数理解析研究所講究録
1153 巻 2000 年 122-131

Shift-like mappings of dynamical degree golden ratio of $\mathbb{C}^3$

Kazutoshi MAEGAWA
(Graduate school of Human and Enviromental Studies, Kyoto University.)
前川 和俊
(京都大学大学院人間・環境学研究科)
km@math.h.kyoto-u.ac.jp

1 Introduction

For polynomial automorphisms of $\mathbb{C}^2$, there is a classification theorem on polynomial conjugation ([FM]). By the theorem, any polynomial automorphism is polynomially conjugate to an affine transformation, an elementary mapping, or a finite composition of generalized Hénon mappings

$$(x, y) \mapsto (y, p(y) + dx),$$

where $p(y)$ is a polynomial of degree at least two and $d \neq 0$. The proof is based on the fact that the group of polynomial automorphisms of $\mathbb{C}^2$ is the amalgamated free product of two subgroups (the group of affine transformations and the group of elementary mappings), called the theorem of Jung. In case of higher dimension, such a theorem on the generators of the group of polynomial automorphisms is not known.

In this note, we will see a classification theorem of quadratic polynomial automorphisms of $\mathbb{C}^3$ which is on polynomial conjugation. In the theorem, an invariant under polynomial conjugation is used to characterize each class. It's called *dynamical (bi)degree*, introduced by Bedford and Smillie ([BS]). The proof is based on the theory of quadratic form or the fact that the determinant of the Jacobian matrix of any polynomial automorphisms is a nonzero constant ([FW][M1]). Next, we will focus our attention on one of normal forms which is of the form

$$(x, y, z) \mapsto (y, z, yz + by + cz + dx),$$

where $d \neq 0$. (If a constant term exists in the third component, it can be vanished by affine conjugation sending one of two fixed points to the origin.) It is one of shift-like mappings of $\mathbb{C}^3$. (Shift-like mappings were introduced by Bedford and Pambuccian [BP] as a generalization of Hénon maps.) But there are some remarkable differences between Hénon maps and our map in their dynamical properties. They will be seen through the proofs of convergence and continuity of Green functions on $\mathbb{C}^3$. For a birational (polynomial) map of $\mathbb{P}^2$

$$(x, y) \mapsto (y, xy + bx + cy) \ (\textit{imhomogeneous coordinate}),$$

Nishimura introduced "another" Green function which does not vanish constantly in [N]. (If we adopt the usual Green function as that in the case of holomorphic maps of $\mathbb{P}^2$, it vanishes in the whole space.) It was shown that there is a domain ($\subset \mathbb{C}^2$) in which the function converges (proposition 4.5, p 199, [N]). It is easy to check that our approach for our map is also valid for this birational map, so the domain of convergence will extend to the whole $\mathbb{C}^2$.

2 Quadratic polynomial automorphisms of $\mathbb{C}^3$

Let $\mathsf{Aut}_2(\mathbb{C}^3)$ be the family of polynomial automorphisms of degree at most 2 of $\mathbb{C}^3$. To identify the map which will be studied in this note, we will see a classification theorem of $\mathsf{Aut}_2(\mathbb{C}^3)$. To begin with, we will define subfamilies of $\mathsf{Aut}_2(\mathbb{C}^3)$.

- **affine transformations**
  A : the group of affine transformations of $\mathbb{C}^3$.
- **skew products (2 dimensional fibers)**
  $\mathsf{E}_1 : (ax+b, z, x^2+L(x,z)+dy), (ad \neq 0,\ \deg L \leq 1)$
  $\mathsf{E}_2 : (ax+b, z, xz+rx^2+L(x,z)+dy), (ad \neq 0,\ \deg L \leq 1)$
  $\mathsf{E}_3 : (ax+b, z, z^2+qxz+rx^2+L(x,z)+dy), (ad \neq 0,\ \deg L \leq 1)$.
- **skew products (1 dimensional fibers)**
  $\mathsf{F}_1 : (ax+b, Q(x)+cy, P(x,y)+dz), (acd \neq 0,\ \max\{\deg P, \deg Q\} = 2)$
  $\mathsf{F}_2 : (y, y^2+a+bx, P(x,y)+dz), (bd \neq 0, \deg P \leq 2)$.
- **shift-like mappings (1-shift)**
  $\mathsf{G}_1 : (y, z, y^2+L(y,z)+dx), (d \neq 0,\ \deg L \leq 1)$
  $\mathsf{G}_2 : (y, z, yz+L(y,z)+dx), (d \neq 0,\ \deg L \leq 1)$
  $\mathsf{G}_3 : (y, z, yz+ry^2+L(y,z)+dx), (rd \neq 0,\ \deg L \leq 1)$
  $\mathsf{G}_4 : (y, z, z^2+L(y,z)+dx), (d \neq 0,\ \deg L \leq 1)$
  $\mathsf{G}_5 : (y, z, z^2+qyz+L(y,z)+dx), (qd \neq 0,\ \deg L \leq 1)$
  $\mathsf{G}_6 : (y, z, z^2+qyz+ry^2+L(y,z)+dx), (rd \neq 0,\ \deg L \leq 1)$.
- **shift-like mappings (2-shift)**
  $\mathsf{H}_1 : (y, z, y^2+L(y,z)+dx) \circ (y, z, p'z^2+q'yz+L'(y,z)+d'x),$
  $(d \neq 0,\ q'd' \neq 0,\ \deg L, \deg L' \leq 1)$
  $\mathsf{H}_2 : (y, z, y^2+L(y,z)+dx) \circ (y, z, p'z^2+q'yz+r'y^2+L'(y,z)+d'x),$
  $(d \neq 0,\ r'd' \neq 0,\ \deg L, \deg L' \leq 1)$.

Skew products are analogous to elementary maps of $\mathbb{C}^2$ and shift-like maps to Hénon maps. In general, shift-like mappings (m-shift) of $\mathbb{C}^N$ are of the form

$$G = G_m \circ \cdots \circ G_1,$$

where

$$(x_1, \cdots, x_N) \overset{G_i}{\mapsto} (x_2, \cdots, x_N, P_i(x_2, \cdots, x_N) + d_i x_1), \ (1 \leq i \leq m),$$

P_i $(1 \leq i \leq m)$ are polynomials and d_i $(1 \leq i \leq m)$ are nonzero constants.

Theorem 2.1. ([M1]) Any element of $\mathsf{Aut}_2(\mathbb{C}^3)$ is polynomially conjugate to an element of $\mathsf{A}, \mathsf{E}_\mathsf{i}(\mathsf{i}=1,2,3), \mathsf{F}_\mathsf{i}(\mathsf{i}=1,2), \mathsf{G}_\mathsf{i}(\mathsf{i}=1,\cdots,6)$ or $\mathsf{H}_\mathsf{i}(\mathsf{i}=1,2)$.

Remark 2.2. In theorem 2.1, we may take an element of $\mathsf{Aut}_2(\mathbb{C}^3)$ as the coordinate transformation for the conjugation.

In the following theorem, $\mathsf{Aut}_2(\mathbb{C}^3)$ is devided into 9 classes. Each class is characterized by an invariant under polynomial conjugation, called dynamical bidegree. Dynamical degree of f is defined by $d(f) = \lim_{n\to\infty} (\deg f^n)^{\frac{1}{n}}$ and the limit exists for any polynomial selfmap f of $\mathbb{C}^N$ ([BS]). In case f is inversible, we'll consider dynamical bidegree $D(f) := (d(f), d(f^{-1}))$.

Denote by ϕ golden ratio $\frac{1+\sqrt{5}}{2}$.

Theorem 2.3. ([M1]) *Elements of* $\mathsf{Aut}_2(\mathbb{C}^3)$ *are classified as follows by dynamical bidegree. (It is also shown which classes the subfamilies belong to.)*

class	dynamical bidegree	family
I	$(1,1)$	$\mathsf{A}, \mathsf{E}_1, \mathsf{E}_2, \mathsf{F}_1$
II	$(\sqrt{2},2)$	G_1
III	(ϕ,ϕ)	G_2
IV	$(\phi,2)$	G_3
V	$(2,\sqrt{2})$	G_4
VI	$(2,\phi)$	G_5
VII	$(2,2)$	$\mathsf{E}_3, \mathsf{F}_2, \mathsf{G}_6$
VIII	$(2,3)$	H_1
IX	$(2,4)$	H_2

Remark 2.4.
(i) Unlike the case of $\mathbb{C}^2$, if a polynomial automorphism is polynomially conjugate to a shift-like mapping (m-shift), then m is not uniquely determined.
(ii) Unlike the case of $\mathbb{C}^2$, there is a shift-like mapping which is conjugate to a skew product.

3 Shift-like mappings of dynamical bidegree (ϕ, ϕ)

Any element of III is conjugate to $f(x,y,z) = (y, z, yz + by + cz + dx)$ where d is a nonzero constant. In this section, we will see that Green function of f converges in $\mathbb{C}^3$. We will adopt the following definition of the function. (If we use the usual definition as that in the case of Hénon maps, then we have $\lim_{n\to\infty} \frac{1}{2^n}\log^+||f^n(w)|| \equiv 0$.)

Definition 3.1. For a polynomial mapping f from $\mathbb{C}^N$ into itself, we will define Green function of f by

$$G^+(w) := \lim_{n\to\infty} \frac{1}{\deg f^n}\log^+||f^n(w)||, \ (\log^+||w|| = \max\{0, \log||w||\} \ ).$$

If the limit exists, then $G^+(f(w)) = d(f)G^+(w)$ holds. (The multiplier of the functional equation is dynamical degree, not algebraic degree $\deg f$.)

When we'd like to see the convergence, there are remarkable differences between the cases of Hénon map $h(x,y) = (y, y^2 + c + dx)$ and our map $f(x,y,z) = (y, z, yz + by + cz + dx)$.

(1) Degree lowering, i.e. $\deg f^n < 2^n$, $(n \geq 2)$. In fact, $\{\deg f^n\}$ is Fibonacci's sequence.

(2) In the case of h, any orbit except bounded orbits converges to a superattracting fixed point $[0:1:0] \in \mathbb{P}^2$. But in our case, there isn't such a fixed point. Any orbit except bounded orbits accumulates at points of indeterminacy $\in \mathbb{P}^3$, i.e. not *nice orbit.* (See [FS] for the definition.) It's not known whether such an orbit tends to infinity or not in general. (See theorem 3.20.)

(3) In the case of h, there is a compact polydisc D such that any bounded orbit gets into D in forward iteration and never leave it. But in our case, for some parameters, the set of periodic points is not bounded. This means that for our map, we cannot take such a polydisk as D. Hence the expected proofs of convergence and continuity of Green functions cannot depend on compactness of such a polydisk.

We will consider the following subsets in $\mathbb{C}^3$.

Definition 3.2. R : a positive number.

$$\begin{aligned}
V_R^+ &= \{|y| > R,\ |z| > R,\ |z| > |x|\}, \\
V_R^- &= \{|y| > R,\ |x| > R,\ |x| > |z|\}, \\
W_R^1 &= \{|x| \geq R,\ |y| \leq R,\ |z| \leq R\}, \\
W_R^2 &= \{|x| \leq R,\ |y| \geq R,\ |z| \leq R\}, \\
W_R^3 &= \{|x| \leq R,\ |y| \leq R,\ |z| \geq R\}, \\
W_R^4 &= \{|x| \geq R,\ |y| \leq R,\ |z| \geq R\}, \\
D_R &= \{|x| \leq R,\ |y| \leq R,\ |z| \leq R\}.
\end{aligned}$$

$$W_R = \left(\bigcup_{i=1}^{4} W_R^i\right) \cup D_R.$$

We will denote $f^n(x,y,z)$ by (x_n, y_n, z_n) for $(x,y,z) \in \mathbb{C}^3$.

Lemma 3.3. *There is a positive number R such that if $w = (x,y,z) \in V_R^+$, then*

$$(1 - \delta_R)|y||z| \leq |z_1| \leq (1 + \delta_R)|y||z|,$$

where $\delta_R = \frac{1}{R}(|b| + |c| + |d|)$.

proof : If R is large enough, then for $w \in V_R^+$,

$$\begin{aligned} \big|\,|z_1| - |yz|\,\big| &\leq |z_1 - yz| \\ &= |by + cz + dx| \\ &\leq |yz|\left(\frac{|b|}{|z|} + \frac{|c|}{|y|} + \frac{|dx|}{|yz|}\right) \\ &\leq \delta_R |yz|. \end{aligned}$$

$\square$

Remark 3.4. Since $f^{-1}(x,y,z) = (\frac{1}{d}(-xy - bx - cy + z), x, y)$, the similar inequality to lemma 3.3 hold for f^{-1}, exchanging the roles of x and z. (But, then the coefficients are not $1 \pm \delta_R$.)

Proposition 3.5. *For any number $\rho > 1$, there is a positive number R such that the following hold.*

(i) $f^n(V_R^+) \subset V_{\rho^{n-1}R}^+$ *for all* $n \geq 1$.

(ii) $f^{-n}(V_R^-) \subset V_{\rho^{n-1}R}^-$ *for all* $n \geq 1$.

(iii) $\mathbb{C}^3 = \bigcup_{n=0}^{\infty} f^n(\mathbb{C}^3 \setminus V_R^+)$. *(the limit of an increasing sequence.)*

(iv) $\mathbb{C}^3 = \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{C}^3 \setminus V_R^-)$. *(the limit of an increasing sequence.)*

(v) *For any compact set K in $\mathbb{C}^3$, there is a natural number N such that $f^{-n}(K) \subset \mathbb{C}^3 \setminus V_R^+$ for all $n \geq N$.*

(vi) *For any compact set K in $\mathbb{C}^3$, there is a natural number N such that $f^n(K) \subset \mathbb{C}^3 \setminus V_R^-$ for all $n \geq N$.*

proof : By the symmetry of f and f^{-1}, we have only to prove (i), (iii), (v). (See remark 3.4.)

(i) By lemma 3.3, if R is large enough,

$$|z_1| \geq \rho|z|, \ \rho|y|.$$

Since $y_1 = z$ and $x_1 = y$, we have $f(V_R^+) \subset V_R^+$. Applying this inequality to (x_1, y_1, z_1), we have $f^2(V_R^+) \subset V_{\rho R}^+$. In the same way, $f^n(V_R^+) \subset V_{\rho^{n-1}R}^+$ hold for all $n \geq 1$.

(iii) By (i), the sequence $\{f^n(\mathbb{C}^3 \setminus V_R^+)\}_{n=0}^{\infty}$ is increasing, and

$$\bigcap_{n=0}^{\infty} f^n(V_R^+) = \emptyset.$$

Hence, we have the equality.

(v) From (iii) and the compactness of K, it follows immediately.

□

Remark 3.6. From the proof of (i), we know $f^2(\overline{V_R^+}) \subset V_R^+$.

Denote Fibonacci sequence by $\{a_n\}_{n=0}^{\infty}$, where $a_0 = 0,\ a_1 = 1,\ a_{n+2} = a_{n+1} + a_n$.

Proposition 3.7. *There is a positive number R such that if $w = (x, y, z) \in V_R^+$, then*

$$(1-\delta_R)^{a_{n+2}-1}|y|^{a_n}|z|^{a_{n+1}} \le |z_n| \le (1+\delta_R)^{a_{n+2}-1}|y|^{a_n}|z|^{a_{n+1}},$$
$$(1-\delta_R)^{a_{n+1}-1}|y|^{a_{n-1}}|z|^{a_n} \le |y_n| \le (1+\delta_R)^{a_{n+1}-1}|y|^{a_{n-1}}|z|^{a_n}$$

hold for all $n \ge 1$.

proof : Take R in lemma 3.3. By lemma 3.3, if $w = (x, y, z) \in V_R^+$, then

$$\begin{aligned}(1-\delta_R)|y||z| \le\ & |z_1| && \le (1+\delta_R)|y||z|,\\ |z| \le\ & |y_1| && \le |z|.\end{aligned}$$

Applying these inequalities to (x_1, y_1, z_1),

$$\begin{aligned}(1-\delta_R)^2|y||z|^2 \le\ & |z_2| && \le (1+\delta_R)^2|y||z|^2,\\ (1-\delta_R)|y||z| \le\ & |y_2| && \le (1+\delta_R)|y||z|.\end{aligned}$$

By induction, then, we have the inequalities for *all* $n \ge 1$.

□

Proposition 3.8. *There is a positive number R such that if $w \in W_R$ and $f^n(w) \in W_R$ for some $n \ge 1$, then*

$$||f^j(w)|| \le k_R^j||w||,\ (j = 1, \cdots, n),$$

where $k_R = R + |b| + |c| + |d|$.

proof : Take $R(>1)$ in proposition 3.5. Suppose $w \in W_R$ and $f(w) \in W_R$.

- if $w \in W_R^1$, then $f(w) \in W_R^3 \cup D_R$ and $||f(w)|| \le \max\{R, R^2 + |b|R + |c|R + |d|||w||\}$. Since $R > 1$ and $||w|| \ge R$, we have $||f(w)|| \le (R + |b| + |c| + |d|)||w||$.

- if $w \in W_R^2$, then $f(w) \in W_R^1 \cup W_R^4$ and $||f(w)|| \le \max\{||w||, ||w||R + |b|||w|| + |c|R + |d|R\}$. Since $R > 1$ and $||w|| \ge R$, we have $||f(w)|| \le (R + |b| + |c| + |d|)||w||$.

- if $w \in W_R^3$, then $f(w) \in W_R^2$ and $||f(w)|| = ||w||$. Since $R > 1$, we have $||f(w)|| \le (R + |b| + |c| + |d|)||w||$.

• if $w \in W_R^4$, then $f(w) \in W_R^2$ and $||f(w)|| \leq ||w||$. Since $R > 1$, we have $||f(w)|| \leq (R + |b| + |c| + |d|)||w||$.

• if $w \in D_R$, then $f(w) \in W_R^3 \cup D_R$ and $||f(w)|| \leq \max\{||w||, \ ||w||^2 + |b|||w|| + |c|||w|| + |d|||w||\}$. Since $R > 1$ and $||w|| \leq R$, we have $||f(w)|| \leq (R + |b| + |c| + |d|)||w||$.

Hence, in any case, we have $||f(w)|| \leq k_R||w||$. If $w \in W_R$ and $f^n(w) \in W_R$ for some $n \geq 1$, then $f^j(w)$, $(j = 1, \cdots, n-1)$ are also contained in W_R by proposition 3.5. Thus, the proposition follows.

□

Theorem 3.9. *Let ϕ be the golden ratio $\frac{1+\sqrt{5}}{2}$.*

(i) *$\{G_n^+\}_{n=1}^{\infty}$ converges uniformly on any compact set in $\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$, and G^+ is positive pluriharmonic on $\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$.*

(ii) *$G^+(x, y, z) = \frac{1}{\phi^2}\log|y| + \frac{1}{\phi}\log|z| + O(1)$ on V_R^+.*

(iii) *$\{G_n^+\}_{n=1}^{\infty}$ converges to zero on $\mathbb{C}^3 \setminus \bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$.*

(iv) *$G^+(f(w)) = \phi G^+(w)$.*

(v) *G^+ is continuous plurisubharmonic on $\mathbb{C}^3$.*

proof : Let R be a large number(> 1).
(i),(ii) By proposition 3.5 (i), for any compact set K in $\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$, there is a natural number m such that $f^m(K) \subset V_R^+$. If $\{G_l^+(w)\}_{l=1}^{\infty}$ converges uniformly on $f^m(K)$, then $G_{m+l}^+(w)$ converges uniformly on K since $G_{m+l}^+(w) = \frac{a_{l+2}}{a_{l+m+2}} G_l^+(f^m(w))$ and $\frac{a_{l+2}}{a_{l+m+2}}$ is convergent. Hence, we will prove uniform convergence on compact sets in V_R^+.

From the proof of proposition 3.5 (i), we know $|z_1| > |y_1|, \ |x_1|, \ R$ for $w = (x, y, z) \in V_R^+$. Hence, $\log^+ ||f(w)|| = \log|z_1|$. From the definition,
$G_n^+(x, y, z) = \frac{a_n}{a_{n+2}}\log|y| + \frac{a_{n+1}}{a_{n+2}}\log|z| + \frac{1}{a_{n+2}}\log\frac{|z_n|}{|y|^{a_n}|z|^{a_{n+1}}}$.
By proposition 3.7,
$\frac{a_{n+2}-1}{a_{n+2}}\log(1-\delta_R) \leq \frac{1}{a_{n+2}}\log\frac{|z_n|}{|y|^{a_n}|z|^{a_{n+1}}} \leq \frac{a_{n+2}-1}{a_{n+2}}\log(1+\delta_R)$. Hence, we know $\{G_n^+\}_{n=1}^{\infty}$ is locally uniformly bounded in V_R^+ since $\{\frac{a_n}{a_{n+2}}\}_{n=1}^{\infty}$ and $\{\frac{a_{n+1}}{a_{n+2}}\}_{n=1}^{\infty}$ is convergent.

Let K be any compact set in V_R^+. For any $\epsilon > 0$, there is a positive number R_ϵ s.t. $\left|\frac{a_{n+2}-1}{a_{n+2}}\log(1 \pm \delta_{R_\epsilon})\right| < \epsilon$ for all $n \geq 1$. Take a natural number m s.t. $f^m(K) \subset V_{R_\epsilon}^+$. For $w_m \in f^m(K)$ and $k, l \geq 1$,

$$\left|G_k^+(w_m) - G_l^+(w_m)\right| \leq \left|\frac{a_k}{a_{k+2}} - \frac{a_l}{a_{l+2}}\right| \log|y_m| + \left|\frac{a_{k+1}}{a_{k+2}} - \frac{a_{l+1}}{a_{l+2}}\right| \log|z_m| + 2\epsilon.$$

By the convergence of $\{\frac{a_n}{a_{n+2}}\}_{n=1}^{\infty}$ and $\{\frac{a_{n+1}}{a_{n+2}}\}_{n=1}^{\infty}$, there is a natural number N_0 s.t. for $k, l \geq N_0$,

$$\left|G_k^+(w_m) - G_l^+(w_m)\right| \leq \epsilon + \epsilon + 2\epsilon = 4\epsilon \quad \text{on } f^m(K).$$

For $w \in K$, $k, l \geq 1$,

$$\begin{aligned}
\left|G_{m+k}^+(w) - G_{m+l}^+(w)\right| &= \left|\frac{a_{k+2}}{a_{m+k+2}} G_k^+(w_m) - \frac{a_{l+2}}{a_{m+l+2}} G_l^+(w_m)\right| \\
&\leq \left|\frac{a_{k+2}}{a_{m+k+2}} G_k^+(w_m) - \frac{a_{k+2}}{a_{m+k+2}} G_l^+(w_m)\right| + \left|\frac{a_{k+2}}{a_{m+k+2}} G_l^+(w_m) - \frac{a_{l+2}}{a_{m+l+2}} G_l^+(w_m)\right| \\
&\leq \frac{a_{k+2}}{a_{m+k+2}} \left|G_k^+(w_m) - G_l^+(w_m)\right| + \left|\frac{a_{k+2}}{a_{m+k+2}} - \frac{a_{l+2}}{a_{m+l+2}}\right| \left|G_l^+(w_m)\right|.
\end{aligned}$$

By the convergence of $\{\frac{a_{n+2}}{a_{m+n+2}}\}_{n=1}^{\infty}$ and the locally uniform boundedness of $\{G_n^+\}_{n=1}^{\infty}$, there is a natural number $N_1 (> N_0)$ s.t. for $k, l \geq N_1$,

$$\left|G_{m+k}^+(w) - G_{m+l}^+(w)\right| \leq 1 \cdot 4\epsilon + \epsilon = 5\epsilon.$$

Thus, it turns out that G^+ is positive pluriharmonic on $\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$, and $G^+(x, y, z) = \frac{1}{\phi^2} \log|y| + \frac{1}{\phi} \log|z| + O(1)$ in V_R^+.

(iii) We have only to prove the claim for $w \in W_R \cap \left(\mathbb{C}^3 \setminus \bigcup_{n=0}^{\infty} f^{-n}(V_R^+)\right)$. By proposition 3.8,

$$||f^n(w)|| \leq k_R^n ||w||$$

for all $n \geq 1$. By definition,

$$G_n^+(w) \leq \frac{1}{a_{n+2}} \log^+ k_R^n ||w||.$$

The right hand side tends to zero as $n \to \infty$.

(iv) $G_n^+(f(w)) = \frac{1}{a_{n+2}} \log^+ ||f^{n+1}(w)|| = \frac{a_{n+3}}{a_{n+2}} \frac{1}{a_{n+3}} \log^+ ||f^{n+1}(w)||$. By $\lim_{n\to\infty} \frac{a_{n+3}}{a_{n+2}} = \phi$, the equation holds.

(v) We will prove the continuity of G^+ at boundary points of $\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)$. We have only to prove the continuity at the boundary points contained in $\mathrm{int} W_R$. Let p be any point in $\partial \left(\bigcup_{n=0}^{\infty} f^{-n}(V_R^+)\right) \cap \mathrm{int} W_R$. Take a neighborhood O of p contained in W_R. $M := \sup_{w \in O} ||w||$. We will estimate G^+ on $B_n = O \cap \left(f^{-n}(\overline{V_R^+}) \setminus f^{-(n-1)}(\overline{V^+}_R)\right)$. Since $f^{n-1}(B_n)$ is contained in W_R, by proposition 3.8, $f^{n-1}(B_n)$ is contained in $\{||w|| < k_R^{n-1} M\}$. Hence, $f^n(B_n)$ is contained in $C_n = \{||w|| < \max\{k_R^{n-1} M,\ (k_R^{n-1} M)^2 + (|b| + |c| + |d|) k_R^{n-1} M\}\} \cap \overline{V_R^+}$. $L_n := \sup_{w \in C_n} G^+(w)$. By (iv), $G^+(w) \leq \frac{1}{\phi^n} L_n$ on B_n. By the estimate (ii), $\frac{1}{\phi^n} L_n$ tends to zero as $n \to \infty$. Since $G^+ = 0$ on $O \setminus (\bigcup_{n=1}^{\infty} B_n)$, the continuity of G^+ at p follows.

□

Remark 3.10. Any point w such that $G^+(w) > 0$ has an orbit which converges to $[0:0:1:0]$ in forward time. ($[0:0:1:0]$ is a point of indeterminacy.)

Since $f^{-1}(x,y,z) = (\frac{1}{d}(-xy - bx - cy + z), x, y)$, by changing the scale, the coefficient of xy becomes 1. Considering to exchange the roles of x and z, we will obtain Green function of f^{-1}. Denote the function by $G^-(w)$. We'll define the following subsets in $\mathbb{C}^3$.

Definition 3.11.

$$\begin{aligned}
U^+ &= \{(x,y,z) \in \mathbb{C}^3 \mid G^+(x,y,z) > 0\}, \\
E^+ &= \mathbb{C}^3 \setminus U^+ = \{(x,y,z) \in \mathbb{C}^3 \mid G^+(x,y,z) = 0\}. \\
U^- &= \{(x,y,z) \in \mathbb{C}^3 \mid G^-(x,y,z) > 0\}, \\
E^- &= \mathbb{C}^3 \setminus U^- = \{(x,y,z) \in \mathbb{C}^3 \mid G^-(x,y,z) = 0\}. \\
E &= E^+ \cap E^-.
\end{aligned}$$

A point in E^+ does not necessarily have a bounded orbit.

Example 3.12. *Let $b = c = 0$. Each of the x, y, z axies is f^3-invariant.*
(1) *If $|d| > 1$, then any point in the axies tends to infinity in forward time although Green function vanishes there. The limit points of the orbit are $[1:0:0:0]$, $[0:1:0:0]$ and $[0:0:1:0]$. (These points are of indeterminacy.) Hence we know that $[0:0:1:0]$ is not only the limit point of orbits with higher rate of escape but also that of orbits with lower rate of escape. (See remark 3.10.)*
(2) *If $|d| = 1$, then any point in the axies has a bounded orbit in foward and backward time.*

Since $G^\pm$ are plurisubharmonic, the current $\mu^\pm = dd^cG^\pm$ are positive, d-closed (1,1)-currents. From theorem 3.9 (iv), $f^*\mu^+ = \phi\mu^+$ and $f^*\mu^- = \frac{1}{\phi}\mu^-$ hold.

Proposition 3.13. supp $\mu^+ = \partial E^+$

proof : G^+ is pluriharmonic on $U^+ \cup (\mathrm{int}E^+)$. Hence, supp $\mu^+ \subset \partial E^+$. Suppose there is a neighborhood O of a point $p \in \partial E^+$ s.t. G^+ puts no mass on O. Then G^+ is pluriharmonic on O. But $G^+|_O$ has a minimal at p though it is not constant. This is a contradiction. □

Since G^+ and G^- are continuous, $\mu = \mu^+ \wedge \mu^- = dd^cG^+ \wedge dd^cG^-$ is a positive (2,2)-current. (See [K], pp.113.)

Proposition 3.14.
(i) *The current μ is f-invariant.*
(ii) $\mu^+ \wedge \mu^+ = 0$

proof :
(i) $f^*(\mu^+ \wedge \mu^-) = f^*(\mu^+) \wedge f^*(\mu^-) = \phi\mu^+ \wedge \frac{1}{\phi}\mu^- = \mu^+ \wedge \mu^-$.
(ii) Since G^+ is pluriharmonic on U^+, we have $\{dd^c \max(G^+, \varepsilon)\}^2 = 0$ for all $\varepsilon > 0$. Then let $\varepsilon \to 0$. □

The following propositions are on the properties of $U^\pm$, $E^\pm$ and E. (The precision and the proofs of them will appear in a forthcoming paper [M2].)

Proposition 3.15. *Let $h(t)$ be a non-constant holomorphic map from $\mathbb{C}$ to $\mathbb{C}^3$. Then, $h(\mathbb{C})$ is contained in a level set $\{G^+(w) = a\}, (a > 0)$ or meets E^+ at infinitely many points.*

Proposition 3.16.
(i) int$E^{\pm}$ *are polynomially convex if they are not empty.*
(ii) $U^{\pm}$ *are pseudoconvex domains.*

Proposition 3.17. *For any complex numbers α, β s.t. $|\beta|$ is greater than R, we have $\{x = \alpha,\ y = \beta\} \cap U^+$ is connected. The slice of U^+ by a hyperplane $\{y = \beta\}$ is also connected.*

Proposition 3.18. *For any complex number β s.t. $|\beta|$ is greater than R, the slice of E by a hyperplane $\{y = \beta\}$ is not empty. In particular, E is an unbounded closed set in $\mathbb{C}^3$.*

Theorem 3.19. $\pi_1(U^{\pm}) = \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 3.20. *Let $|d| < 1$. Then, there is a number $r > 0$ such that for any $(b, c) \in \{|b|, |c| < r\}$, the set of points which have bounded orbits in forward time is precisely equal to E^+ and the non-wandering set is compact.*

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