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Finite Element Approximations of Conformal Mappings

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Abstract. In this paper, finite element approximations of conformal mappings defined on the unit disk to Jordan regions are considered. Three types of the normalization conditions are dealt with. In each case, convergence of the finite element conformal mappings to the exact solution is proved.

1. Introduction.

Let $D \subset \mathbb{R}^2$ be the unit disk, and $\Omega \subset \mathbb{R}^2$ a bounded domain whose boundary $\partial\Omega$ is a closed Jordan curve $\gamma \subset \mathbb{R}^2$ (such bounded regions are called *Jordan regions*). Let $\varphi : \overline{D} \to \overline{\Omega}$, $\varphi \in C(\overline{D} : \mathbb{R}^2) \cap C^1(D : \mathbb{R}^2)$ be C^1-diffeomorphic and orientation-preserving. If φ also preserves any angles of two curves crossing at a point in D, φ is called a ***conformal mapping***. From the theory of complex functions, we know that, if a C^1-diffeomorphism $\varphi = (\varphi_1, \varphi_2) : D \to \Omega$ is conformal, then the complex-valued function $f(z) := \varphi_1(x_1, x_2) + \sqrt{-1}\varphi_2(x_1, x_2)$ is differentiable with respect to the complex value $z := x_1 + \sqrt{-1}x_2$, and vice versa. In this paper, we study finite element approximation of conformal mappings on the unit disk.

For conformal mappings from the unit disk to Jordan regions, the following variational principle has been known (for example, see [3, pp.107–115], [4, Section 4.5]): Define the subset X_γ of $C(\overline{D}; \mathbb{R}^2) \cap H^1(D; \mathbb{R}^2)$ by

$$X_\gamma := \left\{ \psi \in C(\overline{D}; \mathbb{R}^2) \cap H^1(D; \mathbb{R}^2) \;\middle|\; \psi(\partial D) = \gamma \text{ and } \psi|_{\partial D} \text{ is monotone} \right\}, \tag{1.1}$$

where $\psi|_{\partial D}$ being monotone means that $\psi|_{\partial D}$ preserves the orientation of ∂D, and $(\psi|_{\partial D})^{-1}(p)$ is connected for any $p \in \gamma$. We denote the Dirichlet integral (or the energy functional) on D for $\varphi = (\varphi_1, \varphi_2) \in H^1(D; \mathbb{R}^2)$ by

$$D(\varphi) := \int_D |\nabla \varphi|^2 dx = \int_D (|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2) dx. \tag{1.2}$$

Then, we have that *$\varphi \in X_\gamma$ is conformal if and only if $\varphi \in X_\gamma$ is a stationary point of the functional $D(\varphi)$ in X_γ*. The problem of finding stationary points of the Dirichlet integral in X_γ is called the ***Plateau problem*** [3], [4]. Therefore, in this sense, the conformal mappings are minimal surfaces in $\mathbb{R}^2$.

By the Riemann mapping theorem, or the existence proof of solutions of the Plateau problem, there exist conformal mappings $\varphi \in X_\gamma$, and we have

$$D(\varphi) = \inf_{\psi \in X_\gamma} D(\psi) = |\Omega|,$$

where $|\Omega|$ is the area of Ω. It is also well-known that in our case the conformal mappings are homeomorphisms between $\overline{D}$ and $\overline{\Omega}$.

To specify a conformal mapping, we need a normalization condition. Several normalization conditions are known for conformal mappings on the unit disk. In this paper, we deal with the following conditions (see Figure 1):

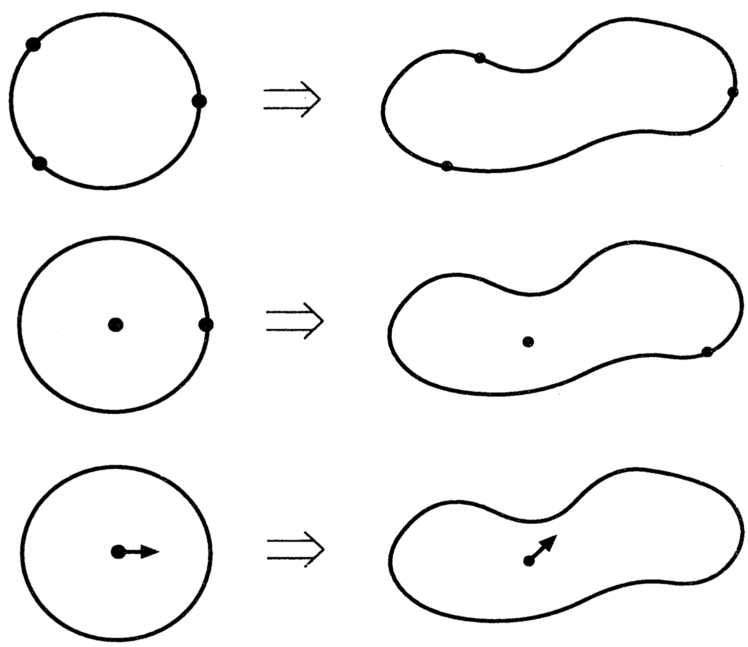

Figure 1: The three normalization conditions.

(a) Specify the image of three points on ∂D.
(b) Specify the image of a point on ∂D and a point in D.

(c) Specify the image of a point in D and the direction of derivative at the point.

In the following sections we discuss finite element approximations of conformal mappings with the above normalization conditions. Since the number of pages is limited, we cannot give any numerical examples in this article. They and a detailed discussion on implementation of finite element conformal mappings will be given in the final version of this paper.

2. Finite element approximation of conformal mappings

In this section, we consider a finite element approximation of conformal mappings. First, we suppose that we have a family of regular triangulation $\{\Delta_h\}$ of the unit disk with triangles (for the definition of *regular triangulation* see Ciarlet [1, p124]), where h stands for the maximum size of triangles in the triangulation Δ_h, and $h \to 0$. Set $D_h := \bigcup_{K \in \Delta_h} K$. Note that $D_h \subset D$ for any h. Let $S_h \subset C^0(D_h)$ be the set of piecewise linear functions on each triangle. We extend the functions of S_h in the following manner. On a point $x \in \partial D_h$, which is not a nodal point, there exists an outer normal half line l. On $y \in l \cap (D - D_h)$, the value of $f_h \in S_h$ is defined by $f_h(y) := f_h(x)$. A straightforward computation yields

$$\int_{D_h} |\nabla v_h|^2 dx \le \int_D |\nabla v_h|^2 dx \le (1 + Ch) \int_{D_h} |\nabla v_h|^2 dx$$

for any $v_h \in S_h$, where C is a positive constant independent of h.

We discretize X_γ as

$$\mathbb{S}_{\gamma,h} := \left\{ \psi_h \in (S_h)^2 \;\middle|\; \psi_h(\partial D \cap N_h) \subset \gamma \text{ and } \psi_h|_{\partial D} \text{ is } d\text{-monotone} \right\},$$

where N_h is the set of nodal points in D_h, and $\psi_h|_{\partial D}$ being d-monotone means that the order of nodes on ∂D is preserved on γ by $\psi_h|_{\partial D}$.

The *finite element* (or *FE*) *conformal mappings* φ_h are defined as the minimizer of the Dirichlet integral $D(v_h)$ in $\mathbb{S}_{\gamma,h}$:

$$D(\varphi_h) = \inf_{v_h \in \mathbb{S}_{\gamma,h}} D(v_h).$$

Since $\mathbb{S}_{\gamma,h}$ is a subset of finite-dimensional Euclidean space, it is obvious that FE conformal mappings exist.

From the definition it is obvious that finite element conformal mappings are "discrete harmonic mappings." That is, if $\varphi_h \in \mathbb{S}_{\gamma,h}$ is a FE conformal mapping, φ_h is the minimizer of the Dirichlet integral in the subset $\left\{ \psi_h \in \mathbb{S}_{\gamma,h} \;\middle|\; \psi_h|_{\partial D} = \varphi_h|_{\partial D} \right\}$. In other wards, φ_h

is the unique solution of the weak problem

$$\int_D (\nabla \varphi_h^1 \cdot \nabla v_h^1 + \nabla \varphi_h^2 \cdot \nabla v_h^2) dx = 0, \quad \forall v_h \in \mathbb{S}_h^0, \tag{2.1}$$

where $\mathbb{S}_h^0$ is defined by

$$\mathbb{S}_h^0 := \Big\{ v_h \in (S_h)^2 \;\Big|\; v_h = \vec{0} \text{ on } \partial D \Big\}.$$

It is well-known that the maximum principle does not hold for discrete harmonic mappings in general. We however have the following *weak maximum principle* for the discrete harmonic mappings proved by Schatz [5]:

Lemma 2.1 *Let the triangulation $\{\Delta_h\}$ be regular and quasi-uniform. Suppose that $\varphi_h \in \mathbb{S}_{\gamma,h}$ satisfies* (2.1). *Then for sufficiently small h there exists a positive $C \geq 1$ independent of h and φ_h such that*

$$\|\varphi_h\|_{L^\infty(D)} \leq C \|\varphi_h\|_{L^\infty(\partial D)}. \qquad \square \tag{2.2}$$

3. The Three Point Condition

In this section we consider the first normalization condition. Let $z_1, z_2, z_3 \in \partial D$ and $\zeta_1, \zeta_2, \zeta_3 \in \gamma$ be taken so that those points define the same orientation on ∂D and γ. Define

$$X_\gamma^{tp} := \Big\{ \psi \in X_\gamma \;\Big|\; \psi(z_i) = \zeta_i,\ i = 1, 2, 3 \Big\},$$

where "*tp*" stands for the *three point condition*. It is obvious that

$$\inf_{\psi \in X_\gamma^{tp}} D(\psi) = \inf_{\psi \in X_\gamma} D(\psi).$$

By the Riemann mapping theorem we have

Theorem 3.1 *There exists a unique $\varphi \in X_\gamma^{tp}$ which is conformal and bijective between D and Ω. Moreover, $\varphi : \overline{D} \to \overline{\Omega}$ is a homeomorphism.* $\square$

To prove Theorem 3.1 we use the following well-known lemma [3, Lemma 3.2], [4, Section 4.3]:

Lemma 3.2 *Let M be a positive constant such that the subset $A \subset X_\gamma^{tp}$ defined by*

$$A := \Big\{ \psi \in X_\gamma^{tp} \;\Big|\; D(\psi) \leq M \Big\}$$

is not empty. Then the functions $\big\{ \psi|_{\partial D} \big\}_{\psi \in A}$ are equicontinuous on ∂D. $\square$

Therefore, taking a minimizing sequence $\{\psi_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} D(\psi_n) = \inf_{X_\gamma^{tp}} D(\phi)$, it follows from Ascoli-Arzelà's theorem that there exists a subsequence $\{\psi_{n_i}\}$ such that $\{\psi_{n_i}|_{\partial D}\}$ converges some $g \in C(\partial D; \mathbb{R}^2)$ uniformly. Moreover, from the lower semi-continuity of the Dirichlet integral, we conclude that the solution $\varphi \in X_\gamma^{tp}$ of the Laplace equation

$$\Delta\varphi = 0 \quad \text{in } D, \qquad \varphi = g \quad \text{on } \partial D$$

is the desired conformal mapping.

We follow the above scenario to define the finite element conformal mappings. Let $\{\Delta_h\}_{h>0}$ be a family of regular triangulation of the unit disk such that $h \to 0$. In this section we always suppose that the point $z_1, z_2, z_3 \in \partial D$ are nodal points of Δ_h for each $h > 0$. Define the subset $\mathbb{S}_{\gamma,h}^{tp}$ by

$$\mathbb{S}_{\gamma,h}^{tp} := \left\{ \psi_h \in \mathbb{S}_{\gamma,h} \;\middle|\; \psi_h(z_i) = \zeta_i, \ i = 1, 2, 3 \right\}.$$

Definition 3.3 *A map $\varphi_h \in \mathbb{S}_{\gamma,h}^{tp}$ is called the **finite element conformal mapping** if it is a minimizer of the Dirichlet integral in $\mathbb{S}_{\gamma,h}^{tp}$:*

$$D(\varphi_h) = \inf_{\psi_h \in \mathbb{S}_{\gamma,h}^{tp}} D(\psi_h). \qquad \square$$

We now consider the convergence of the finite element conformal mappings. For each $h > 0$ there exists a finite element conformal mapping $\varphi_h \in \mathbb{S}_{\gamma,h}^{tp}$. The following is the most crucial [8, Corollary 8]:

Lemma 3.4 *Let $\{\Delta_h\}_{h>0}$ be a family of regular triangulation of the unit disk such that $h \to 0$ and $z_1, z_2, z_3 \in \partial D$ are nodal points in Δ_h for each $h > 0$. Suppose that the Jordan curve γ is rectifiable. Then, for the sequence of the finite element conformal mappings $\{\varphi_h\}_{h>0}$, the function $\{\varphi_h|_{\partial D}\}_{h>0}$ are equicontinuous on ∂D.* $\square$

By the exactly same manner as in [7] we obtain

Theorem 3.5 *Let $\{\Delta_h\}_{h>0}$ be a family of regular quasi-uniform triangulation of the unit disk such that $h \to 0$ and $z_1, z_2, z_3 \in \partial D$ are nodal points in Δ_h for each $h > 0$. Suppose that the Jordan curve γ is rectifiable. Then, the sequence of the finite element conformal mappings $\{\varphi_h\}_{h>0}$ converges to the exact conformal mapping $\varphi \in X_\gamma^{tp}$ in the following sense:*

$$\lim_{h\to 0} \|\varphi - \varphi_h\|_{H^1(D;\mathbb{R}^2)} = 0.$$

and if $\varphi \in W^{1,p}(D; \mathbb{R}^2)$, $p > 2$, then

$$\lim_{h\to 0} \|\varphi - \varphi_h\|_{C(D;\mathbb{R}^2)} = 0. \qquad \square$$

Remark In [7], [8], the family $\{\Delta_h\}_{h>0}$ of triangulation was supposed to be of *non-negative type* (see [2]) to ensure the maximum principle for the discrete harmonic mappings. However, since the weak maximum principle proved by Schatz (Lemma 2.1) is good enough for our proof, we only need to assume that $\{\Delta_h\}_{h>0}$ is quasi-uniform instead. □

4. The One Point Condition

In this section we consider another normalization condition: the *one point condition*. Let $z_0 \in D$ and $\zeta_0 \in \Omega$. Define X_γ^{op} by

$$X_\gamma^{op} := \Big\{\psi \in X_\gamma \;\Big|\; \psi(z_0) = \zeta_0\Big\}.$$

We know that the degree of "freedom" of conformal mappings in X_γ^{op} is just one, and if rotation around z_0 is specified, then the conformal mapping is determined uniquely. As is stated in Section 1 we specify either the correspondence of boundary points or the direction of the derivative at z_0 to fix rotation.

To take the same strategy as in Section 3, we would have to prove that, for a positive constant $M > |\Omega|$ and the subset $A := \{\psi \in X_\gamma^{op} \mid D(\psi) \leq M\}$, the functions $\{\psi|_{\partial D}\}_{\psi \in A}$ are equicontinuous. It seems, however, that this may be a wrong statement. Thus, we must impose an additional condition to make it valid.

A mapping $\psi \in X_\gamma$ is called *monotone* if $\psi^{-1}(p) \subset D$ is connected for any point $p \in \psi(D)$. We redefine X_γ^{op} by

$$X_\gamma^{op} := \Big\{\psi \in X_\gamma \;\Big|\; \psi(z_0) = \zeta_0 \text{ and } \psi \text{ is monotone}\Big\}.$$

Then we have

Lemma 4.1 *Let M be a positive constant such that $M > |\Omega|$. Let $A := \{\psi \in X_\gamma^{op} \mid D(\psi) \leq M\}$. Then the functions $\{\psi|_{\partial D}\}_{\psi \in A}$ are equicontinuous.*

Proof. First, we recall the famous Courant-Lebesgue lemma [3, pp.101-102], [4, Section4.4]. For any $z \in \mathbb{R}^2$ and any $r > 0$ we define

$$S_{r,z} := D \cap \{w \in \mathbb{R}^2 : |w - z| < r\}, \quad C_{r,z} := \overline{D} \cap \{w \in \mathbb{R}^2 : |w - z| = r\}.$$

If $z \in \partial D$, then we can write

$$C_{r,z} = \{z + re^{i\theta} : \theta_1(r) \leq \theta \leq \theta_2(r)\} \quad \text{with} \quad 0 < \theta_1(r) - \theta_2(r) < \pi.$$

Lemma 4.2 *Let $z \in \partial D$ and set $Z(r, \theta) := f(z + re^{i\theta})$ where r, θ denotes polar coordinates about z. Then, for arbitrary $\delta \in (0, R)$, $0 < R < 1$, there exists $\rho \in (\delta, \sqrt{\delta})$*

depending on f and z such that, for any pair θ, θ' with $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, we obtain the estimate

$$\int_\theta^{\theta'} |Z_\theta(\rho,\theta)| d\theta \le \eta(\delta, R)|\theta - \theta'|^{1/2}$$

with

$$\eta(\delta, R) := \left\{ \frac{2}{\log(1/\delta)} \int_{S_{R,z}} |\nabla f|^2 dx \right\}^{1/2},$$

and in particular

$$|Z(\rho,\theta) - Z(\rho,\theta')| \le \eta(\delta, R)|\theta - \theta'|^{1/2}.$$

From a topological argument the following statement is valid: *for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, taking any two distinct points $b_1, b_2 \in \gamma$ with $|b_1 - b_2| < \alpha$, the diameter of the smaller connected component of $\gamma - \{b_1, b_2\}$ is less than ε.*

Let $\varepsilon > 0$ be taken arbitrarily. Let $\alpha > 0$ be the real in the above statement and $\alpha_0 := \min(\alpha, \frac{1}{2}\text{dist}(\zeta_0, \gamma))$. Let $z \in \partial D$. We now take $\delta > 0$ such that

$$\left(\frac{2M\pi}{\log(1/\delta)} \right)^{1/2} < \alpha_0.$$

Then from Lemma 4.2 there exists $\rho \in (\delta, \sqrt{\delta})$ such that

$$|Z(\rho, \theta_1(\rho)) - Z(\rho, \theta_2(\rho))| \le \left(\frac{2M\pi}{\log(1/\delta)} \right)^{1/2} < \alpha.$$

Set $b_i := Z(\rho, \theta_i(\rho))$, $i = 1, 2$. From the above statement the diameter of the smaller connected component of $\gamma - \{b_1, b_2\}$ is less than ε. The proof, therefore, will be completed if we show that, for any $f \in A$ and $z \in \partial D$, the smaller connected component of $\partial D - \{c_1, c_2\}$, $c_i := z + \rho e^{i\theta_i(\rho)}$ $(i = 1, 2)$, is mapped to the smaller connected component of $\gamma - \{b_1, b_2\}$ by f.

We prove it by contradiction. Suppose that for any sufficiently small $\delta > 0$ there exist $f \in A$ and $z \in \partial D$ (and ρ in Lemma 4.2) such that the smaller component of $\partial D - \{c_1, c_2\}$ is mapped to the bigger component of $\gamma - \{b_1, b_2\}$. Define $l_{\rho,z} := f(C_{\rho,z})$. From the definitions the length of $l_{\rho,z}$ is less then $\frac{1}{2}\text{dist}(\zeta_0, \gamma)$. This implies that $\zeta_0 \notin l_{\rho,z}$. By a topological argument we conclude that $\zeta_0 \in f(S_{\rho,z})$. Hence we have $f^{-1}(\zeta_0) \cap S_{\rho,z} \ne \emptyset$ and $f^{-1}(\zeta_0) \cap C_{\rho,z} = \emptyset$. Since $z_0 \in f^{-1}(\zeta_0)$ and $z_0 \in D - S_{\rho,z}$ this means that the set $f^{-1}(\zeta_0)$ has at least two connected components. This contradicts the assumption that f is monotone. $\square$

We consider the finite element discretization of X_γ^{op}. Let $\{\Delta_h\}_{h>0}$ be a family of regular triangulation of the unit disk such that $h \to 0$. In this section we always suppose that the point $z_0 \in D$ is a nodal point of Δ_h for each $h > 0$. Define the subset $\mathbb{S}_{\gamma,h}^{op}$ by

$$\mathbb{S}_{\gamma,h}^{op} := \left\{ \psi_h \in \mathbb{S}_{\gamma,h} \mid \psi_h(z_0) = \zeta_0 \text{ and } \psi_h \text{ is monotone} \right\}.$$

Recall that N_h is the set of nodal points in Δ_h. We number the nodal points on the boundary ∂D in counter-clockwise order: $\{c_1, c_2, \cdots, c_m, c_{m+1}\} = N_h \cap \partial D$, where we assume $c_1 = c_{m+1}$. For $\psi_h \in \mathbb{S}^{op}_{\gamma,h}$ define

$$L_h(\psi_h) := \big\{|\psi_h(c_i) - \psi_h(c_{i+1})| : c_i \in N_h \cap \partial D, i = 1, \cdots m\big\}.$$

The following lemma is valid.

Lemma 4.3 *Let $\{\Delta_h\}$ be a family of regular triangulation as above. Let $M > |\Omega|$ be a constant and $\psi_h \in \mathbb{S}^{op}_{\gamma,h}$ taken so that $D(\psi_h) \le M$ for each h. Then $\lim_{h\to 0} L_h(\psi_h) = 0$.*

Proof. Since the proof of this lemma is very similar to that of [8, Lemma 3], we omit it here. (The proof will be given in the final version of this article.) □

Corollary 4.4 *Let $\{\Delta_h\}$ be a family of regular triangulation as in Lemma 4.3. Let $M > |\Omega|$ be a constant and $\psi_h \in \mathbb{S}^{op}_{\gamma,h}$ taken so that $D(\psi_h) \le M$ for each h. Then the functions $\{\psi_h|_{\partial D}\}$ are equicontinuous.*

Proof. See the proof of [8, Corollary 5]. □

5. The Other Normalization Conditions

Using the results obtained in Section 4, we now consider the normalization conditions other than the three point condition. The following lemma is obvious.

Lemma 5.1 *Let $z_0 \in D$, $z_1 \in \partial D$, $\zeta_0 \in \Omega$, and $\zeta_1 \in \gamma$. Define*

$$X^1_\gamma = X^1_\gamma(\zeta_1) := \big\{\psi \in X^{op}_\gamma \mid \psi(z_1) = \zeta_1\big\}.$$

Then the minimizer $\varphi \in X^1_\gamma$ of the Dirichlet integral $D(\psi)$ in X^1_γ is the unique conformal mapping from D onto Ω with $\varphi(z_i) = \zeta_i$, $i = 1, 2$.

We are now in a position to define the finite element conformal mappings corresponding $\varphi \in X^1_\gamma$ in Lemma 5.1. Let $\{\Delta_h\}_{h>0}$ be a family of regular triangulation of the unit disk such that $h \to 0$. We here assume that z_0, z_1 are nodal points of Δ_h for each $h > 0$. Define the subset $\mathbb{S}^1_{\gamma,h}$ by

$$\mathbb{S}^1_{\gamma,h} := \big\{\psi_h \in \mathbb{S}^{op}_{\gamma,h} \mid \psi_h(z_1) = \zeta_1\big\}.$$

Definition 5.2 *A map $\varphi_h \in \mathbb{S}^1_{\gamma,h}$ is called a finite element conformal mapping if it is the minimizer of the Dirichlet integral in $\mathbb{S}^1_{\gamma,h}$.* □

Using Corollary 4.4, we obtain the following theorem as in [7], [8].

Theorem 5.3 *Let $\{\Delta_h\}_{h>0}$ be a family of regular quasi-uniform triangulation of the unit disk such that $h \to 0$ and $z_0 \in D$, $z_1 \in \partial D$ are nodal points in Δ_h for each $h > 0$. Suppose that the Jordan curve γ is rectifiable. Then, the sequence of the finite element conformal mappings $\{\varphi_h \in \mathbb{S}^1_{\gamma,h}\}_{h>0}$ converges to the exact conformal mapping $\varphi \in X^1_\gamma$ in the following sense:*

$$\lim_{h\to 0} \|\varphi - \varphi_h\|_{H^1(D;\mathbb{R}^2)} = 0,$$

and if $\varphi \in W^{1,p}(D;\mathbb{R}^2)$, $p > 2$, then

$$\lim_{h\to 0} \|\varphi - \varphi_h\|_{C(D;\mathbb{R}^2)} = 0. \qquad \square$$

The third normalization condition is treated in a similar manner with the technique called "recovered gradient". The detailed discussion will be given in the final version of this article.

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