lambda)$ *in* (11) *is the best bound of* $S_0(\lambda)$. *In particular, if* $\lambda = 1$, *then*

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} - \langle ABx, x\rangle \leq M_1M_2F_0(1), \tag{12}$$

and if $F_0(\lambda_0) = 0$, *then*

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} \leq \lambda_0 \langle ABx, x\rangle, \tag{13}$$

where the constant $F_0(1)$ *and* λ_0 *are defined by* (7) *and* (9), *respectively.*

Proof. Let a and b be n- tuples with the same conditions of Theorem B and $w = (w_1, \ldots, w_n)$ be an n-tuple of nonnegative numbers with $w = \sum w_k$. Then by the same method as [10, Theorem 4.1] , we hold the weighted version of Theorem B, for $\lambda > 0$

$$(\sum w_k a_k^p)^{\frac{1}{p}} (\sum w_k b_k^q)^{\frac{1}{q}} - \lambda \sum w_k a_k b_k \leq w M_1M_2F_0(\lambda). \tag{14}$$

Next let μ be a positive measure on the rectangle $X = [m_1, M_1] \times [m_2, M_2]$ with $\mu(X) = 1$, and let $L^r(X)$ $(r > 1)$ be the set of functions f such that $|f|^r$ is integrable on X. Suppose that $f \in L^p(X)$ and $g \in L^q(X)$ satisfying $0 < m_1 \leq f \leq M_1$ and $0 < m_2 \leq g \leq M_2$. Furthermore let $X_1, X_2, \ldots, X_n$ be a decomposition of X and let $x_k \in X_k$ $(k = 1, 2, \ldots, n)$. Then from (14)

$$\left\{\sum f(x_k)^p \mu(X_k)\right\}^{\frac{1}{p}} \left\{\sum f(x_k)^q \mu(X_k)\right\}^{\frac{1}{q}} - \lambda \sum f(x_k) g(x_k) \mu(X_k) \leq M_1M_2F_0(\lambda).$$

Taking the limit of the decomposition, we obtain

$$\left(\int f^p d\mu\right)^{\frac{1}{p}} \left(\int g^q d\mu\right)^{\frac{1}{q}} - \lambda \int fg d\mu \leq M_1M_2F_0(\lambda). \tag{15}$$

Now A and B are commuting, so there exist commuting spectral families $E_A(\cdot)$ and $E_B(\cdot)$ corresponding to A and B such that for a polynomial $p(A, B)$ (or a uniform limit of polynomials) in A and B,

$$\langle p(A, B)x, x\rangle = \int_{-\infty}^{\infty} p(s, t) d\langle E_A(s)E_B(t)x, x\rangle \qquad \text{for} \quad x \in H,$$

[20, p.287]. Let $d\mu = d\langle E_A(s)E_B(t)x, x\rangle = d\|E_A(s)E_B(t)x\|^2$. Hence from (15) we have

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} - \lambda\langle ABx, x\rangle = \left(\iint_X s^p d\mu\right)^{\frac{1}{p}} \left(\iint_X t^q d\mu\right)^{\frac{1}{q}} - \lambda \iint_X st d\mu \leq M_1M_2F_0(\lambda).$$

Furthermore we easily see inequalities (12) and (13) from Theorem B and the remark after it. □

We remark that the difference inequality (12) is given directly in [10, Theorem 4.3] as operator version of (8) and the ratio inequality (13) is the same inequality as commutative version in [4, Theorem 4].

3. Application to difference and ratio operator inequalities

In this section as application of Theorem 1, we deduce three corollaries which consider special inequalities as the cases of $p = q = 2$ or $\beta \to 1$, and give explicit expressions of their estimations. In particular we shall show that for $\lambda = 1$ they correspond with difference inequalities given in [10]. Furthermore we point out that the obtained ratio inequalities are the operator version of the well-known numerical inequalities.

Now we take $\beta \to 1$ in Theorem 1. Then obtained inequalities in the following corollary are the complementary operator inequalities of the arithmetic and power mean inequality.

Corollary 2. *Let A be a positive operator on H satisfying $0 < m_1 \le A \le M_1$, $m_1 < M_1$. Suppose that $\alpha = m_1/M_1$. Put $K_1 = \left(\frac{K_\alpha}{p}\right)^{\frac{1}{p}}$ and $\tilde{K}_1 = \frac{K_1}{\alpha^{1/q}}$. Then for any $\lambda > 0$ and any unit vector $x \in H$,*

$$S_1(\lambda) := \langle A^p x, x\rangle^{\frac{1}{p}} - \lambda\langle Ax, x\rangle \le M_1 F_1(\lambda), \tag{16}$$

where $F_1(\lambda)$ is a constant defined as follows:

$$F_1(\lambda) = \begin{cases} \alpha(1-\lambda) & \text{if } \tilde{K}_1^p < \lambda \\ \{\frac{\alpha^p}{K_\alpha} + \frac{1}{q}(\frac{K_1}{\lambda})^q - \alpha\}\lambda & \text{if } \tilde{K}_1 < \lambda \le \tilde{K}_1^p \\ (\frac{1}{K_\alpha} - 1)\lambda + \frac{1}{q}(\frac{K_1}{\lambda^{1/p}})^q & \text{if } K_1 \le \lambda \le \tilde{K}_1 \\ \{\frac{1}{K_\alpha} + \frac{1}{q}(\frac{K_1}{\lambda})^q - 1\}\lambda & \text{if } K_1^p \le \lambda < K_1 \\ 1-\lambda & \text{if } 0 < \lambda < K_1^p. \end{cases}$$

Furthermore for each λ, the constant $M_1F_1(\lambda)$ in (16) is the best bound of $S_1(\lambda)$. In particular, if $\lambda = 1$, then

$$\langle A^p x, x\rangle^{\frac{1}{p}} - \langle Ax, x\rangle \le M_1 \left[\frac{1}{q}\left\{\frac{1-\alpha^p}{p(1-\alpha)}\right\}^{q-1} - \frac{\alpha-\alpha^p}{1-\alpha^p}\right], \tag{17}$$

and if $F_1(\lambda) = 0$, then there exists a unique solution $\lambda = \lambda_1 = \frac{1-\alpha^p}{p^{1/p}q^{1/q}(1-\alpha)^{1/p}(\alpha-\alpha^p)^{1/q}}$ ($\in [K_1, \tilde{K}_1]$) and the following inequality holds

$$\langle A^p x, x\rangle^{\frac{1}{p}} \le \lambda_1\langle Ax, x\rangle. \tag{18}$$

Proof. We obtain the inequality (16) by using the same method as [13, Theorem 4.1] in Theorem 1. The difference inequality (17) is trivial by $K_1 \le 1 \le \tilde{K}_1$. The constant $\lambda = \lambda_1$ with $F_1(\lambda_1) = 1$ is in $[K_1, \tilde{K}_1]$, so the ratio inequality (18) is hold by elementary calculation. □

The inequality (17) is given directly in [10]. The inequality (18) is a complementary inequality of Hölder-McCarthy inequality [16] corresponds to Jensen's inequality with respect to the convex function $f(x) = x^p$ $(p > 1)$ and is given directly in [4]. The constant λ_1 coincides with the p-th root of the constant defined by Ky Fan [1].

Next we take $p = q = 2$ in Theorem 1. For the convenience of representation, we may assume $\alpha < \beta$ in it.

Corollary 3. *Let A and B be two commuting positive operators on H satisfying $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$, $m_1 < M_1$ and $m_2 < M_2$. Suppose that $\alpha = \min\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}$ and $\beta = \max\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}$. Put $K_2 = \frac{(1+\alpha)^{1/2}(1+\beta)^{1/2}}{2}$ and $\tilde{K}_2 = \frac{K_2}{\alpha^{1/2}\beta^{1/2}}$. Write c'_λ the constant of (5) with respect to $p = q = 2$. Then for any $\lambda > 0$ and any unit vector $x \in H$,*

$$S_2(\lambda) := \langle A^2x, x\rangle^{\frac{1}{2}}\langle B^2x, x\rangle^{\frac{1}{2}} - \lambda\langle ABx, x\rangle \leq M_1M_2F_2(\lambda), \tag{19}$$

where $F_2(\lambda)$ is a constant defined as follows:

$$F_2(\lambda) = \begin{cases} \alpha\beta(1-\lambda) & \text{if } \frac{1+\alpha}{2\alpha} < \lambda \\ \left(\frac{1+\alpha}{4\lambda^2} - \frac{\alpha}{1+\alpha}\right)\beta\lambda & \text{if } \tilde{K}_2 < \lambda \leq \frac{1+\alpha}{2\alpha} \\ \frac{(1-\alpha\beta)\lambda}{(1+\alpha)(1+\beta)} - c'_\lambda\left\{\frac{1-\alpha^2\beta^2}{(1-\alpha^2)(1-\beta^2)}\right\} & \text{if } K_2 \leq \lambda \leq \tilde{K}_2 \\ \left(\frac{1+\alpha}{4\lambda^2} - \frac{\alpha}{1+\alpha}\right)\lambda & \text{if } \frac{1+\alpha}{2} \leq \lambda < K_2 \\ 1-\lambda & \text{if } 0 < \lambda < \frac{1+\alpha}{2}. \end{cases}$$

Furthermore for each λ, the constant $M_1M_2F_2(\lambda)$ in (19) is the best bound of $S_2(\lambda)$. In particular, if $\lambda = 1$, then

$$\langle A^2x, x\rangle^{\frac{1}{2}}\langle B^2x, x\rangle^{\frac{1}{2}} - \langle ABx, x\rangle \leq M_1M_2\frac{(1-\alpha\beta)^2}{2(1+\alpha)(1+\beta)}, \tag{20}$$

and if $F_2(\lambda) = 0$, then there exists a unique solution $\lambda = \lambda_2 = \frac{1+\alpha\beta}{2\alpha^{1/2}\beta^{1/2}} (\in [K_2, \tilde{K}_2])$ and the following inequality holds

$$\langle A^2x, x\rangle^{\frac{1}{2}}\langle B^2x, x\rangle^{\frac{1}{2}} \leq \lambda_2\langle ABx, x\rangle. \tag{21}$$

Proof. Let put $p = q = 2$ in Theorem 1. Then we obtain these desired inequalities by using the same method as [13, Theorem 4.2]. □

The inequality (20) is given directly in [10]. The inequality (21) is commutative operator version of the Pólya-Szegö inequality [19]

$$\sum a_k^2 \sum b_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2}(\sum a_kb_k)^2.$$

and Greub-W.Rheinboldt inequality [8]

$$\sum p_ka_k^2 \sum p_kb_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2}(\sum p_ka_kb_k)^2.$$

under the same assumptions of Theorem B.

In particular, we obtain the following corollary for $p = q = 2$ in Corollary 2.

Corollary 4. *Let A be a positive operator on H satisfying $0 < m_1 \leq A \leq M_1$, $m_1 < M_1$. Put $\alpha = m_1/M_1$. Then for any $\lambda > 0$ and a unit vector $x \in H$,*

$$S_3(\lambda) := \langle A^2x, x\rangle^{\frac{1}{2}} - \lambda\langle Ax, x\rangle \leq M_1F_3(\lambda), \tag{22}$$

where $F_3(\lambda)$ *is a constant defined as follows:*

$$F_3(\lambda)=\begin{cases}\alpha(1-\lambda) & \text{if } \frac{1+\alpha}{2\alpha}<\lambda\\ (\frac{1+\alpha}{4\lambda^2}-\frac{\alpha}{1+\alpha})\lambda & \text{if } \left(\frac{1+\alpha}{2\alpha}\right)^{\frac{1}{2}}<\lambda\le\frac{1+\alpha}{2\alpha}\\ \frac{1+\alpha}{4\lambda}-\frac{\alpha}{1+\alpha}\lambda & \text{if } (\frac{1+\alpha}{2})^{\frac{1}{2}}\le\lambda\le(\frac{1+\alpha}{2\alpha})^{\frac{1}{2}}\\ (\frac{1+\alpha}{4\lambda^2}-\frac{\alpha}{1+\alpha})\lambda & \text{if } \frac{1+\alpha}{2}\le\lambda<(\frac{1+\alpha}{2})^{\frac{1}{2}}\\ 1-\lambda & \text{if } 0<\lambda<\frac{1+\alpha}{2}.\end{cases}$$

Furthermore for each λ, *the constant* $M_1F_3(\lambda)$ *in* (22) *is the best bound of* $S_3(\lambda)$. *In particular, if* $\lambda=1$, *then*

$$\langle A^2x,x\rangle^{\frac{1}{2}}-\langle Ax,x\rangle\le\frac{(M-m)^2}{4(M-m)},\tag{23}$$

and if $F_3(\lambda)=0$, *then there exists a unique solution* $\lambda=\lambda_3=\frac{1+\alpha}{2\alpha^{1/2}}$ $\left(\in[(\frac{1+\alpha}{2})^{\frac{1}{2}},(\frac{1+\alpha}{2\alpha})^{\frac{1}{2}}]\right)$ *and the following inequality holds*

$$\langle A^2x,x\rangle^{\frac{1}{2}}\le\lambda_3\langle Ax,x\rangle.\tag{24}$$

The inequalityies (23) and (24) are well-known inequalities related to the following celebrated Kantorovich inequality

$$\langle Ax,x\rangle\langle A^{-1}x,x\rangle\le\frac{(M+m)^2}{4mM}.$$

As application of Theorem 1 we shall show some operator inequalities without commutativity. In [14], F. Kubo and T. Ando introduced the s-geometric mean $A\sharp_sB$ defined by

$$A\sharp_sB=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^sA^{\frac{1}{2}}\quad(0<s\le1).$$

for positive invertible operators A and B.

By using it Corollary 2 implies the noncommutative version of Theorem 1.

Theorem 5. *Let* A *and* B *be two positive invertible operators on* H *satisfying* $0<m_1\le A\le M_1$, $0<m_2\le B\le M_2$, $m_1<M_1$ *and* $m_2<M_2$. *Suppose that* $\alpha=m_1/M_1$, $\beta=m_2/M_2$ *and* $\gamma=\alpha\beta^{q-1}=\frac{m_1m_2^{q-1}}{M_1M_2^{q-1}}$. *Put* $K_\gamma=\frac{1-\alpha^p\beta^q}{1-\alpha\beta^{q-1}}$, $K_\sharp=(\frac{K_\gamma}{p})^{1/p}$ *and* $\tilde{K}_\sharp=\frac{K_\gamma}{\gamma^{1/q}}$. *Then for any* $\lambda>0$ *and any unit vector* $x\in H$

$$S_\sharp(\lambda):=\langle A^px,x\rangle^{\frac{1}{p}}\langle B^qx,x\rangle^{\frac{1}{q}}-\lambda\langle B^q\sharp_{1/p}A^px,x\rangle\le\frac{M_1M_2}{\beta^{q-1}}F_\sharp(\lambda),\tag{25}$$

where $F_\sharp(\lambda)$ *is a constant defined as follows:*

$$F_\sharp(\lambda)=\begin{cases}\gamma(1-\lambda) & \text{if } \tilde{K}_\sharp^p<\lambda\\ \{\frac{\gamma^p}{K_\gamma}+\frac{1}{q}(\frac{K_\sharp}{\lambda})^q-\gamma\}\lambda & \text{if } \tilde{K}_\sharp<\lambda\le\tilde{K}_\sharp^p\\ (\frac{1}{K_\gamma}-1)\lambda+\frac{1}{q}(\frac{K_\sharp}{\lambda^{1/p}})^q & \text{if } K_\sharp\le\lambda\le\tilde{K}_\sharp\\ \{\frac{1}{K_\gamma}+\frac{1}{q}(\frac{K_\gamma}{\lambda})^q-1\}\lambda & \text{if } K_\sharp^p<\lambda<K_\sharp\\ 1-\lambda & \text{if } 0<\lambda\le K_\sharp^p.\end{cases}\tag{26}$$

Furthermore for each λ, *the constant* $\frac{M_1M_2}{\beta^{q-1}}F_\sharp(\lambda)$ *in* (25) *is the best bound of* $S_\sharp(\lambda)$.

Proof. In Corollary 2, $F_1(\lambda)$ is determined by λ, α (and p), and hence we may put $F_1(\lambda) = F_1(\lambda, \alpha)$. Corollary 2 says that if C is a positive operator such that $0 < m \leq C \leq M$, then for $\gamma_0 = \frac{m}{M}$ and any vector $x \in H$,

$$\langle C^p x, x\rangle^{\frac{1}{p}} \langle x, x\rangle^{\frac{1}{q}} - \lambda\langle Cx, x\rangle \leq MF_1(\lambda, \gamma_0)\langle x, x\rangle. \tag{27}$$

Here we remark that the constants K_α, K_1 and $\tilde{K}_1$ in Corollary 2 are given as follows, respectively,

$$\frac{1-\gamma_0^p}{1-\gamma_0}, \quad \left(\frac{1-\gamma_0^p}{p(1-\gamma_0)}\right)^{\frac{1}{p}} \quad \text{and} \quad \left(\frac{1-\gamma_0^p}{p(1-\gamma_0)\gamma_0^{1/q}}\right)^{\frac{1}{p}}. \tag{28}$$

We replace C by $(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})^{\frac{1}{p}}$ and x by $B^{\frac{q}{2}}x$ in (27). Then we see that from $q-1>0$,

$$0 < \frac{m_1}{M_2^{q-1}} \leq (B^{-q/2}A^pB^{-q/2})^{1/p} \leq \frac{M_1}{m_2^{q-1}}.$$

Hence we have for $\gamma = \gamma_0 = \alpha\beta^{q-1}$

$$\begin{aligned}\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} &- \lambda\langle B^{\frac{q}{2}}(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})^{\frac{1}{p}}B^{\frac{q}{2}}x, x\rangle \\ &\leq \frac{M_1}{m_2^{q-1}}F_1(\lambda,\gamma)\langle B^q x, x\rangle \leq \frac{M_1M_2}{\beta^{q-1}}F_1(\lambda,\gamma).\end{aligned} \tag{29}$$

Here we denote the constants γ_0, K_α, K_1 and $\tilde{K}_1$ by γ, K_γ, $K_\sharp$ and $\tilde{K}_\sharp$ respectively in (28). Moreover putting $F_\sharp(\lambda) = F_1(\lambda,\gamma)$, we obtain the desired inequalty (25) by the definition of s-geometric mean. □

In Theorem 5, if $\lambda = 1$, then (25) is equivalent to the following inequality which is the noncommutative version of (12): For any unit vector $x \in H$,

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} - \langle B^q \sharp_{1/p} A^p x, x\rangle \leq \frac{M_1M_2}{\beta^{q-1}}\left\{\left(\frac{1}{K_\gamma}-1\right)+\frac{1}{q}K_\sharp^q\right\}. \tag{30}$$

This inequality is given directly in [10, Theorem 4.5]

Furthermore we see that if $F_\sharp(\lambda) = 0$ in (25), there exist a unique solution $\lambda = \lambda_\sharp$ ($\in [K_\sharp, \tilde{K}_\sharp]$) from Corollary 2. So we hold a ratio inequality in the following corollary which is the noncommutative version of (13).

Corollary 6. *Let A and B be positive invertible operators on H satisfying $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$, $m_1 < M_1$ and $m_2 < M_2$. Put $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then for any $\lambda > 0$ and any unit vector $x \in H$,*

$$\langle A^p x, x\rangle^{\frac{1}{p}} \langle B^q x, x\rangle^{\frac{1}{q}} \leq \lambda_0\langle B^q \sharp_{1/p} A^p x, x\rangle, \tag{31}$$

where $\lambda_0 \in [K_\sharp, \tilde{K}_\sharp]$ is the constant defined by (9).

Proof. The equation $F_\sharp(\lambda) = 0$ has a unique solution $\lambda = \lambda_\sharp$ ($\in [K_\sharp, \tilde{K}_\sharp]$). So we have from (25)

$$\left(\frac{1}{K_\gamma}-1\right)\lambda_0 + \frac{1}{q}\left(\frac{K_\sharp}{\lambda_0^{1/p}}\right)^q = 0.$$

So,

$$\lambda_0 = \left\{ \frac{K_\sharp^q K_\gamma}{q(K_\gamma - 1)} \right\}^{\frac{1}{q}} = \frac{K_\gamma}{p^{1/p} q^{1/q} (K_\gamma - 1)^{1/q}} = \frac{1 - \alpha^p \beta^q}{p^{1/p} q^{1/q} (\beta - \alpha\beta^q)^{1/p} (\alpha - \alpha^p \beta)^{1/q}}.$$

Hence $\lambda_\sharp$ coincides with λ_0 which is the constant defined by (9). □

We remark that (31) is given the same inequality in [4, Theorem 4] and is the noncommutative operator version of Gheorghiu's inequality.

Remark 7. *From the ratio inequalities* (13) *and* (31), *these estimations are eqivalent to* λ_0 *and are the best, so the ratio inequalities derived from Hölder's type operator inequality have the same best estimations regardless of commutativity of operators* A *and* B. *On the other hand, the differential estimations* $M_1 M_2 F_0(1)$ *in Theorem* 1 *and* $\frac{M_1 M_2}{\beta^{q-1}} F_\sharp(1)$ *in Theorem* 6 *are not equivalent. Indeed, if* $p = q = 2$, *then* $M_1 M_2 F_0(\lambda) = M_1 M_2 \frac{(1-\alpha\beta)^2}{2(1+\alpha)(1+\beta)}$, $\frac{M_1 M_2}{\beta^{q-1}} F_\sharp(\lambda) = M_1 M_2 \frac{(1-\alpha\beta)^2}{4\beta(1+\alpha\beta)}$, *and hence by* $2(1+\alpha)(1+\beta) > 4\beta(1+\alpha) > 4\beta(1+\alpha\beta)$ *we see that* $M_1 M_2 F(\lambda) \le \frac{M_1 M_2}{\beta^{q-1}} F_\sharp(\lambda; \gamma, 1, p)$. *So the estimations of the difference inequalities derived from Hölder's type operator inequality depend on the commutativity of operators* A *and* B.

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