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ラインダイグラフの底グラフにおける完全独立全域木

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Completely independent spanning trees in the underlying graph of a line digraph

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Abstract

In this note, we define completely independent spanning trees. We say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees in a graph H if for any vertex r of H, they are independent spanning trees rooted at r. We present a characterization of completely independent spanning trees. Also, we show that for any k-vertex-connected line digraph $L(G)$, there are k completely independent spanning trees in the underlying graph of $L(G)$. At last, we apply our results to de Bruijn graphs, Kautz graphs, and wrapped butterflies.

1 Introduction

In a graph, two paths P_1 and P_2 from a vertex x to another vertex y are called *openly disjoint* if P_1 and P_2 are edge-disjoint and have no common vertex except for x and y. (Note that if x is adjacent to y, and both P_1 and P_2 only have the edge (x, y), then P_1 and P_2 have no common vertex except for x and y, but they are not edge-disjoint.) Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. Let r be a vertex of H. If for any vertex $v(\neq r)$ of H, the paths from r to v in $T_1, T_2, \ldots, T_k$, are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are k independent spanning trees rooted at r. (When we treat digraphs instead of graphs, a rooted tree is defined as an acyclic digraph in which there is a unique vertex (root) with indegree 0 such that for any other vertex, the indegree is 1. Independent spanning trees in a digraph are similarly defined.) For independent spanning trees, the following conjecture is well-known; "Let H be a k-vertex-connected graph. Then, for any vertex r of H, there are k independent spanning trees rooted at r." This conjecture was proved for $k \leq 3$ ([12] [3] [16]). Also, it has been shown that the conjecture holds for the class of planar graphs ([11]). The directed version of the conjecture was proved for $k = 2$ ([15]) and also for any $k \geq 1$ if we restrict ourselves to the class of line digraphs ([8]). However, in general, the directed version of the conjecture does not hold for $k \geq 3$ ([9]).

Independent spanning trees have been studied from not only the theoretical point of view but also the practical point of view because of their application to fault-tolerant broadcasting in parallel computers ([12]). Until now, independent spanning trees in several interconnection networks have been studied; product graphs ([14]), de Bruijn and Kautz digraphs ([6] [8]), and chordal rings ([13]).

Many papers have presented constructions of independent spanning trees for a given root vertex. However, if one set of spanning trees is always a set of independent spanning trees rooted at any given vertex, then we do not need to reconstruct independent spanning trees when the root is changed with another vertex. Motivated by this point of view, we define the following notion.

Definition 1.1 *Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. If for any two vertices u, v of H, the paths from u to v in $T_1, T_2, \ldots, T_k$, are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent.*

Note that completely independent spanning trees must be edge-disjoint although independent spanning trees are not always edge-disjoint. It is known that edge-disjoint spanning trees have an application to worm-hole routing in parallel computers ([1]). In this note, we present a characterization of completely independent spanning trees.

Unless otherwise stated, a digraph may have loops but not multiarcs. Let G be a digraph. Then, $V(G)$ and $A(G)$ denote the vertex set and the arc set of G, respectively. The *line digraph* $L(G)$ of G is defined as follows. The vertex set of $L(G)$ is the arc set of G, i.e., $V(L(G)) = A(G)$. Then, there is an arc from a vertex (u, v) to a vertex (x, y) in $L(G)$ iff $v = x$, i.e., $A(L(G)) = \{((u, v), (v, w)) \mid (u, v), (v, w) \in A(G)\}$. When we regard "$L$" as an operation on digraphs, the operation is called the *line digraph operation.* The *m-iterated line digraph* $L^m(G)$ of G is the digraph obtained from G by iteratively applying the line digraph operation m times. The underlying graph $U(G)$ of G is the graph obtained from G by replacing each arc with the corresponding edge and deleting loops. Note that $U(G)$ may have a 2-multiedge because G may have a pair of opposite arcs.

It has been shown in [8] that if a line digraph $L(G)$ is k-vertex-connected, then for any vertex r of $L(G)$, there are k independent spanning trees rooted at r in $L(G)$, thus, in $U(L(G))$ too. In this note, we strengthen such a result, i.e., we show that if a line digraph $L(G)$ is k-vertex-connected, then there are k completely independent spanning trees in $U(L(G))$. Since the class of the underlying graphs of line digraphs contains de Bruijn graphs, Kautz graphs, and wrapped butterflies which are known as interconnection networks of massively parallel computers, we finally apply our results to these interconnection networks.

The set of vertices adjacent from a vertex v in G is denoted by $\Gamma_G^+(v)$, and the outdegree of v in G, i.e., $|\Gamma_G^+(v)|$, is denoted by $deg_G^+ v$. Analogously, $\Gamma_G^-(v)$ and $deg_G^- v$ are defined. If for any vertex u of G, $deg_G^+ u = deg_G^- u = d$, then we say that G is d-regular. Let B be a subset of $A(G)$. Then, the subdigraph of G induced by B is denoted by $\langle B \rangle_G$. For a graph H and $v \in V(H)$, $deg_H v$ denotes the degree of v in H. A rooted tree of depth 1 is called a *star*. Let T be a rooted tree. The depth of T is the maximum length of paths from the root in T. The trees obtained from T by deleting the root are called the subtrees of T.

2 A characterization of completely independent spanning trees

The notion of completely independent spanning trees can be characterized as follows.

Theorem 2.1 *Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. Then, $T_1, T_2, \ldots, T_k$ are completely independent if and only if $T_1, T_2, \ldots, T_k$ are edge-disjoint and for any vertex v of H, there is at most one spanning tree T_i such that $deg_{T_i} v > 1$.*

Proof. ($\Leftarrow$): Let $T_1, T_2, \ldots, T_k$ be spanning trees such that they satisfy the right condition in the proposition. Now assume that $T_1, T_2, \ldots, T_k$ are not completely independent. Then, there exist two vertices r, v and two spanning trees T_i, T_j such that the paths from r to v in T_i and T_j are not openly disjoint. Since T_i and T_j are edge-disjoint, the paths from r to v have a common vertex w except for r and v. This means that $deg_{T_i} w > 1$ and $deg_{T_j} w > 1$, which produces a contradiction.
($\Rightarrow$): Suppose that $T_1, T_2, \ldots, T_k$ are completely independent. Clearly, $T_1, T_2, \ldots, T_k$ must be edge-disjoint. Now assume that there exists a vertex w such that $deg_{T_i} w > 1$ and $deg_{T_j} w > 1$. Without loss of generality, we can set $i = 1$ and $j = 2$. Let v be a vertex different from w. Let $\{w, t_l\}$ be the first edge on the path from w to v in T_l for $l = 1, 2$. Let x_l be a vertex such that the path from w to x_l in T_l does not contain the edge $\{w, t_l\}$ for $l = 1, 2$. Such vertices exist since $deg_{T_1} w > 1$ and $deg_{T_2} w > 1$. Both the path from x_1 to v in T_1 and the path from x_2 to v in T_2 contain w. Thus, $x_1 \neq x_2$. Since the paths from x_1 to v in T_1 and T_2 are openly disjoint, the path from x_1 to v in T_2 does not contain w. Now, we regard T_2 as a tree rooted at w. Then, x_1 and v are in the same subtree of T_2. On the other hand, x_2 and v are not in the same subtree of T_2. Thus, x_1 and x_2 are not in the same subtree of T_2. Similarly, when we regard T_1 as a tree rooted at w, x_1 and x_2 are not in the same subtree of T_1. Therefore, the paths from x_1 to x_2 in T_1 and T_2 have w as a common vertex, which contradicts our assumption that T_1 and T_2 are completely independent. Hence, for any vertex v, there is at most one T_i such that $deg_{T_i} v > 1$.
$\square$

3 Completely independent spanning trees in the underlying graph of a line digraph

Definition 3.1 *Let F be a unicyclic spanning subdigraph of H. If for any vertex of F, the indegree is one, then F is called a cycle-rooted tree, and the cycle is denoted by $C(F)$.*

Lemma 3.2 *Let F be a cycle-rooted tree. Then, $L(F) \cong F$.*

Proof. Define a bijection φ from $V(L(F))$ to $V(F)$ as $\varphi((u,v)) = v$. Then, for any arc $((u,v),(v,w)) \in A(L(F))$, $(\varphi((u,v)), \varphi((v,w))) = (v,w) \in A(F)$. Suppose that $((u,v),(x,y)) \notin A(L(F))$, i.e., $v \neq x$. Then, $(\varphi((u,v)), \varphi((x,y))) = (v,y)$. Since the indegree of y in F is one, $\Gamma_F^-(y) = \{x\}$. Hence, $(v,y) \notin A(F)$. Therefore, φ is an isomorphism from $L(F)$ to F. $\square$

Lemma 3.3 *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees $G_1, G_2, \ldots, G_k$ in G. Then, there are k arc-disjoint spanning cycle-rooted trees $F_1, F_2, \ldots, F_k$ in $L(G)$ such that for any F_i and any vertex v of $L(G)$, $deg^+_{F_i} v = deg^+_{L(G)} v$, or $deg^+_{F_i} v = 0$.*

Proof. Let $G_1, G_2, \ldots, G_k$ be arc-disjoint spanning cycle-rooted trees in G. For each G_i, we consider the following set of arcs of $L(G)$.

$$A_i = \{((u,v),(v,w)) \mid (u,v) \in A(G_i), (v,w) \in A(G)\}.$$

Clearly, $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq k$ since $A(G_i) \cap A(G_j) = \emptyset$ for $1 \leq i < j \leq k$. Now we divide A_i into two subsets A'_i and A''_i as follows;

$$\begin{cases} A'_i = \{((u,v),(v,w)) \mid (u,v),(v,w) \in A(G_i)\}, \\ A''_i = \{((u,v),(v,w)) \mid (u,v) \in A(G_i), (v,w) \notin A(G_i)\}. \end{cases}$$

From Lemma 3.2, $\langle A_i' \rangle_{L(G)} \cong G_i$. Clearly, $\langle A_i'' \rangle_{L(G)}$ is a union of stars such that each root is a vertex of $\langle A_i' \rangle_{L(G)}$ and each leaf is not a vertex of $\langle A_i' \rangle_{L(G)}$. Hence, $\langle A_i \rangle_{L(G)} = \langle A_i' \cup A_i'' \rangle_{L(G)}$ is also a cycle-rooted tree. Since G_i is spanning, it is easily checked that $\langle A_i \rangle_{L(G)}$ is also spanning. Here, let $F_i = \langle A_i \rangle_{L(G)}$ for $i = 1, 2, \ldots, k$.

Now, consider a vertex (u, v) of $L(G)$. Suppose that (u, v) is contained in G_j, i.e., (u, v) is a vertex of $\langle A_j' \rangle_{L(G)}$. Then, for any $(v, w) \in A(L(G))$, $((u, v), (v, w))$ is contained in F_j, i.e., $deg^+_{F_j}(u, v) = deg^+_{L(G)}(u, v)$. Thus, in this case, for any $F_i, i \neq j$, $deg^+_{F_i}(u, v) = 0$. Suppose that (u, v) is not contained in any G_i. In this case, $deg^+_{F_i}(u, v) = 0$ for any F_i. □

Lemma 3.4 *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees in G. Then, there are k completely independent spanning trees in $U(L(G))$.*

Proof. Let $G_1, G_2, \ldots, G_k$ be k arc-disjoint spanning cycle-rooted trees in G. Then, let F_i be the digraph defined as $\langle A_i \rangle_{L(G)}$ in the proof of Lemma 3.3 for $i = 1, 2, \ldots, k$. Let T_i be the spanning tree in $U(L(G))$ obtained from $U(F_i)$ by deleting one edge in $U(C(F_i))$ for $i = 1, 2, \ldots, k$. Then, clearly $T_1, T_2, \ldots, T_k$ are edge-disjoint. Also, for any vertex v of $U(L(G))$,

$$deg_{T_i} v \leq deg^+_{F_i} v + deg^-_{F_i} v = deg^+_{F_i} v + 1.$$

From Lemma 3.3, there is at most one F_j such that $deg^+_{F_j} v \geq 1$. Therefore, from Theorem 2.1, $T_1, T_2, \ldots, T_k$ are completely independent spanning trees in $U(L(G))$. □

Theorem 3.5 [4] *Let G be a k-arc-connected digraph. Then, for any vertex r of G, there are k arc-disjoint spanning trees rooted at r in G.*

Edmonds' Theorem is corresponding to the arc-version of the conjecture mentioned in the introduction.

Theorem 3.6 *Let $L(G)$ be a k-vertex-connected line digraph. Then, there are k completely independent spanning trees in $U(L(G))$.*

Proof. It is easily checked that if $L(G)$ is k-vertex-connected, then G is k-arc-connected. From Edmonds' Theorem, there are k arc-disjoint spanning trees rooted at any vertex r. Since G is k-arc-connected, $deg^-_G r \geq k$. Adding an arc adjacent to the root to each spanning tree disjointly, we can obtain k arc-disjoint spanning cycle-rooted trees in G. Hence, by Lemma 3.4, there are k completely independent spanning trees in $U(L(G))$. □

4 Applications to de Bruijn graphs, Kautz graphs, and wrapped butterflies

Applying Lemma 3.3 iteratively and discussing similarly to the proof of Lemma 3.4, we can see that the following proposition holds.

Proposition 4.1 *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees in G. Then, there are k completely independent spanning trees in $U(L^m(G))$.*

In the above proposition, if we add some conditions, then we can obtain a more interesting result. The depth of a cycle-rooted tree T is the maximum depth of the trees obtained from T by deleting all the edges in the cycle.

Proposition 4.2 *Let G be a regular digraph. Suppose that there are k isomorphic arc-disjoint spanning cycle-rooted trees of cycle-length r and depth c in G. Then, there are k isomorphic completely independent spanning trees of depth at most $2(m+c)+r-1$ in $U(L^m(G))$.*

Proof. Let G be d-regular. We use the same notations introduced in the proof of Lemma 3.3. By the assumption, $\langle A'_i \rangle_{L(G)} \cong \langle A'_j \rangle_{L(G)}$ for $1 \leq i < j \leq k$. By adding arcs in A''_i to $\langle A'_i \rangle_{L(G)}$, for any vertex of $\langle A'_i \rangle_{L(G)}$, if the outdegree is not equal to d, then it becomes d in $\langle A_i \rangle_G$. Thus, we can see that $F_i \cong F_j$ for $1 \leq i < j \leq k$. From this observation, the isomorphic property in the proposition is induced.

By adding arcs in A''_i, the depth of each subtree of a spanning cycle-rooted tree increases by one. On the other hand, the cycle-length is invariant with respect to the line digraph operation. Since we consider the underlying graph of a spanning cycle-rooted tree and delete one edge in the cycle, the upper bound on the depth shown in the proposition is obtained. □

Let K^*_d denote the complete symmetric digraph with d vertices. Also, let K°_d denote the complete digraph with d vertices, i.e., the digraph obtained from K^*_d by adding a loop to each vertex. Then, the de Bruijn digraph $B(d,D)$ and the Kautz digraph $K(d,D)$ are defined as $B(d,D) = L^{D-1}(K^\circ_d)$ and $K(d,D) = L^{D-1}(K^*_{d+1})$. We abbreviates $U(B(d,D))$ and $U(K(d,D))$ to $UB(d,D)$ and $UK(d,D)$, respectively. It is easily checked that K°_d and K^*_{d+1} have d isomorphic arc-disjoint spanning cycle-rooted trees. Hence, from Proposition 4.2, the following corollaries are obtained. The fact of Corollary 4.3 has been shown in [6].

Corollary 4.3 [6] *There are d isomorphic completely independent spanning trees of depth $2D$ in $UB(d,D)$.*

Corollary 4.4 *There are d isomorphic completely independent spanning trees of depth $2D$ in $UK(d,D)$.*

The wrapped butterfly $wb(k,r)$ can be defined by the underlying graph of $L^{r-1}(K^\circ_k \otimes C_r)$ ([7]), where C_r is the cycle of length r, and $\otimes$ is the Kronecker product, i.e., for two digraphs G_1 and G_2,

$$\begin{cases} V(G_1 \otimes G_2) = V(G_1) \times V(G_2), \\ A(G_1 \otimes G_2) = \{((u_1,u_2),(v_1,v_2)) \mid (u_1,v_1) \in A(G_1) \\ \qquad\qquad\qquad\qquad \text{and } (u_2,v_2) \in A(G_2)\}. \end{cases}$$

Since $K^\circ_k \otimes C_r$ has k isomorphic arc-disjoint spanning cycle-rooted trees, the next corollary follows from Proposition 4.2.

Corollary 4.5 *There are k isomorphic completely independent spanning trees of depth $3r-1$ in $wb(k,r)$.*

Note that the numbers of completely independent spanning trees in $UB(d,D), UK(d,D)$ and $wb(k,r)$ shown in the corollaries are best possible. In fact, there is no remaining edge in $UB(d,D)$. Also, there are only d (resp, k) remaining edges in $UK(d,D)$ (resp, $wb(k,r)$).

5 Concluding remarks

In this note, we have shown that there are k completely independent spanning trees in the underlying graph of a k-vertex-connected line digraph. It is well-known that there are k edge-disjoint spanning trees in a $2k$-edge-connected graph. We have the following conjecture on completely independent spanning trees.

Conjecture: There are k completely independent spanning trees in a $2k$-vertex-connected graph.

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