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Iterative Methods for Eigenvalue Problems of a General Complex Matrix

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1 Introduction

Let C be the set of complex numbers. We consider the following eigenvalue problem of a general complex $n\times n$ matrix A

(1) $$Az=\lambda z, \quad \|z\|^{2}=1.$$

Here $A=(a_{i,j}),\ a_{i,j}\in C,\ 1\leq i,\ j\leq n,\ \lambda\in C,\ z=(z_{1},z_{2},\cdots,z_{n})^{T}\in C^{n},\ \|z\|=\sqrt{z^{H}z},$ T and H denote transpose and conjugate transpose, respectively. In order to obtain approximate eigenvalue λ and its corresponding eigenvector z, (1) can be written as a system of complex nonlinear equations

(2) $$F(Z)=F(z,\lambda)=\left(\begin{array}{c} Az-\lambda z\\ -\frac{1}{2}(\|z\|^{2}-1) \end{array}\right)=0.$$

Here $Z=(z_{1},z_{2},\cdots,z_{n},\lambda)^{T}\in C^{n+1}$.

Remark 1. We note that $\|z\|^{2}=\sum_{j=1}^{n}|z_{j}|^{2}$ is not a differentiable function of complex variables $z_{1},z_{2},\cdots,z_{n}$.

We use the following notations ($t>0,\ d\in C^{n+1},\ I_{n}$ denotes the $n\times n$ identity matrix):

$$g(Z)=\frac{1}{2}\|F(Z)\|^{2}, \quad g'(Z,d)=\lim_{t\to+0}\frac{g(Z+td)-g(Z)}{t},$$

$$J(Z)=J(z,\lambda)=\left(\begin{array}{cc} A-\lambda I_{n} & -z\\ -z^{H} & 0 \end{array}\right)$$

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Remark 2. When A, λ, z are real-valued, $J(Z)$ represents the Jacobian matrix of $F(Z)$. However, when A, λ, z are complex-valued, $J(Z)$ does not represent the Jacobian matrix of $F(Z)$ because of the nondifferentiable term.

2 Iteration and Convergence

For solving (2) by iteration with line search, we put $Z_k = (z_{1,k}, \cdots, z_{n,k}, \lambda_k)^T$, $k = 0, 1, 2, \cdots$. Then we have the following iterative method GDNM(generalized damped Newton method) by using initial vector Z_0 and constants β, $\sigma \in (0, 1)$.

$$(3) \quad \begin{cases} \textit{step 1}: \text{By assuming that } F(Z_k) \neq 0 \text{ and } J(Z_k) \text{ is nonsingular,} \\ \qquad\qquad \text{solve } F(Z_k) + J(Z_k)d_k = 0 \text{ to get } d_k. \\ \textit{step 2}: \text{Let } m_k \text{ be the smallest nonnegative integer } m \text{ such that} \\ \qquad\qquad g(Z_k + \beta^m d_k) - g(Z_k) \leq \sigma\beta^m g'(Z_k, d_k), \quad m = 0, 1, 2, \cdots. \\ \qquad\qquad \text{Set } Z_{k+1} = Z_k + \alpha_k d_k \text{ with } \alpha_k = \beta^{m_k}. \\ \textit{step 3}: \text{Test } Z_{k+1} \text{ for convegence.} \end{cases}$$

Furthermore, when $J(Z_k)$ is singular we also have the following iterative method GDGNM(generalized damped Gauss-Newton method) by using initial vector Z_0 and constants β, $\sigma \in (0, 1)$, $\mu > 0$.

$$(4) \quad \begin{cases} \textit{step 1}: \text{By assuming that } J(Z_k)^H F(Z_k) \neq 0, \text{ solve} \\ \qquad\qquad J(Z_k)^H F(Z_k) + \left(J(Z_k)^H J(Z_k) + \mu I_{n+1}\right) d_k = 0 \text{ to get } d_k. \\ \textit{step 2}: \text{Let } m_k \text{ be the smallest nonnegative integer } m \text{ such that} \\ \qquad\qquad g(Z_k + \beta^m d_k) - g(Z_k) \leq \sigma\beta^m g'(Z_k, d_k), \quad m = 0, 1, 2, \cdots. \\ \qquad\qquad \text{Set } Z_{k+1} = Z_k + \alpha_k d_k \text{ with } \alpha_k = \beta^{m_k}. \\ \textit{step 3}: \text{Test } Z_{k+1} \text{ for convegence.} \end{cases}$$

Some lemmas are prepared.

Lemma 1 [5]. *Let λ be an eigenvalue and z be a corresponding eigenvector of A. Then λ is simple if and only if $J(z, \lambda)$ is nonsingular.*

Lemma 2 . *In GDNM, we have*

$$g'(Z_k, d_k) = -\|F(Z_k)\|^2.$$

In GDGNM, we have

$$g'(Z_k, d_k) = -\left(J(Z_k)^H F(Z_k)\right)^H \left(J(Z_k)^H J(Z_k) + \mu I_{n+1}\right)^{-1} J(Z_k)^H F(Z_k).$$

Lemma 3 . *Suppose that $F(Z_k) \neq 0$ in GDNM(or $J(Z_k)^H F(Z_k) \neq 0$ in GDGNM). Then there exists a scalar $s_0 > 0$ such that for all $s \in [0, s_0]$*

$$g(Z_k + sd_k) - g(Z_k) \leq \sigma s g'(Z_k, d_k).$$

We are now in a position to obtain the following results.

Theorem 1 . *Let $Z_* = (z_*, \lambda_*)$ be a solution of $F(Z) = 0$, where λ_* is a simple eigenvalue. Suppose that Z_0 is sufficiently close to Z_*. Let $\{Z_k\}$ be a sequence given by GDNM (3). If $\tilde{Z}$ is any accumulation point of $\{Z_k\}$, then we have $F(\tilde{Z}) = 0$ and $\tilde{Z} = (\delta z_*, \lambda_*)$, $\delta \in C$.*

Remark 3. When A, λ, z are real-valued, GDNM (3) with $\alpha_k = 1$ ($m_k = 0$) reduces to the usual Newton method which locally converges quadratically. When A, λ, z are complex-valued, we are not able to establish its rate of convergence because of the nondifferentiable term $\|z\|^2$.

Theorem 2 . *Assume that $\{Z_k\}$ given by GDGNM (4) is bounded. If $\tilde{Z} = (\tilde{z}, \tilde{\lambda})$ is any accumulation point of $\{Z_k\}$, then we have $J(\tilde{Z})^H F(\tilde{Z}) = 0$ and $\|F(\tilde{Z})\| = \frac{1}{2}\sqrt{1 - \|\tilde{z}\|^4}$.*

3 Numerical Results

Some numerical examples will be shown to indicate the effectiveness by using the following matrices([2], [5]) with $i = \sqrt{-1}$.

Example 1:

$$\begin{pmatrix} 5+9i & 5+5i & -6-6i & -7-7i \\ 3+3i & 6+10i & -5-5i & -6-6i \\ 2+2i & 3+3i & -1+3i & -5-5i \\ 1+i & 2+2i & -3-3i & 4i \end{pmatrix}$$

eigenpair $\{\lambda_*, z_*\}$;

$$\left\{\lambda_{*,1} = 1+5i,\ \ z_{*,1} = \frac{1}{\sqrt{7}}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\},\quad \left\{\lambda_{*,2} = 2+6i,\ \ z_{*,2} = \frac{1}{\sqrt{7}}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}\right\},$$

$$\left\{\lambda_{*,3} = 3+7i,\ \ z_{*,3} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\},\quad \left\{\lambda_{*,4} = 4+8i,\ \ z_{*,4} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Example 2:

$$\begin{pmatrix} 7 & 3 & 1+2i & -1+2i \\ 3 & 7 & 1-2i & -1-2i \\ 1-2i & 1+2i & 7 & -3 \\ -1-2i & -1+2i & -3 & 7 \end{pmatrix}$$

eigenpair $\{\lambda, z_*\}$;

$$\left\{ \lambda_{*,1} = 0, \quad z_{*,1} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -i \\ -i \end{pmatrix} \right\},$$

$$\left\{ \lambda_{*,2} = \lambda_{*,3} = 8, \quad z_{*,2} = \frac{1}{2} \begin{pmatrix} -1+i \\ 0 \\ 1 \\ i \end{pmatrix}, \quad z_{*,3} = \frac{1}{2} \begin{pmatrix} i \\ 1 \\ 0 \\ 1+i \end{pmatrix} \right\},$$

$$\left\{ \lambda_{*,4} = 12, \quad z_{*,4} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Example 3:

$$\begin{pmatrix} 14 & 9 & 6 & 4 & 2 \\ -9 & -4 & -3 & -2 & -1 \\ -2 & -2 & 0 & -1 & -1 \\ 3 & 3 & 3 & 5 & 3 \\ -9 & -9 & -9 & -9 & -4 \end{pmatrix}$$

eigenpair $\{\lambda_*, z_*\}$;

$$\left\{ \lambda_{*,1} = 5, \quad z_{*,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \left\{ \lambda_{*,2} = \lambda_{*,3} = 2, \quad z_{*,2} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$\left\{ \lambda_{*,4} = 1+\sqrt{2}\, i, \quad z_{*,4} = \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2-\sqrt{2}\, i \\ -1+2\sqrt{2}\, i \end{pmatrix} \right\},$$

$$\left\{ \lambda_{*,5} = 1-\sqrt{2}\, i, \quad z_{*,5} = \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2+\sqrt{2}\, i \\ -1-2\sqrt{2}\, i \end{pmatrix} \right\}$$

Table 1. Number of iterations for Example 1 ($\beta = 0.8$, $\sigma = 0.4$, $\mu = 10^{-7}$).

$Z_0 = (z_0, \lambda_0)$	GDNM(3)	GDGNM(4)	$\lim_{k\to\infty} \lambda_k$
$(1+i, 1+i, 1+i, 1+i, 0)^T$	8	8	$\lambda_{*,1} = 1+5i$
$(1+i, 1+i, 1+i, 1+i, 2.5+2.5i)^T$	7	7	$\lambda_{*,2} = 2+6i$
$(1+i, 1+i, 1+i, 1+i, 3.5+6.5i)^T$	8	8	$\lambda_{*,3} = 3+7i$
$(1+i, 1+i, 1+i, 1+i, 4.5+7.5i)^T$	7	7	$\lambda_{*,4} = 4+8i$

Table 2. Number of iterations for Example 2 ($\beta = 0.8$, $\sigma = 0.4$, $\mu = 10^{-7}$).

$Z_0 = (z_0, \lambda_0)$	GDNM(3)	GDGNM(4)	$\lim_{k\to\infty} \lambda_k$
$(1+i, 1+i, 1+i, 1+i, 1)^T$	8	8	$\lambda_{*,1} = 0$
$(1+i, 1+i, 1+i, 1+i, 5)^T$	8	7	$\lambda_{*,2} = \lambda_{*,3} = 8$
$(1+i, 1+i, 1+i, 1+i, 15)^T$	7	7	$\lambda_{*,4} = 12$

Table 3. Number of iterations for Example 3 ($\beta = 0.8$, $\sigma = 0.4$, $\mu = 10^{-15}$).

$Z_0 = (z_0, \lambda_0)$	GDNM(3)	GDGNM(4)	$\lim_{k\to\infty} \lambda_k$
$(1, 1, 1, 1, 1, 6)^T$	8	8	$\lambda_{*,1} = 5$
$(1, 1, 1, 1, 1, 1)^T$	27	29	$\lambda_{*,2} = \lambda_{*,3} = 2$
$(1+i, 1+i, 1+i, 1+i, 1+i, 2+2i)^T$	9	9	$\lambda_{*,4} = 1+\sqrt{2}\,i$
$(1+i, 1+i, 1+i, 1+i, 1+i, 2-2i)^T$	9	9	$\lambda_{*,5} = 1-\sqrt{2}\,i$

Table 4. Number of iterations of GDGNM(4) for Example 3 ($Z_0 = (1+i, 1+i, 1+i, 1+i, 1+i, 2-2i)^T$, $\beta = 0.8$, $\sigma = 0.4$).

μ	10^{-1}	10^{-2}	10^{-3}	10^{-5}	10^{-7}	10^{-15}
Number of iterations	164	28	13	10	9	9

Table 5. Numerical solutions of Example 3 for GDNM(3) ($Z_0 = (1, 1, 1, 1, 1, 6)^T$, $\beta = 0.8$, $\sigma = 0.4$).

k	m_k	λ_k	$g(Z_k)$
0	19	6.000000	1.925500×10^{3}
1	0	5.833238	1.897355×10^{3}
2	0	5.722243	3.030650×10^{0}
3	0	5.385764	1.896446×10^{-1}
4	0	5.113088	6.961577×10^{-3}
5	0	5.007389	2.275923×10^{-5}
6	0	5.000017	9.753440×10^{-11}
7	0	5.000000	4.455883×10^{-22}
8		5.000000	5.825032×10^{-31}
Exact $\lambda_{*,1}$		5.000000	

Table 6. Numerical solutions of Example 3 for GDNM(3) ($Z_0 = (1,1,1,1,1,1)^T$, $\beta = 0.8$, $\sigma = 0.4$).

k	m_k	λ_k	$g(Z_k)$
0	3	1.000000	1.773000×10^3
1	0	1.170667	8.189538×10^2
2	0	1.284823	3.243613×10^1
3	0	1.555609	3.970212×10^0
4	0	1.696398	1.982624×10^{-1}
5	0	1.825814	5.118973×10^{-3}
6	0	1.919700	4.259145×10^{-5}
7	0	1.961583	6.686242×10^{-7}
8	0	1.980819	4.275822×10^{-8}
9	0	1.990409	2.676738×10^{-9}
10	0	1.995205	1.672907×10^{-10}
11	0	1.997602	1.045567×10^{-11}
12	0	1.998801	6.534791×10^{-13}
13	0	1.999401	4.084245×10^{-14}
14	0	1.999700	2.552653×10^{-15}
15	0	1.999850	1.595408×10^{-16}
16	0	1.999925	9.971308×10^{-18}
17	0	1.999963	6.232045×10^{-19}
18	0	1.999981	3.895013×10^{-20}
19	0	1.999991	2.434478×10^{-21}
20	0	1.999995	1.521745×10^{-22}
21	0	1.999998	9.519367×10^{-24}
22	0	1.999999	5.922877×10^{-25}
23	0	1.999999	3.783560×10^{-26}
24	0	2.000000	2.251309×10^{-27}
25	1	2.000000	1.300824×10^{-28}
26	0	2.000000	3.098399×10^{-29}
27		2.000000	2.197792×10^{-31}
Exact $\lambda_{*,2} = \lambda_{*,3}$		2.000000	

Table 7. Numerical solutions of Example 3 for GDNM(3) ($Z_0 = (1+i,1+i,1+i,1+i,1+i,2+2i)^T$, $\beta = 0.8$, $\sigma = 0.4$).

k	m_k	λ_k	$g(Z_k)$
0	2	$2.000000 + 2.000000i$	3.613125×10^3
1	0	$1.653234 + 2.274796i$	1.246445×10^3
2	0	$1.333469 + 1.998749i$	9.134617×10^1
3	0	$1.200091 + 1.736889i$	5.682852×10^0
4	0	$1.098347 + 1.556285i$	2.915130×10^{-1}
5	0	$1.030216 + 1.455280i$	7.324111×10^{-3}
6	0	$1.002658 + 1.417781i$	2.143398×10^{-5}
7	0	$1.000012 + 1.414230i$	2.790953×10^{-10}
8	0	$1.000000 + 1.414214i$	2.839812×10^{-20}
9		$1.000000 + 1.414214i$	5.926901×10^{-31}
Exact $\lambda_{*,4}$		$1.000000 + 1.414214i$	

Table 8. Numerical solutions of Example 3 for GDGNM(4)
($Z_0 = (1+i, 1+i, 1+i, 1+i, 1+i, 2-2i)^T$, $\beta = 0.8$, $\sigma = 0.4$, $\mu = 10^{-15}$).

k	m_k	λ_k	$g(Z_k)$
0	2	$2.000000 - 2.000000i$	3.613125×10^{3}
1	0	$1.653234 - 2.274796i$	1.246445×10^{3}
2	0	$1.333469 - 1.998749i$	9.134617×10^{1}
3	0	$1.200091 - 1.736889i$	5.682852×10^{0}
4	0	$1.098347 - 1.556285i$	2.915130×10^{-1}
5	0	$1.030216 - 1.455280i$	7.324111×10^{-3}
6	0	$1.002658 - 1.417781i$	2.143398×10^{-5}
7	0	$1.000012 - 1.414230i$	2.790953×10^{-10}
8	0	$1.000000 - 1.414214i$	2.839797×10^{-20}
9		$1.000000 - 1.414214i$	3.827295×10^{-31}
Exact	$\lambda_{*,5}$	$1.000000 - 1.414214i$	

Thus, we can see that the iterative methods are effective and Theorems 1 and 2 are valid.

References

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