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LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential $\mathbf{a}=(a_j(x))_{j=1,\ldots,d}$ is expressed by the following form

$$H(\mathbf{a})=\sum_{j=1}^{d}\gamma_j(P_j-a_j)+m\gamma_{d+1}+V, \tag{1.1}$$

where $P_j=\frac{1}{i}\partial_{x_j}$, V is a multiplication of an Hermitian matrix $V(x)$. m is the mass of electron. The matrices $\{\gamma_j\}$ satisfy the following relations

$$\gamma_j\gamma_k+\gamma_k\gamma_j=2\delta_{jk}\mathbf{1} \quad (j,k=1,\ldots,d+1). \tag{1.2}$$

Here δ_{jk} is Kronecker's delta and $\mathbf{1}$ is an identity matrix. We assume that the speed of the light $c=1$. When $V\equiv 0$, the square of $H(\mathbf{a})$ has the form

$$H(\mathbf{a})^2=\sum_{j=1}^{d}(P_j-a_j)^2+m^2+\frac{1}{i}\sum_{1\leq j<k\leq d}b_{jk}(x)\gamma_j\gamma_k, \tag{1.3}$$

where

$$b_{jk}(x)=\partial_{x_k}a_j(x)-\partial_{x_j}a_k(x). \tag{1.4}$$

It is called Pauli's Hamiltonian. The skew symmetric matrix $(b_{jk}(x))$ is the magnetic field associated with $\mathbf{a}$. We say the magnetic field is asymptotically constant if it satisfies the following conditions as $|x|\to\infty$:

$$b_{jk}(x)\to {}^{\exists}\Lambda_{jk} \quad (1\leq j,k\leq d), \tag{1.5}$$

where $(\Lambda_{jk})_{j,k}$ is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for $H(\mathbf{a})$ with a constant magnetic field $(b_{jk}(x))$ and a long-range electric potential $V(x)$ when $d=3$. Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for $d=2,3$. As can be infered from (1.3), the spectrum of $H(\mathbf{a})$ is closely related with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2} a_1(x) - \partial_{x_1} a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1(P_1 + \frac{\lambda}{2}x_2) + \sigma_2(P_2 - \frac{\lambda}{2}x_1) + m\sigma_3, \tag{1.6}$$

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + m^2 - \lambda\sigma_3. \tag{1.7}$$

The right hand side is a de-coupled 2 dimensional magnetic Schödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{\pm\sqrt{2\lambda n + m^2} \quad | n = 0, 1, 2 \ldots\}.$$

In fact we have

$$\sigma(h(\lambda)) = \{\sqrt{2\lambda n + m^2}, \quad -\sqrt{2\lambda(n+1) + m^2} | n = 0, 1, 2 \ldots\}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$\mathbf{a}_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along x_3-axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1(P_1 + \frac{\lambda x_2}{2}) + \alpha_2(P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m\beta, \tag{1.8}$$

where $\{\alpha_j\}$ and β are 4×4 Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.9}$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \tag{1.10}$$

with $D_0 = \sigma \cdot (P - \mathbf{a}_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

By using Foldy-Wouthuysen transform, explained in detail in the following section, $H_0(\lambda)$ can be diagonalized by a unitary operator U_{FW}.

$$U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2+m^2} & 0 \\ 0 & -\sqrt{D_0^2+m^2} \end{pmatrix}. \tag{1.11}$$

From the commutation relation (1.2) we have

$$D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda\beta. \tag{1.12}$$

We can easily see that $\sigma(D_0^2) = [0, \infty)$. So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of $H_0(\lambda)$ is absolutely continuous.

Let us consider the perturbation of $H_0(\lambda)$: We put

$$H(\lambda) = H_0(\lambda) + V. \tag{1.13}$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent $(z - H(\lambda))^{-1}$ on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

$$\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x). \tag{1.14}$$

The existence of the boundary values

$$\langle x_3\rangle^{-s}(\tilde{H} - \mu \mp i0)^{-1}\langle x_3\rangle^{-s}$$

is proved for $s > 1/2$ and $\mu \in \mathbb{R}\backslash(\{\lambda(2n+1)|n = 0, 1, 2, \dots\} \cup \sigma_{pp}(\tilde{H}))$.

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as $|x| \to \infty$. (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential $V(x)$. Roughly speaking, his assumption means that the absolute value of each components of V is dominated from above by $C\langle x'\rangle^{-1-\epsilon}\langle x\rangle^{-\epsilon}$ ($x' = (x_2, x_3)$) for sufficiently large x. We remark that $\epsilon > 0$ is used as a sufficiently small parameter throughout this paper. To be accurate, $\langle x'\rangle^{1+\epsilon}V(x)$ is required to be a $H_0(\lambda)$-compact operator.

In this paper we treat directly the following operator

$$H(\lambda) = \alpha_1(P_1 + \frac{\lambda}{2}x_2) + \alpha_2(P_2 - \frac{\lambda}{2}x_1) + \alpha_3 P_3 + m\beta + V(x), \tag{1.15}$$

where $V(x)$ is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for $V(x)$. In this case it seems that an appropriate choice of the conjugate operator is

$$\frac{P_3}{\langle P_3\rangle}\cdot x_3 + x_3\cdot\frac{P_3}{\langle P_3\rangle},$$

which is inspired by [9], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [9]. Namely we rewrite $H_0(\lambda)$ by a direct integral and the conjugate operator A acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. Conjugate operator

Let us recall

$$Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - \mathbf{a}_0), \tag{2.1}$$

with

$$\mathbf{a}_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0). \tag{2.2}$$

The Dirac Hamiltonian $Q_0 + m\beta$ can be diagonalized by sandwiching it between a unitary operator U and $U^* = U^{-1}$. In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator $H_0(\lambda)$. Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians $H_0(\lambda)$ and $H(\lambda)$.

Let Q_0 be the self-adjoint operator as in (2.1) and $|Q_0| = \sqrt{Q_0^2}$, $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$. We define a unitary operator U_{FW}, which diagonalize $H_0(\lambda)$, in the following way.

Definition 2.1. (i). *At first we define a signiture function associated with Q_0 by*

$$sgnQ_0 = \begin{cases} \frac{Q_0}{|Q_0|}, & on\ (kerQ_0)^{\perp} \\ 0, & on\ (kerQ_0) \end{cases} \tag{2.3}$$

We note that $sgnQ_0$ is isometory on $(kerQ_0)^{\perp}$.

(ii). *We can easily see that $m/|H_0(\lambda)| \leq 1$. So we denote the square root of $\frac{1}{2}(1 \pm \frac{m}{|H_0(\lambda)|})$ as $a_{\pm}$. i.e.*

$$a_{\pm} = \frac{1}{\sqrt{2}}\sqrt{1 \pm m/|H_0(\lambda)|}. \tag{2.4}$$

(iii). *Combining these operators we define the operator U_{FW} as*

$$U_{FW} = a_+ + \beta(sgnQ_0)a_-. \tag{2.5}$$

Lemma 2.2. (i). *U_{FW} is a unitary operator on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. Further,*

$$U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(sgnQ_0)a_-. \tag{2.6}$$

(ii). *$H_0(\lambda)$ can be diagonalized by U_{FW} as follows.*

$$U_{FW}H_0(\lambda)U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2+m^2} & 0 \\ 0 & -\sqrt{D_0^2+m^2} \end{pmatrix}. \quad (2.7)$$

Proof. See 5.6.1 in [8]. □

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW}H_0(\lambda)U_{FW}^{-1}.$$

We rewrite (1.12) as follows.

$$D_0^2 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}.$$

Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_\pm = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + P_3^2 \pm \lambda.$$

It is well-known that $(P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2$ has eigenvalues

$$\{\lambda(2n+1)|n = 0,1,2,\dots\}.$$

We denote the eigenprojection on each eigenspace as Π_n. With these projections, $\sqrt{D_0^2+m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2+m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix}, \quad (2.8)$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) =$$

$$\sum_{n=0}^{\infty} \begin{pmatrix} f(d_n)\otimes\Pi_n & & & \\ & f(d_{n+1})\otimes\Pi_n & & \\ & & f(-d_n)\otimes\Pi_n & \\ & & & f(-d_{n+1})\otimes\Pi_n \end{pmatrix},$$

for any Borel function f.

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2}\{\frac{P_3}{\langle P_3\rangle}\cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3\rangle}\}. \quad (2.9)$$

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson's commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A}\beta$ between U_{FW}^{-1} and U_{FW} :

$$A = U_{FW}^{-1}(\hat{A}\beta)U_{FW}. \quad (2.10)$$

Before we show Mourre's inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.

Suppose that $f \in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0 \in \mathbb{R}$.

$$|f^{(k)}(t)| \leq C_k(1+|t|)^{m_0-k}, \quad {}^\forall k \in \mathbb{N} \cup \{0\}. \tag{2.11}$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R},$$
$$supp\tilde{f} \subset \{z; |Imz| \leq 1 + |Rez|\},$$

$$|\partial_{\bar{z}}\tilde{f}(z)| \leq C_N |Imz|^N \langle z \rangle^{m_0-1-N}, \quad {}^\forall N \in \mathbb{N}. \tag{2.12}$$

Then for all f, satisfying (2.11) for $m_0 < 0$ and a self-adjoint operator H, we have

$$f(H) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z-H)^{-1} dz \wedge d\bar{z}. \tag{2.13}$$

3. Limiting absorption principle for long-range potentials

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing A defined in the previous section as the conjugate operator.

Lemma 3.1. *Let $\mathbb{R}_{\mathbb{N}}$ be the following discrete subset of $\mathbb{R}$*

$$\mathbb{R}_{\mathbb{N}} = \{\pm\sqrt{2\lambda n + m^2} \quad |n = 0, 1, 2, \dots\} \subset \mathbb{R}.$$

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C_0^\infty(I)$

$$f(H_0(\lambda))i[H_0(\lambda), A]f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \tag{3.1}$$

Proof. By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \tag{3.2}$$

We rewrite the commutator as follow.

$$i[\hat{H}_0(\lambda), \hat{A}\beta] = \begin{pmatrix} i[\sqrt{D_0^2+m^2}, \hat{A}] & \\ & i[\sqrt{D_0^2+m^2}, \hat{A}] \end{pmatrix}. \tag{3.3}$$

We proceed the calculus more precisely to see that

$$i[\sqrt{D_0^2+m^2}, \hat{A}] = \sum_{n=0}^{\infty} \begin{pmatrix} i[d_n, \hat{A}] \otimes \Pi_n & \\ & i[d_{n+1}, \hat{A}] \otimes \Pi_n \end{pmatrix} \tag{3.4}$$

by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4 \end{pmatrix} \tag{3.5}$$

where

$$I_1 = \sum_{n=0}^{\infty} f(d_n)i[d_n, \hat{A}]f(d_n) \otimes \Pi_n,$$

$$I_2 = \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_n,$$

$$I_3 = \sum_{n=0}^{\infty} f(-d_n)i[d_n, \hat{A}]f(-d_n) \otimes \Pi_n,$$

$$I_4 = \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_n.$$

We note that all the sum in $I_1, \cdots, I_4$ are finite since f is a compactly supported function. By an elementary caluculus, we have

$$i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2}\langle P_3\rangle} \quad (l \in \mathbb{N} \cup \{0\}). \tag{3.6}$$

Since $suppf \subset I \subset \mathbb{R}\backslash\mathbb{R}_{\mathbb{N}}$, P_3 is away from zero when $P_3 \in suppf(d_l(P_3))$ or $P_3 \in suppf(-d_l(P_3))$. So there exist $C_l > 0$ such that

$$f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l \geq C_l f(d_l)^2 \otimes \Pi_l,$$

$$f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l \geq C_l f(-d_l)^2 \otimes \Pi_l.$$

Since only a finite number of $l = l_j \quad (j = 1, \ldots, N)$ is concerned, we have (3.2) with $\alpha = \inf_{j=1,\ldots,N} C_{l_j}$. □

Now we give the assumption for the potential, which is necessary to Mourre's inequality associated to $H(\lambda)$. After that we give an example of V satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means $V(x) = O(\langle x\rangle^{-\epsilon}\langle x_3\rangle^{-1-\epsilon})$ as $|x| \to \infty$. And long-range part is a multiplication of a real valued function $\varphi(x)$ such that $\varphi(x) = O(\langle x\rangle^{-\epsilon})$ as $|x| \to \infty$. More precisely we assume that V satisfies the following.

Assumption 3.2. *$V = V(x)$ is a multiplicative operator of a 4×4 Hermitian matrix satisfying the following properties.*

(i). *V is a $H_0(\lambda)$-compact operator.*
(ii). *The form $[V, A]$ can be extended to a $H_0(\lambda)$-compact operator.*

For example a 4×4 matrix $V(x)$ satisfying the following inequality is $H_0(\lambda)$-compact.

$$|V(x)| \leq C\langle x\rangle^{-\epsilon} \quad (x \in \mathbb{R}^3). \tag{3.7}$$

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for $H(\lambda)$.

Lemma 3.3. *Suppose V satisfies Assumption 3.2.*

(i). *We take $\mu \in \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ and $\delta > 0$ so that the closed interval $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$. There exist $\alpha > 0$ and a compact operator K such that the following inequality holds for all $f \in C_0^\infty(I)$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K. \tag{3.8}$$

(ii). *There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$. For $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu - \delta_0, \mu + \delta_0])$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2. \tag{3.9}$$

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

Theorem 3.4. *Suppose V satisfies Assumption 3.2. Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, the following limits*

$$R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3\rangle^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}\langle x_3\rangle^{-s} \tag{3.10}$$

exist and $R^{\pm}(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$.

Sketch of proof

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value $\langle A\rangle^{-s}(H(\lambda) - \mu \mp i0)^{-1}\langle A\rangle^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient if we show the boundness of $\langle A\rangle^{s}\langle x_3\rangle^{-s}$. Since $\langle \hat{A}\rangle^{s}\langle x_3\rangle^{-s}$ is bounded, it is sufficient to show $\langle x_3\rangle^{s} U_{FW}\langle x_3\rangle^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let $\chi(t) \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1/\sqrt{2} & (t > -m^2/3) \\ 0 & (t < -2m^2/3). \end{cases} \tag{3.11}$$

With this function we define $F_\pm(t)$ and $F_{\chi,\pm}$ as follows.

$$F_+(t) = \chi(t)\sqrt{1 + \frac{m}{\sqrt{t + m^2}}}$$

$$F_-(t) = \chi(t)\left(\sqrt{1 + \frac{m}{\sqrt{t + m^2}}}\right)^{-1} \frac{1}{\sqrt{t + m^2}}$$

$$F_{\chi,+}(t) = F_+(t) - \chi(t)$$

$$F_{\chi,-}(t) = \sqrt{t + m^2}F_-(t) - \chi(t)$$

Then we can easily verify that

$$a_+ = F_+(Q_0^2),$$
$$a_- sgn Q_0 = F_-(Q_0^2)Q_0 = Q_0 F_-(Q_0^2).$$

As for the proof of $[Q_0, F_-(Q_0^2)] \equiv 0$, see 5.2.4 in [8]. By the construction of these functions, we can also see that $F_{\chi,\pm}(t)$ satisfy (2.11) with $m_0 < 0$. So we apply the functional calculus in section 2 to $F_{\chi,\pm}(t)$ and see the following properties hold.

Lemma 3.5. *Suppose $0 \leq s \leq 2$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

(i). *For $0 < s \leq 1$, there exists $C_s > 0$ such that*

$$\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C_s(|Imz|^{-1} + |Imz|^{-2}\langle z \rangle). \tag{3.12}$$

(ii). *For $1 < s \leq 2$, there exists $C'_s > 0$ such that*

$$\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C'_s(|Imz|^{-1} + |Imz|^{-2}\langle z \rangle + |Imz|^{-3}\langle z \rangle^2). \tag{3.13}$$

(iii). *$\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}$ and $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$ are bounded operators.*

Proof. For the proof of (i) and (ii), we use the resolvent equation. Suppose $0 < s \leq 1$. Then

$$\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1}(Q_0^2 + 1) \tag{3.14}$$
$$\times (Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0^2](z - Q_0^2)^{-1}\langle x \rangle^{-s}. \tag{3.15}$$

From the boundness of $(Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0^2]$ and the following estimate

$$\|(z - Q_0^2)^{-1}(Q_0^2 + 1)\|_{\mathbb{H}} \leq C(|Imz|^{-1}\langle z \rangle + 1), \tag{3.16}$$

we obtain (i). As for the case $1 < s \leq 2$, we rewrite the last term $(z - Q_0^2)^{-1}[\langle x \rangle^s, Q_0^2](z - Q_0^2)^{-1}\langle x \rangle^{-s}$ as

$$(z - Q_0^2)^{-1}(Q_0^2 + 1)(Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0^2]\langle x \rangle^{-s+1} \tag{3.17}$$
$$\times \langle x \rangle^{s-1}(z - Q_0^2)^{-1}\langle x \rangle^{-s+1}\langle x \rangle^{-1}. \tag{3.18}$$

By using the result for $0 < s \leq 1$, we have the inequality for $1 < s \leq 2$.

With these estimates, we prove (iii). Since $\chi(Q_0^2) \equiv 1$, we can easily see that

$$\langle x \rangle^s F_+(Q_0^2)\langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2)\langle x \rangle^{-s} + I. \tag{3.19}$$

Since $F_{\chi,+}(t)$ satisfies (2.12) for $m_0 = -1/2$, $F_{\chi,+}(Q_0^2)$ can be rewritten as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z)\langle x \rangle^s (z - Q_0^2)^{-1}\langle x \rangle^{-s} dz \wedge d\bar{z}. \tag{3.20}$$

From this formula and (i) (ii) we have

$$\|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)\langle x \rangle^s (z - Q_0^2)^{-1}\langle x \rangle^{-s}\|_{\mathbb{H}}$$
$$\leq C|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)|(|Imz|^{-1} + |Imz|^{-2}\langle z \rangle + |Imz|^{-3}\langle z \rangle^2).$$

From (2.12) we have

$$\|\partial_{\bar z}\tilde F_{\chi,+}(z)\langle x\rangle^s(z-Q_0^2)^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}} \le C\langle z\rangle^{-5/2}.$$

This implies the boundness of $\langle x\rangle^s F_+(Q_0^2)\langle x\rangle^{-s}$.

In a similar way, we rewrite $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$ as

$$\langle x\rangle^s F_{\chi,-}(Q_0^2)\langle x\rangle^{-s}\langle x\rangle^s \frac{Q_0}{\sqrt{Q_0^2+m^2}}\langle x\rangle^{-s} + \langle x\rangle^s \frac{Q_0}{\sqrt{Q_0^2+m^2}}\langle x\rangle^{-s}. \tag{3.21}$$

It is sufficient to show the boundness of $\langle x\rangle^s Q_0/\sqrt{Q_0^2+m^2}\langle x\rangle^{-s}$. To see this, we denote $\chi(t)/\sqrt{t+m^2} \in C^\infty(\mathbb{R}^3)$ as $S(t)$ and its almost analytic extension as $\tilde S(z)$. We can easily see that $S(Q_0^2)\langle x\rangle^s Q_0\langle x\rangle^{-s}$ is bounded. So we obtain the boundness of $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$ if we show that $[\langle x\rangle^s, S(Q_0^2)]Q_0\langle x\rangle^{-s}$ is bounded. We rewrite it as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde S(z)\frac{Q_0^2+1}{z-Q_0^2}(Q_0^2+1)^{-1}[\langle x\rangle^s, Q_0^2]Q_0\langle x\rangle^{-s}\langle x\rangle^s(z-Q_0^2)^{-1}\langle x\rangle^{-s}dz\wedge d\bar z.$$

By an elementary calculus, we have $(Q_0^2+1)^{-1}[\langle x\rangle^s, Q_0^2]Q_0\langle x\rangle^{-s}$ bounded. Combining (i) and (ii), we have

$$\begin{aligned}\|[\langle x\rangle^s, S(Q_0^2)]Q_0\langle x\rangle^{-s}\|_{\mathbb{H}} &\le C\int_{\mathbb{C}}|\partial_{\bar z}\tilde S(z)|\{1+|Imz|^{-1}\}\\ &\times\{|Imz|^{-1}+|Imz|^{-2}\langle z\rangle+|Imz|^{-3}\langle z\rangle^2\}dz\wedge d\bar z<\infty.\end{aligned}$$

This implies the boundness of $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$. □

Next we give an example of V. It requires smoothness, but allows long-range part in its diagonal components.

Lemma 3.6. *Let V be a 4×4 Hermitian matrix of the form*

$$V(x) = (v_{ij}(x)) + \varphi(x)I_4 \equiv V_s(x) + V_l(x) \tag{3.22}$$

where $V_s(x) = (v_{ij}(x))$ is an Hermitian matrix and I_4 is an identity matrix. Suppose the following conditions hold. Then $V(x)$ satisfies Assumption 3.2.

There exist $\delta > 0$ such that the following inequalities hold for all multi-index α.

$$|\partial_x^\alpha v_{ij}(x)| \le C_\alpha\langle x\rangle^{-\delta-|\alpha|}\langle x_3\rangle^{-1} \quad (1\le i,j\le 4). \tag{3.23}$$

$\varphi(x)\in C^\infty(\mathbb{R}^3)$ is real valued and satisfies

$$|\partial_x^\alpha\varphi(x)| \le C'_\alpha\langle x\rangle^{-\delta-|\alpha|}. \tag{3.24}$$

The relatively compactness of $V(x)$ itself is clear since V satisfies (3.7). So we only have to show the relatively compactness of $[V, A]$. We prove the relatively compactness of $[V_s, A] = [V_s, U_{FW}^{-1}\hat A\beta U_{FW}]$ at first. From the boundness of $\langle x_3\rangle^{-1}\hat A\beta$ and the relatively compactness of $V_s\langle x_3\rangle$, it is sufficient to show that $\langle x_3\rangle U_{FW}\langle x_3\rangle^{-1}$ and $\langle x_3\rangle U_{FW}^{-1}\langle x_3\rangle^{-1}$ are bounded operators in $\mathbb{H}$. We have already proved it in Lemma 3.5.

Next we treat the long-range term. The conjugate operator A can be decomosed into the sum of $J_1, \cdots, J_4$ where

$$J_1 = F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2),$$
$$J_2 = F_+(Q_0^2)\hat{A}\beta^2 F_-(Q_0^2)Q_0,$$
$$J_3 = \beta F_-(Q_0^2)Q_0\hat{A}\beta F_+(Q_0^2),$$
$$J_4 = \beta F_-(Q_0^2)Q_0\hat{A}\beta^2 F_-(Q_0^2)Q_0.$$

We prove that the $H_0(\lambda)$ - compactness holds for each of $[V_l, J_1], \cdots, [V_l, J_4]$. To see this we use the functional calculus again and rewrite J_1 as follows.

$$\begin{aligned} F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2) &= \hat{A}\beta F_+(Q_0^2)^2 + [F_{\chi,+}(Q_0^2), \hat{A}\beta]F_+(Q_0^2) \\ &\equiv J_1' + J_1'' \end{aligned}$$

At first we prove the boundness of J_1'' and conseqently the relatively compactness of $[V_l, J_1'']$. By using (2.13), we rewrite $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{F}_{\chi,+}(z)(z - Q_0^2)^{-1}[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1} dz \wedge d\bar{z}. \tag{3.25}$$

From (3.16) we have $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |Imz|\}$. So we have

$$\|[F_{\chi,+}(Q_0^2), \hat{A}\beta]\| \leq C\int_{\mathbb{C}} |\partial_{\bar{z}}\tilde{F}_{\chi,+}(z)|\{|Imz|^{-1} + |Imz|^{-2}\langle z\rangle\} dz \wedge d\bar{z}. \tag{3.26}$$

Since the almost analytic extension $\tilde{F}_{\chi,+}(z)$ satisfies

$$|\partial_{\bar{z}}\tilde{F}_+(z)| \leq C_N |Imz|^N \langle z\rangle^{-3/2-N} \quad ({}^\forall N \in \mathbb{N}), \tag{3.27}$$

we have $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ is bounded and inductively $[V_l, J_1'']$ is $H_0(\lambda)$-compact. So we only have to show the relatively compactness of $[V_l, J_1']$.

$$[V_l, J_1'] = [V_l, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta[V_l, F_+(Q_0^2)^2]. \tag{3.28}$$

Clearly $[V_l, \hat{A}\beta]F_+(Q_0^2)$ is $H_0(\lambda)$-compact. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $\langle x\rangle^{1+\delta}[V_l, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_l, J_1]$.

As for the commutator $[V_l, J_2], \cdots, [V_l, J_4]$ we also replace $F_\pm$ by $F_{\chi,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_l, J_2]$ and $[V_l, J_3]$ are almost the same. We only give the proof for J_2. We also estimate the 'principle' part before we compute the commutator with V_l.

$$J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0 \tag{3.29}$$

It is sufficient to show that $[V_l, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]$ is a $H_0(\lambda)$-compact operator. We decompose it into the following sum.

$$\begin{aligned}[V_l, \hat{A}]F_+(Q_0^2)F_-(Q_0^2)Q_0 \\ +\hat{A}[V_l, F_+(Q_0^2)F_-(Q_0^2)]Q_0 \\ +\hat{A}F_+(Q_0^2)F_-(Q_0^2)[V_l, Q_0].\end{aligned}$$

We can easily see that the first and the third term is relatively compact since $\langle x\rangle^{1+\delta}[V_l, Q_0]$ is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for J_4, the proof is similar. We rewrite it as

$$\hat{A}F_-(Q_0^2)^2Q_0^2 + [F_-(Q_0^2)Q_0, \hat{A}]F_-(Q_0^2)Q_0 \tag{3.30}$$

We can also obtain the relatively compactness by estimating the term $[V, \hat{A}F_-(Q_0^2)^2Q_0^2]$.

Corollary 3.7. *Let V be a 4×4 Hermitian matrix. Suppose V satisfies the condition in Lemma 3.6. Then the following limits*

$$R^{\pm}(\mu) = \lim_{\epsilon\downarrow 0}\langle x_3\rangle^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}\langle x_3\rangle^{-s} \tag{3.31}$$

exist for $\mu \in \mathbb{R}\setminus(\mathbb{R}_{\mathbb{N}}\cup\sigma_{pp}(H(\lambda)))$ and $R^{\pm}(\mu)$ are continuous with respect to μ.

References

[1] Avron, J., Herbst, I., Simon, B. : Schrödinger operators with magnetic fields. 1. General interactions, *Duke Math. J.*, vol **45** (4) (1978), p.847-883.

[2] Boutet de Monvel-Berthier, A., Manda, D., Purice, R. : Limiting absorption principle for the Dirac operator, *Ann. Inst. Henri Poincaré*, vol **58** (4), (1993), p.413-431.

[3] Hachem, G. : Effet Zeeman pour un électron de Dirac, *Ann. Inst. Henri Poincaré*, vol **58** (1), (1993), p.105-123.

[4] Iwashita, H., : On the long-range scattering for one- and two- particle Schrödinger operators with constant magnetic fields, *Tsukuba J. Math.*, vol **19** (2) (1995) p.369-376.

[5] Jensen, A., Mourre, E., Perry, P. : Multiple commutator estimates and resolvent smoothness in quantum scattering theory, *Ann. Inst. Henri Poincaré*, vol **41** (2), (1984), p.207-225.

[6] Mourre, E. : Absence of singular continuous spectrum for certain self-adjoint operators, *Commun. Math. Phys.*, **78** (1981), 391-408.

[7] Reed, M., Simon, B. : Methods of Modern Mathematical Physics 2, Academic Press, New York - San Francisco - London.

[8] Thaller, B. : The Dirac Equation, Texts Monographs Phys. *Springer-Verlag*, (1992).

[9] Yokoyama, K. : Mourre theory for time-periodic systems, *Nagoya Math. J.*, vol **149**, (1998), p.193-210.

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