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# On the Ground State Energy of the Spin-Boson Model without Infrared Cutoff and the Superradiant Ground State of the Wigner-Weisskopf Model.

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## 1 Introduction and Preliminaries

We give new upper bounds for the ground state energy of the spin-boson (SB) model without infrared cutoff. Using it we argue how an effect by the spin appears in the ground state energy without infrared cutoff. We first investigate spectral properties of the Wigner-Weisskopf (WW) model, and apply them to SB model to achieve our purpose. Then, as extra results of the spectral analysis for WW model, we show two kinds of phase transition: (i) there exists a phase transition in the expectation of the number of (massive) photons at the ground state, which occurs from the reverse between the expectations of the number of photons at the ground and first excited states; and (ii) there exists another phase transition such that a non-perturbative ground state appears, and its ground state energy is so low that we cannot conjecture it by using the regular perturbation theory.

We take a Hilbert space of bosons to be

$$\mathcal{F}_b := \mathcal{F}\left(L^2\left(\mathbf{R}^d\right)\right) \equiv \bigoplus_{n=0}^{\infty}\left[\otimes_s^n L^2\left(\mathbf{R}^d\right)\right] \tag{1.1}$$

($d \in \mathbf{N}$) the symmetric Fock space over $L^2(\mathbf{R}^d)$ ($\otimes_s^n \mathcal{K}$ denotes the $n$-fold symmetric tensor product of a Hilbert space $\mathcal{K}$, $\otimes_s^0 \mathcal{K} \equiv \mathbf{C}$). In this paper, we set both of $\hbar$ (the Planck constant divided by $2\pi$) and $c$ (the speed of light) one, i.e., $\hbar = c = 1$.

Let $\omega : \mathbf{R}^d \to [0, \infty)$ be a Borel measurable function such that $0 \le \omega(k) < \infty$ for all $k \in \mathbf{R}^d$ and $\omega(k) \ne 0$ for almost everywhere (a.e.) $k \in \mathbf{R}^d$ with respect to the

$d$-dimensional Lebesgue measure. We here assume that

$$\inf_{k \in \mathbf{R}^d} \omega(k) = 0 \tag{1.2}$$

because we are interested in the case without infrared cutoff. Let $\hat{\omega}$ be the multiplication operator by the function $\omega$, acting in $L^2(\mathbf{R}^\nu)$. We denote by $d\Gamma(\hat{\omega})$ the second quantization of $\hat{\omega}$ [RS2, §X.7] and set

$$H_b = d\Gamma(\hat{\omega}) = \int_{\mathbf{R}^d} dk\, \omega(k) a(k)^* a(k),$$

where $a(k)$ is the operator-valued distribution kernels of the smeared annihilation operator $a(f)$, so $a(k)^*$ is that of creation operator $a(f)^*$:

$$a(f) = \int_{\mathbf{R}^d} dk\, a(k) \overline{f(k)}, \tag{1.3}$$

$$a(f)^* = \int_{\mathbf{R}^d} dk\, a(k)^* f(k) \tag{1.4}$$

for every $f \in L^2(\mathbf{R}^d)$ on $\mathcal{F}_b$. Let $\Omega_0$ be the Fock vacuum in $\mathcal{F}_b$:

$$\Omega_0 := \{1, 0, 0, \cdots\} \in \mathcal{F}_b. \tag{1.5}$$

The Segal field operator $\phi_S(f)$ $(f \in L^2(\mathbf{R}^d))$ is given by

$$\phi_S(f) := \frac{1}{\sqrt{2}} \left( a(f)^* + a(f) \right). \tag{1.6}$$

The inner product (resp. norm) of a Hilbert space $\mathcal{K}$ is denoted $(\cdot\,,\,\cdot)_{\mathcal{K}}$, complex linear in the second variable (resp. $\|\cdot\|_{\mathcal{K}}$). For each $s \in \mathbf{R}$, we define a Hilbert space

$$\mathcal{M}_s = \left\{ f : \mathbf{R}^d \to \mathbf{C}, \text{Borel measurable} \mid \omega^{s/2} f \in L^2(\mathbf{R}^\nu) \right\}$$

with inner product $(f, g)_s := (\omega^{s/2} f, \omega^{s/2} g)_{L^2(\mathbf{R}^\nu)}$ and norm

$$\|f\|_s := \|\omega^{s/2} f\|_{L^2(\mathbf{R}^d)}, \quad f \in \mathcal{M}_s.$$

We shall assume the following (A.1) to obtain upper bounds for the ground state energy:

**(A.1)** The function $\lambda(k)$ of $k \in \mathbf{R}^d$ satisfies that $\lambda \in \mathcal{M}_{-1} \cap \mathcal{M}_0$.

We call the following condition the *infrared singularity condition* (see [Da2, p153], [AH2])

$$\|\lambda\|_{-2} = \infty, \qquad \left(\text{i.e.,} \quad \lambda/\omega \notin L^2\left(\mathbf{R}^d\right)\right). \tag{1.7}$$

**Remark 1.1** *Recently, Bach, Fröhlich and Sigal showed in [BFS2] that the Pauli-Fierz model has a ground state even under the infrared singularity condition. Moreover, Arai and the author proved that if the ground state energy of a Hamiltonian has a condition, then its ground state exists even under the infrared singularity condition [AH2].*

The Hamiltonian of the spin-boson model is defined by

$$\begin{aligned} H_{\rm SB} &:= \frac{\mu}{2}\sigma_3 \otimes I + I \otimes H_b + \sqrt{2}\alpha\sigma_1 \otimes \phi_S(\lambda) \\ &= \begin{pmatrix} H_b + \frac{\mu}{2} & \sqrt{2}\alpha\phi_S(\lambda) \\ \sqrt{2}\alpha\phi_S(\lambda) & H_b - \frac{\mu}{2} \end{pmatrix} \end{aligned} \tag{1.8}$$

acting in the Hilbert space

$$\mathcal{F} := \mathbf{C}^2 \otimes \mathcal{F}_b = \mathcal{F}_b \oplus \mathcal{F}_b, \tag{1.9}$$

where $0 < \mu$ is a splitting energy which means the gap of the ground and first excited state energy of uncoupled chiral molecule to a radiation field, $\alpha \in \mathbf{R}$ a coupling constant, and $\sigma_1, \sigma_3$ the standard Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For simplicity, we denote the decoupled free Hamiltonian ($\alpha = 0$) by $H_0$:

$$\begin{aligned} H_0 &:= \frac{\mu}{2}\sigma_3 \otimes I + I \otimes H_b \\ &= \begin{pmatrix} H_b + \frac{\mu}{2} & 0 \\ 0 & H_b - \frac{\mu}{2} \end{pmatrix}. \end{aligned} \tag{1.10}$$

For the above $H_{\rm SB}$, we temporally introduce an infrared cutoff $\nu > 0$ such that the *infrared regularity condition*

$$\lambda/\omega_\nu \in L^2(\mathbf{R}^d), \qquad \nu > 0, \tag{1.11}$$

holds, which raise the bottom of the frequency $\omega(k)$ of bosons (see [AH2]):

$$\omega_\nu(k) := \omega(k) + \nu, \qquad \nu > 0, \tag{1.12}$$

$$H_b(\nu) := d\Gamma(\widehat{\omega}_\nu), \tag{1.13}$$

$$H_{\rm SB}(\nu) := \frac{\mu}{2}\sigma_3 \otimes I + I \otimes H_b(\nu) + \sqrt{2}\alpha\sigma_1 \otimes \phi_S(\lambda), \tag{1.14}$$

where $\nu$ is something like a 'pad' of the frequency $\omega(k)$, namely $\nu$ means the lower bound of the frequency which we can observe precisely by an equipment. Of course, we shall remove '$\nu$' later by taking the limit $\nu \downarrow 0$ such as making the precision better.

For simplicity, we put

$$H_{\mathrm{SB}}(0) := H_{\mathrm{SB}}. \tag{1.15}$$

For a linear operator $T$ on a Hilbert space, we denote its domain by $D(T)$. It is well-known that $H_{\mathrm{SB}}(\nu)$ is self-adjoint on

$$D\left(H_{\mathrm{SB}}\left(\nu\right)\right) = D\left(I \otimes H_b\left(\nu\right)\right), \tag{1.16}$$

$$\text{and bounded from below for all } \alpha \in \mathbf{R} \tag{1.17}$$

for every $\nu \geq 0$ by [AH1, Proposition 1.1(i)] since $\sigma_1$ is bounded now.

For a self-adjoint operator $T$ bounded from below, we denote by $E_0(T)$ the infimum of the spectrum $\sigma(T)$ of $T$:

$$E_0(T) = \inf \sigma(T).$$

In this paper, when $T$ is a Hamiltonian, we call $E_0(T)$ the *ground state energy* of $T$ *even if $T$ has no ground state.*

For $H_{\mathrm{SB}}(\nu)$ $(\nu \geq 0)$ we set

$$E_{\mathrm{SB}}(\nu) := E_0\left(H_{\mathrm{SB}}\left(\nu\right)\right).$$

It is well known that for $\nu > 0$

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \leq E_{\mathrm{SB}}(\nu) \leq -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega_\nu\|_0^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \tag{1.18}$$

by easy estimation and the variational principle ( [Ar2, Theorem 2.4] and [Da2, p.161]). So we have for every $\nu > 0$

$$E_{\mathrm{SB}}(\nu) = -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega_\nu\|_0^2 G_\nu} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_\nu}} \right\|_0^2 \tag{1.19}$$

for some $G_\nu \in [0, 1]$. Under a condition we know a concrete expression of $G_\nu$ [Hm2, Theorems 1.5 and 1.6]. We can prove that

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_0^2 \leq \lim_{\nu \downarrow 0} E_{\mathrm{SB}}(\nu) = E_{\mathrm{SB}}(0) \leq -\alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_0^2 \tag{1.20}$$

even under the infrared singularity condition (1.7) (see [AH2, Proposition 3.2(iii)]). Under (1.7) we have the infrared divergence

$$\lim_{\nu\downarrow 0}\left\|\frac{\lambda}{\omega_\nu}\right\|_0 = \infty \tag{1.21}$$

appearing in the van Hove model. On the other hand, we have

$$0 \le G_\nu \le 1, \qquad \nu > 0. \tag{1.22}$$

Then, the problem of expressing the $E_{\mathrm{SB}}(0)$ in the case without infrared cutoff is as follows: Although $\lim_{\nu\downarrow 0}\|\lambda/\omega_\nu\|_0^2 G_\nu$ is apparently infinite (except for the fortunate case $\lim_{\nu\downarrow 0} G_\nu = 0$) and the term of $\mu$ is seemingly removed under the limit $\nu \downarrow 0$, we cannot believe $E_{\mathrm{SB}}(0) = -\alpha^2\|\lambda/\sqrt{\omega}\|_0^2$. So, *how does the term of $\mu$ from the effect by the spin survive in $E_{\mathrm{SB}}(0)$?* This is what the author would like to consider, so this work is the sequel to his in [Hm2].

Moreover, this work is also the first step for another scheme: Considering the result in [BFS2], there is a possibility that the generalized spin-boson (GSB) model [AH1] has a ground state even under the infrared singularity condition. Actually, as we showed in [AH2, §6.2], a model of a quantum harmonic oscillator coupled to a Bose field with the rotating wave approximation has a ground state, and the Wigner-Weisskopf model [WW] has also a ground state under certain conditions even if we assume the infrared singularity condition [AH2, §6.3].By our recent theory in [AH2], we know that if the right differential $E'_{\mathrm{SB}}(0+)$ of $E_{\mathrm{SB}}(\nu)$ at $\nu = 0$ is less than 1, then we have a ground state of $H_{\mathrm{SB}}$ in the standard state space $\mathcal{F}$. It may be worth pointing out, in passing, that Spohn discovered a critical criterion between the existence and absence of a ground state in $\mathcal{F}$ for the spin-boson model [Sp2, Sp3] by a method of the functional integration. Our goal of the scheme is to characterize the existence and absence of ground states of GSB model in terms of the ground state energy or correlation functions [AHH, AH2] by methods of functional analysis.

The estimation (1.18) is not suitable to check whether $E'_{\mathrm{SB}}(0+) < 1$ or not. Because (1.18) is obtained by regarding $H_{\mathrm{SB}}(\nu)$ as the van Hove model $H_{\mathrm{VH}}(\nu)$ perturbed by bounded operator:

$$U_0^* H_{\mathrm{SB}}(\nu) U_0 = H_{\mathrm{VH}}(\nu) - \frac{\mu}{2}\sigma_1, \tag{1.23}$$

where

$$H_{\mathrm{VH}}(\nu) = I \otimes H_b(\nu) + \sqrt{2}\alpha\sigma_3 \otimes \phi_s(\lambda) = \begin{pmatrix} H_b(\nu) + \sqrt{2}\alpha\phi_S(\lambda) & 0 \\ 0 & H_b(\nu) - \sqrt{2}\alpha\phi_S(\lambda) \end{pmatrix},$$

$$U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$

And, under the infrared singularity condition (1.7), the right differential of the ground state energy $E_{\mathrm{VH}}(\nu) = -\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$ $(\nu \geq 0)$ of $H_{\mathrm{VH}}(\nu)$ is infinite [AH2, §6.1], i.e.,

$$E'_{\mathrm{VH}}(0+) = \left\|\frac{\lambda}{\omega}\right\|_0^2 = \infty.$$

So, we need another estimation which is not influenced by the van Hove model.

We show in Theorem 2.1 that the term of $\mu$ influenced by the spin remains, moreover, the spin may make $\mu/2$ play a role such as the lower bound of frequency (a mass) of bosons.

## 2 Ground State Energy of Spin-Boson Model

In this subsection, we give an answer for the first problem above by using the *variational principle*. To do it, we have to assume the following (A.2) in addition to (A.1):

Fix arbitrarily $\delta$ with

$$0 < \delta < 1/3. \tag{2.1}$$

**(A.2)** The splitting energy $\mu$ and the coupling constant $\alpha$ satisfy

$$4\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k)} < \mu, \tag{2.2}$$

$$\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\left(\omega(k) + \dfrac{\mu}{2}\right)^2} < \frac{1-3\delta}{\delta^2} =: \gamma_\delta. \tag{2.3}$$

**Theorem 2.1** *(without infrared cutoff) Assume (A.1). For the Hamiltonian $H_{\mathrm{SB}}$ of the spin-boson model without infrared cutoff (i.e., even under the infrared singularity condition (1.7)), upper bounds and an equality are given as follows:*

**(a)** *(upper bound)*

*(a-1)* $$E_{\mathrm{SB}}(0) \leq \min\left\{-\frac{\mu}{2}, \inf_{f\in D(\widehat{\omega})} \frac{2\alpha\Re(f, \lambda)_0 + (f, \omega f)_0}{1 + \|f\|_0^2}\right\},$$

*(a-2)* $$E_{\mathrm{SB}}(0) \leq -\frac{\mu}{2} + \inf_{f\in D(\widehat{\omega})} \frac{2\alpha\Re(f, \lambda)_0 + (f, \omega f)_0 + \mu\|f\|_0^2}{1 + \|f\|_0^2}.$$

**(b)** *(equality) Let $\mu\alpha \neq 0$. Then, there exists $c_{\mu,\alpha} > \delta$ such that*

$$E_{\mathrm{SB}}(0) = -\frac{\mu}{2} - c_{\mu,\alpha}\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \dfrac{\mu}{2}}. \tag{2.4}$$

*Moreover, assume (A.2) in addition to (A.1). Then,*

$$-\frac{\mu}{2}-\alpha^2\int_{\mathbf{R}^d}dk\,\frac{|\lambda(k)|^2}{\omega(k)}\le E_{\mathrm{SB}}(0)<-\alpha^2\int_{\mathbf{R}^d}dk\,\frac{|\lambda(k)|^2}{\omega(k)}, \tag{2.5}$$

*and $G_\nu$ in (1.21) renormalizes the infrared divergence (1.22) in the following sense:*

$$\begin{aligned}\lim_{\nu\downarrow 0}\left\|\frac{\lambda}{\omega_\nu}\right\|_0^2 G_\nu &= -\frac{1}{2\alpha^2}\ln\left\{1+\frac{2\alpha^2}{\mu}\left(c_{\mu,\alpha}-1\right)\int_{\mathbf{R}^d}dk\,\frac{|\lambda(k)|^2}{\omega(k)+\frac{\mu}{2}}\right.\\ &\qquad\left.-\alpha^2\int_{\mathbf{R}^d}dk\,\frac{|\lambda(k)|^2}{\omega(k)\left(\omega(k)+\frac{\mu}{2}\right)}\right\}<\infty.\end{aligned} \tag{2.6}$$

**Remark 2.1** *By the equality in Theorem 2.1 (b), we know that*

$$E_{\mathrm{SB}}(0)<E_0(H_0). \tag{2.7}$$

*So, considering the diamagnetic inequality by Hiroshima [Hf1, Theorem 5.1], (2.7) means that there is a difference between the spin-boson model and the Pauli-Fierz model as far as concerning the ground state energy though the spin-boson model is regarded as an approximation of the Pauli-Fierz model in physics.*

Since we use Skibsted's result to make comment on a lower bound, we have to assume the following (A.3) at present because of the reason coming Proposition 3.2:

**(A.3)** $\lambda^{(1)}$, $\lambda^{(1)}/\omega\in L^2(\mathbf{R}^d)$, where

$$\lambda^{(1)}(k):=\frac{\partial}{\partial|k|}\lambda(k)+\frac{(d-1)\lambda(k)}{2|k|},\qquad k\in\mathbf{R}^d \tag{2.8}$$

considered as a distribution on $C_0^\infty(\mathbf{R}^d\setminus\{0\})$.

**Remark 2.2** *Assuming (A.3) practically amounts to assuming the infrared regularity condition, namely not the infrared singularity condition:*

$$\lambda/\omega\in L^2(\mathbf{R}^d). \tag{2.9}$$

**Proposition 2.2** *Let $\omega(k)=|k|$. Assume (A.1), (A.3), (2.2) and (2.9). Then, for all $\alpha\in\mathbf{R}$ with*

$$\alpha^2<\frac{1}{12\|\lambda^{(1)}\|_0^2}, \tag{2.10}$$

*(a) (lower bound)*

$$E_{\mathrm{SB}}(0) > -\frac{\mu}{2} - 2\alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) + \dfrac{\mu}{2}} \tag{2.11}$$

*(b) Assume (2.3) in addition. Then $c_{\mu,\alpha}$ in Theorem 2.1(b) is given as*

$$c_{\mu,\alpha} \in (\delta\, , \, 2)\, . \tag{2.12}$$

# 3 Spectral Properties of Wigner-Weisskopf Model

To prove Theorem 2.1 we use the properties of the Wigner-Weisskopf model [WW, Da1, HüS2, AH2]. So, in this section, we describe fundamental properties of the Wigner-Weisskopf model.

We define a matrix $c$ by

$$c := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

And let

$$H_b(0) := H_b, \tag{3.2}$$

$$\omega_0(k) := \omega(k), \qquad k \in \mathbf{R}^d. \tag{3.3}$$

Then, for every $\varepsilon_0 \in \mathbf{R}$ and $\varepsilon_1, \nu \geq 0$, we define two Hamiltonians $H_\alpha^{\pm}(\varepsilon_0\, , \, \varepsilon_1\, ; \, \nu)$ of the Wigner-Weisskopf model by

$$\begin{aligned} & H_\alpha^{+}(\varepsilon_0\, , \, \varepsilon_1\, ; \, \nu) \\ := \; & (\varepsilon_0 c^* c + \varepsilon_1 c c^*) \otimes I + I \otimes H_b(\nu) + \alpha\, (c^* \otimes a\,(\lambda) + c \otimes a\,(\lambda)^*) \\ = \; & \begin{pmatrix} H_b(\nu) + \varepsilon_0 & \alpha a\,(\lambda) \\ \alpha a\,(\lambda)^* & H_b(\nu) + \varepsilon_1 \end{pmatrix}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & H_\alpha^{-}(\varepsilon_0\, , \, \varepsilon_1\, ; \, \nu) \\ := \; & (\varepsilon_1 c^* c + \varepsilon_0 c c^*) \otimes I + I \otimes H_b(\nu) + \alpha\, (c^* \otimes a\,(\lambda)^* + c \otimes a\,(\lambda)) \\ = \; & \begin{pmatrix} H_b(\nu) + \varepsilon_1 & \alpha a\,(\lambda)^* \\ \alpha a\,(\lambda) & H_b(\nu) + \varepsilon_0 \end{pmatrix}. \end{aligned} \tag{3.5}$$

We call $H_\alpha^{\pm}(\varepsilon_0\, , \, \varepsilon_1\, ; \, \nu)$ the *Wigner-Weisskopf Hamiltonian.* We may put for $\nu = 0$

$$H_\alpha^{\#}(\varepsilon_0\, , \, \varepsilon_1) := H_\alpha^{\#}(\varepsilon_0\, , \, \varepsilon_1\, ; \, 0) \qquad \# \text{ is } + \text{ or } -. \tag{3.6}$$

**Remark 3.1** *The Wigner-Weisskopf model is one of several examples of the generalized spin-boson model. We know it if we put* $B_1 \equiv (c^* + c)/\sqrt{2}$, $B_2 \equiv i(c^* - c)/\sqrt{2}$; $\lambda_1 \equiv \lambda$ *and* $\lambda_2 \equiv i\lambda$. *This model is very simple, but it has an unusual property contrary to our expectation (see Remarks 3.4 and 3.6).*

It is easy to prove that $H_\alpha^{\pm}(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu)$ is self-adjoint on

$$D\left(H_\alpha^{\pm}(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu)\right) = D\left(I \otimes H_b(\nu)\right), \tag{3.7}$$

and bounded from below (3.8)

for every $\nu \geq 0$ by [AH1, Proposition 1.1(i)] since each $B_j$ is bounded, and

$$U_1^* H_\alpha^-(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu) U_1 = H_\alpha^+(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu) \quad \text{for every } \nu \geq 0, \tag{3.9}$$

where the unitary operator $U_1$ is given by

$$U_1 := \sigma_1 \otimes I = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.10}$$

So, we have only to deal with the case # is +. For simplicity, we put

$$H_\alpha(\varepsilon_0\,,\,\varepsilon_1) := H_\alpha^+(\varepsilon_0\,,\,\varepsilon_1\,;\,0) \tag{3.11}$$
$$H_\alpha(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu) := H_\alpha^+(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu), \qquad \nu \geq 0. \tag{3.12}$$

Let

$$\mu_0 := \varepsilon_0 - \varepsilon_1, \tag{3.13}$$

and we may put

$$H_\alpha(\mu_0;\nu) := H_\alpha(\mu_0, 0, ;\nu), \qquad \nu \geq 0, \tag{3.14}$$
$$H_\alpha(\mu_0) := H_\alpha(\mu_0;0) \equiv H_\alpha(\mu_0, 0, ;0) \qquad (\nu = 0). \tag{3.15}$$

We have

$$H_\alpha(\varepsilon_0\,,\,\varepsilon_1\,;\,\nu) = H_\alpha(\mu_0\,;\,\nu) + \varepsilon_1 I \otimes I \quad \text{for every } \nu \geq 0. \tag{3.16}$$

**Remark 3.2** *For* $\mu_0 < 0$, *the above Wigner-Weisskopf Hamiltonian* $H_\alpha(\mu_0\,;\,\nu)$ *was treated in [AH2, Theorem 6.15]. On the other hand, for* $\mu_0 \geq 0$, $H_\alpha(\mu_0\,;\,\nu)$ *was treated in [HüS2, §6] with* $\nu > 0$, *and [AH2, Theorem 6.14] with* $\nu \geq 0$.

As we did in [AH2, §6.2], we introduce a function $D^{\alpha}_{\varepsilon_0,\varepsilon_1,\nu}$ for $\varepsilon_0 \in \mathbf{R}$ and $\varepsilon_1, \nu \geq 0$ by

$$D^{\alpha}_{\varepsilon_0,\varepsilon_1,\nu}(z) := -z + \varepsilon_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) + \varepsilon_1 - z}, \tag{3.17}$$

defined for all $z \in \mathbf{C}$ such that $|\lambda(k)|^2/|z - \varepsilon_1 - \omega_\nu(k)|$ is Lebesgue integrable on $\mathbf{R}^d$.

We put

$$D^{\alpha}_{\mu_0,\nu}(z) := D^{\alpha}_{\mu_0,0,\nu}(z) \equiv -z + \mu_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - z}. \tag{3.18}$$

In particular, as we mentioned in [AH2, §6.2], $D^{\alpha}_{\mu_0,\nu}(z)$ is defined in the cut plane

$$\mathbf{C}_\nu := \mathbf{C} \setminus [\nu, \infty), \qquad \nu \geq 0 \tag{3.19}$$

and analytic there. It is easy to see that $D^{\alpha}_{\mu_0,\nu}(x)$ is monotone decreasing in $x < \nu$. Hence, the limit

$$\begin{aligned} d^{\alpha}_{\nu}(\mu_0) &:= \lim_{x \uparrow \nu} D^{\alpha}_{\mu_0,\nu}(x) \\ &= -\nu + \mu_0 - \alpha^2 \lim_{t \downarrow 0} \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - \nu + t} \end{aligned} \tag{3.20}$$

exists.

We have

$$D^{\alpha}_{\varepsilon_0,\varepsilon_1,\nu}(z) = -(z - \varepsilon_1) + \mu_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - (z - \varepsilon_1)} \tag{3.21}$$

$$= D^{\alpha}_{\mu_0,\nu}(z - \varepsilon_1) \tag{3.22}$$

for ever $\nu \geq 0$.

We may put for $\nu = 0$

$$D^{\alpha}_{\varepsilon_0,\varepsilon_1}(z) := D^{\alpha}_{\varepsilon_0,\varepsilon_1,0}(z), \tag{3.23}$$

$$D^{\alpha}_{\mu_0}(z) := D^{\alpha}_{\mu_0,0}(z), \tag{3.24}$$

$$d^{\alpha}(\mu_0) := d^{\alpha}_0(\mu_0). \tag{3.25}$$

The Wigner-Weisskopf model has a conservation law for a kind of the particle number in the following sense:

We define

$$N_P^{\pm} := \frac{1 \pm \sigma_3}{2} \otimes I + I \otimes N_b, \tag{3.26}$$

which appeared in [HüS2, §6], where $N_b$ is the boson number operator,

$$N_b := d\Gamma(1) = \sum_{\ell=0} \ell P^{(\ell)}. \tag{3.27}$$

Here (3.27) is the spectral resolution of $N_b$, and $P^{(\ell)}$ is the orthogonal projection onto the $\ell$-particle space in $\mathcal{F}_b$ for each $\ell \in \{0\} \cup \mathbf{N}$. The spectral resolution of $N_P^{\pm}$ is given as

$$N_P^{\pm} \;=\; \sum_{\ell=0} \ell P_\ell^{\pm}, \tag{3.28}$$

where

$$P_\ell^{\pm} = \begin{cases} \dfrac{1 \mp \sigma_3}{2} \otimes P^{(0)} & \text{if } \ell = 0, \\ \\ \dfrac{1 \pm \sigma_3}{2} \otimes P^{(\ell-1)} + \dfrac{1 \mp \sigma_3}{2} \otimes P^{(\ell)} & \text{if } \ell \in \mathbf{N}. \end{cases} \tag{3.29}$$

$H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)$ is reduced by $P_\ell^{\pm}\mathcal{F}$ for every $\alpha \in \mathbf{R}$ and each $\ell \in \{0\} \cup \mathbf{N}$, i.e.,

$$P_\ell^{\pm} H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu) \subset H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu) P_\ell^{\pm} \tag{3.30}$$

which means that

$$\begin{aligned} &D\left(P_\ell^{\pm} H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)\right) \subset D\left(H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu) P_\ell^{\pm}\right), \\ &P_\ell^{\pm} H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)\Psi = H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu) P_\ell^{\pm}\Psi \quad \text{for } \Psi \in D\left(P_\ell^{\pm} H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)\right) \end{aligned}$$

(see [Ka, p.278]). So, for every $\alpha \in \mathbf{R}$, $H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)$ is decomposed to the direct sum of $H_{\ell,\alpha}^{\pm}(\varepsilon_0, \varepsilon_1; \nu)$'s as

$$H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu) = \bigoplus_{\ell=0}^{\infty} H_{\ell,\alpha}^{\pm}(\varepsilon_0, \varepsilon_1; \nu), \tag{3.31}$$

where $H_{\ell,\alpha}^{\pm}(\varepsilon_0, \varepsilon_1; \nu)$ is self-adjoint on the closed subspace $\mathcal{F}_\ell^{\pm}$ defined by

$$\mathcal{F}_\ell^{\pm} := P_\ell^{\pm}\mathcal{F} \tag{3.32}$$

for each $\ell \in \{0\} \cup \mathbf{N}$ and

$$\mathcal{F} = \bigoplus_{\ell=0}^{\infty} \mathcal{F}_\ell^{\pm}. \tag{3.33}$$

The proof of the above statement is that, for instance, we have only to extend [Ka, Problem 3.29] to its infinite version by repeating [Ka, Problem 3.29] with the closedness of $H_\alpha^{\pm}(\varepsilon_0, \varepsilon_1; \nu)$.

We call $\mathcal{F}_\ell^\pm$ the $\ell$ *sector.*

We define vectors $\Omega_\pm^0 \in \mathcal{F}_0^\pm$ by

$$\Omega_+^0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega_0 = \begin{pmatrix} 0 \\ \Omega_0 \end{pmatrix}, \tag{3.34}$$

$$\Omega_-^0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \Omega_0 = \begin{pmatrix} \Omega_0 \\ 0 \end{pmatrix}. \tag{3.35}$$

Then, we have

$$\|\Omega_\pm^0\| = 1. \tag{3.36}$$

For every $f \in D(\widehat{\omega})$, we define vectors $\Omega_\pm^1(f) \in \mathcal{F}_1^\pm$ by

$$\Omega_+^1(f) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \Omega_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes a(f)^*\Omega_0 = \begin{pmatrix} \Omega_0 \\ a(f)^*\Omega_0 \end{pmatrix}, \tag{3.37}$$

$$\Omega_-^1(f) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes a(f)^*\Omega_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega_0 = \begin{pmatrix} a(f)^*\Omega_0 \\ \Omega_0 \end{pmatrix}. \tag{3.38}$$

Then, we have

$$\|\Omega_\pm^1(f)\| = \left(1 + \|f\|_0^2\right)^{1/2}. \tag{3.39}$$

When a zero $E^\alpha_{\varepsilon_0,\varepsilon_1,\nu}$ of $D^\alpha_{\varepsilon_0,\varepsilon_1,\nu}(z)$ exists in $(-\infty\,,\,\nu+\varepsilon_1)$, we define a function by

$$g^\alpha_{\varepsilon_0,\varepsilon_1,\nu}(k) := -\alpha \frac{\lambda(k)}{\omega_\nu(k) + \varepsilon_1 - E^\alpha_{\varepsilon_0,\varepsilon_1,\nu}} \in D(\widehat{\omega}_\nu), \qquad k \in \mathbf{R}^d. \tag{3.40}$$

Especially, we may put

$$g^\alpha_{\varepsilon_0,\varepsilon_1}(k) := g^\alpha_{\varepsilon_0,\varepsilon_1,0}(k) \qquad k \in \mathbf{R}^d \qquad (\nu = 0), \tag{3.41}$$

$$E^\alpha_{\varepsilon_0,\varepsilon_1} := E^\alpha_{\varepsilon_0,\varepsilon_1,0} \qquad (\nu = 0), \tag{3.42}$$

and

$$g^\alpha_{\mu_0,\nu}(k) := g^\alpha_{\mu_0,0,\nu}(k), \qquad k \in \mathbf{R}^d; \qquad \nu \geq 0, \tag{3.43}$$

$$g^\alpha_{\mu_0} := g^\alpha_{\mu_0,0}, \qquad (\nu = 0), \tag{3.44}$$

$$E^\alpha_{\mu_0,\nu} := E^\alpha_{\mu_0,0,\nu}, \qquad \nu \geq 0, \tag{3.45}$$

$$E^\alpha_{\mu_0} := E^\alpha_{\mu_0,0}, \qquad (\nu = 0). \tag{3.46}$$

For a self-adjoint operator $T$, we denote the set of all essential spectra of $T$ by $\sigma_{ess}(T)$, and pure point spectra by $\sigma_{pp}(T)$.

By the definition (3.14) of the Hamiltonian $H_\alpha(\mu_0;\nu)$, the free Hamiltonian of the Wigner-Weisskopf model is $H_0(\mu_0;\nu)$ for every $\mu_0 \in \mathbf{R}$ and $\nu \geq 0$. Then, it is clear that

$$\sigma_{pp}\left(H_0\left(\mu_0;\nu\right)\right) = \left\{0\,,\ \mu_0\right\}, \tag{3.47}$$

$$\sigma_{ess}\left(H_0\left(\mu_0;\nu\right)\right) = \left[\min\left\{0\,,\ \mu_0\right\}\,, \infty\right), \tag{3.48}$$

$$0 \text{ and } \mu_0 \text{ are simple,} \tag{3.49}$$

$$\text{the unique eigenvector of } 0 \text{ is } \Omega_+^0 \in \mathcal{F}_0, \tag{3.50}$$

$$\text{and} \quad \text{the unique eigenvector of } \mu_0 \text{ is } \Omega_+^1(0) \in \mathcal{F}_1. \tag{3.51}$$

The following theorem follows from [AH2, Proposition 6.13, Theorems 6.14 and 6.15]. We note here that the proof of [AH2, Theorem 6.15] had already proved part (c) below:

**Theorem 3.1** **(a)** *Let $\nu, d_\nu^\alpha(\mu_0) \geq 0$. Then,*

$$0 \in \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{3.52}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = \left[\nu\,,\,\infty\right). \tag{3.53}$$

*In particular, $0$ is the ground state energy of $H_\alpha\left(\mu_0\,;\,\nu\right)$ with its unique ground state $\Omega_+^0$.*

**(b)** *Let $d_\nu^\alpha(\mu_0) < 0 < \nu$ and $\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 \leq \mu_0$. Then,*

$$\left\{0\,,\ E_{\mu_0,\nu}^\alpha\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{3.54}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = \left[\nu\,,\,\infty\right), \tag{3.55}$$

*with $0 \leq E_{\mu_0,\nu}^\alpha < \nu$. In particular, $0$ is the ground state energy of $H_\alpha(\mu_0\,;\,\nu)$. Moreover,*

*if $\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 < \mu_0$, then $0 < E_{\mu_0,\nu}^\alpha$; $0$ is simple, and $\Omega_+^0$ is the unique ground state of $H_\alpha(\mu_0\,;\,\nu)$,* (3.56)

*if $\alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2 = \mu_0$, then $0 = E_{\mu_0,\nu}^\alpha$; $\Omega_+^0$ and $\Omega_+^1(g_{\mu_0,\nu}^\alpha)$ are the degenerate ground states of $H_\alpha(\mu_0\,;\,\nu)$.* (3.57)

**(c)** *Let $d_\nu^\alpha(\mu_0) < 0 < \nu$ and $\mu_0 < \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$. Suppose that*

$$2\nu - \mu_0 > \alpha^2\left(\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2 - M\left(\alpha, \mu_0, \omega_\nu\right)\right) + \frac{\|\lambda\|_0^2}{M\left(\alpha, \mu_0, \omega_\nu\right)}, \tag{3.58}$$

*where*

$$M(\alpha, \mu_0, \omega_\nu) := \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega_\nu(k) - \mu_0 + \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2}. \tag{3.59}$$

*Then,*

$$\left\{E^\alpha_{\mu_0,\nu}, 0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right), \tag{3.60}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\,;\,\nu\right)\right) = \left[E^\alpha_{\mu_0,\nu} + \nu\,,\,\infty\right), \tag{3.61}$$

*with* $E^\alpha_{\mu_0,\nu} < 0$. *In particular,* $E^\alpha_{\mu_0,\nu}$ *is the ground state energy of* $H_\alpha(\mu_0\,;\,\nu)$ *with its ground state* $\Omega^1_+(g^\alpha_{\mu_0,\nu})$.

**Remark 3.3** *We are also interested in the case for large absolute value of the coupling constant(i.e.,* $|\alpha| \gg 1$*). Fix* $\mu_0$ *and make* $|\alpha|$ *so large. Then, we have* $d^\alpha_\nu(\mu_0) < 0$. *Thus, we have to investigate the case for* $d^\alpha_\nu(\mu_0) < 0$ *to know the case for large* $|\alpha|$. *See Theorem 3.5 below.*

**Remark 3.4** *In* $[\nu\,,\,\infty)$ *for* $\nu \geq 0$, *we can make a different eigenvalue from both of* $E^\alpha_{\mu_0,\nu}$ *and* 0 *by adding some conditions to* $\omega(k)$ *and* $\lambda(k)$ *as we mentioned in [AH2, Remark 6.4]. Namely, as an effect of the scalar Bose field, a new eigenvalue appears in* $(\nu\,,\,\infty)$.

**Remark 3.5** *It is easy to check that*

$$\left\|\frac{\lambda}{\sqrt{\omega_\nu}}\right\|_0^2 - M(\alpha, \mu_0, \omega_\nu) > 0. \tag{3.62}$$

*Let* $\mu_0 \geq 0$. *Then, if* $\nu = 0$, *then (3.58) does not hold by (3.62). Let* $\mu_0 < 0$. *Then, by the definition (3.59), we get*

$$M(\alpha, \mu_0, \omega_\nu) < \int_{\mathbf{R}^d} dk\, \frac{|\lambda(k)|^2}{-\mu_0} = \frac{\|\lambda\|_0^2}{-\mu_0}$$

*since* $\mu_0 < 0$ *now, which implies that*

$$\text{the left hand side of (3.58)} > -\mu_0$$

*since* $\mu_0 < 0 < M(\alpha, \mu_0, \omega_\nu)$. *Thus, (3.58) is meaningful for the case of massive bosons only.*

We note here that, if $d^\alpha(\mu_0) < 0$, then

$$\mu_0 < \alpha^2 \lim_{t \downarrow 0} \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k) + t} \leq \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)} \tag{3.63}$$

since

$$\int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)+t} < \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k)}$$

for all $t > 0$.

In Theorem 3.1(c) for the case $d^\alpha(\mu_0) < 0$, we cannot show the ground state energy of $H_\alpha(\mu_0)$ for the massless bosons as we remarked in Remark 3.5, but *if we add the condition (A.3)*, then we can determine the pure point spectra of $H_\alpha(\mu_0)$ completely for the massless bosons by using [Sk, Theorem 3.1]:

**Proposition 3.2** *Assume (A.1), (A.3) and (2.9). Let $\omega(k) = |k|$ and $d^\alpha(\mu_0) < 0$. Then,*

$$\sigma_{pp}\left(H_\alpha\left(\mu_0\right)\right) = \left\{E^\alpha_{\mu_0}\,,\, 0\right\}, \tag{3.64}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0\right)\right) = \left[E^\alpha_{\mu_0}\,,\, \infty\right) \tag{3.65}$$

*for all $\alpha \in \mathbf{R}$ with*

$$\alpha^2 < \frac{1}{4\|\lambda^{(1)}\|_0^2}. \tag{3.66}$$

*Especially, $E^\alpha_{\mu_0}$ is the simple ground state energy with its unique ground state $\Omega^1_+(g^\alpha_{\mu_0})$, and 0 is the simple first excited state energy with its unique first excited state $\Omega^0_+$.*

**Remark 3.6** *For the generalized spin-boson model, in a generic situation, we hope that the ground state will be unique and that the rest of the spectrum will be pure absolutely continuous as it is mentioned in [DJ, p.11]. However, we have to note that there is a counter-example but familiar to us in physics as one of generalized spin-boson models. Namely, in the case of Proposition 3.2, 0 is sitting very still as an excited state at the same place for all coupling constant $\alpha$, so 0 is not a resonance pole. It means that the rest of the spectrum except the ground state energy is not only pure absolutely continuous spectrum but also the other eigenvalues. Moreover, see Remark 3.4 above and Theorem 3.5 below, and we can find more interesting eigenvalues. This is a remark for our usual expectation of the above spectral property for the generalized spin-boson model.*

Here, we set the following condition

**(D)$_\nu$** The function $\dfrac{|\lambda(k)|^2}{|\omega_\nu(k) - x|}$ is not Lebesgue integrable for all $x \in (\nu\,,\,\infty)$,

and we prove the following lemma:

In the following proposition, we use the result of Hübner and Spohn [HüS2], so we employ the conjugate operator $D_{\mathrm{HS}}$ in [HüS2, (2.9)]:

$$D_{\mathrm{HS}} := \frac{1}{2}\left(\frac{1}{|\nabla_k\omega_\nu|^2}\nabla_k\omega_\nu\cdot\nabla_k + \nabla_k\cdot\nabla_k\omega_\nu\frac{1}{|\nabla_k\omega_\nu|^2}\right). \tag{3.67}$$

**Proposition 3.3** *Let $\omega(k) = |k|$ and $\nu > 0$. Assume*

$$\int_{\mathbf{R}^d} dk\, |\lambda(k)|^2 \delta\left(\omega_\nu(k) - \mu_0\right) > 0, \tag{3.68}$$

$$D^j_{\mathrm{HS}}\lambda \in L^2(\mathbf{R}^d) \qquad j = 1, 2, \tag{3.69}$$

*and $d^\alpha_\nu(\mu_0) < 0$. Then,*

**(a)**

$$\sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right) = \left\{E^\alpha_{\mu_0,\nu}\,,\, 0\right\}, \tag{3.70}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0;\nu\right)\right) = \left[\min\left\{E^\alpha_{\mu_0}\,,\, 0\right\} + \nu\,,\, \infty\right) \tag{3.71}$$

*for all $\alpha \in \mathbf{R}$ with*

$$|\alpha|\|D_{\mathrm{HS}}\lambda\|_0 < 1. \tag{3.72}$$

**(b)** *If $\mu_0 > \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$, then $0$ is the simple ground state energy with its unique ground state $\Omega^0_+$, and $E^\alpha_{\mu_0,\nu}$ is the simple first excited state energy with its unique first excited state $\Omega^1_+(g^\alpha_{\mu_0,\nu})$ for all $\alpha \in \mathbf{R}$ with (3.72).*

**(c)** *If $\mu_0 < \alpha^2\|\lambda/\sqrt{\omega_\nu}\|_0^2$, then $E^\alpha_{\mu_0,\nu}$ is the simple ground state energy with its unique ground state $\Omega^1_+(g^\alpha_{\mu_0,\nu})$, and $0$ is the simple first excited state energy with its unique first excited state $\Omega^0_+$ for all $\alpha \in \mathbf{R}$ with (3.72).*

**(d)** *Assume $\mu_0 > 0$ and $\sqrt{\mu_0}\|D_{\mathrm{HS}}\lambda\|_0 < \|\lambda/\sqrt{\omega_\nu}\|_0$, then $H_\alpha\left(\mu_0;\nu\right)$ has degenerate ground states for $\alpha_c = \sqrt{\mu_0}/\|\lambda/\sqrt{\omega_\nu}\|_0$ with ground state energy $0 = E^\alpha_{\mu_0,\nu}$, and ground states are given by $\Omega^0_+$ and $\Omega^1_+(g^\alpha_{\mu_0,\nu})$.*

We define expectations, $\overline{n}_{grd}$ and $\overline{n}_{1st}$, of the number of (massive) photons at the ground and first excited states, respectively, as follows:

$$\overline{n}_{grd} := (\Psi_{grd}\,,\, I \otimes N_b\Psi_{grd})_{\mathcal{F}}, \tag{3.73}$$

$$\overline{n}_{1st} := (\Psi_{1st}\,,\, I \otimes N_b\Psi_{1st})_{\mathcal{F}}, \tag{3.74}$$

where $\Psi_{grd}$ and $\Psi_{1st}$ denote the ground and first excited states of $H_\alpha(\mu_0;\nu)$, respectively.

By Proposition 3.3, we obtain the following corollary:

**Corollary 3.4** *Let $\omega(k) = |k|$ and $\nu > 0$. Assume (3.68), (3.69) and $d_\nu^\alpha(\mu_0) < 0$. Then, for all $\alpha \in \mathbf{R}$ with (3.72),*

**(a)**

$$\overline{n}_{grd} = \begin{cases} 0 & \text{if} \quad \mu_0 > \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2, \\ \\ \|g_{\mu_0,\nu}^\alpha\|_0^2 & \text{if} \quad \mu_0 < \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2. \end{cases}$$

**(b)** *A reverse between $\overline{n}_{grd}$ and $\overline{n}_{1st}$ occurs as follows:*

$$\begin{cases} \overline{n}_{grd} < \overline{n}_{1st} & \text{if} \quad \mu_0 > \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2, \\ \\ \overline{n}_{1st} < \overline{n}_{grd} & \text{if} \quad \mu_0 < \alpha^2 \|\lambda/\sqrt{\omega_\nu}\|_0^2. \end{cases}$$

We use the following condition in Theorem 3.5 (b) and (c) below:

**(A.4)** The functions, $\omega(k)$, $\lambda(k)$, are continuous, and

$$\lim_{|k|\to\infty} \omega(k) = \infty. \tag{3.75}$$

Moreover, there exist constants $\gamma_\omega > 0$ and $C_\omega > 0$ such that

$$|\omega(k) - \omega(k')| \leq C_\omega |k - k'|^{\gamma_\omega} \left(1 + \omega(k) + \omega(k')\right), \qquad k, k' \in \mathbf{R}^d. \tag{3.76}$$

**Theorem 3.5** *Let $\nu \geq 0$. Assume (A.1). Then,*

**(a)** *there exists $\alpha_{\rm ww}(\nu) > 0$ such that*

$$\left\{E_{\mu_0,\nu}^\alpha \,, 0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right) \tag{3.77}$$

$$\text{with } E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) < \min\left\{E_{\mu_0,\nu}^\alpha \,, 0\right\}, \tag{3.78}$$

$$\sigma_{ess}\left(H_\alpha\left(\mu_0;\nu\right)\right) = \left[E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) + \nu \,, \infty\right) \tag{3.79}$$

*for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\rm ww}(\nu)$.*

**(b)** *Let $\nu > 0$ (massive bosons). Assume (A.4) in addition. Then, there exists a ground state $\Psi_{\rm ww} \in \mathcal{F}$ of $H_\alpha(\mu_0;\nu)$, namely*

$$H_\alpha\left(\mu_0;\nu\right)\Psi_{\rm ww} = E_0\left(H_\alpha\left(\mu_0;\nu\right)\right)\Psi_{\rm ww},$$

*such that*

$$\left\{E_0\left(H_\alpha\left(\mu_0;\nu\right)\right) \,, E_{\mu_0,\nu}^\alpha \,, 0\right\} \subset \sigma_{pp}\left(H_\alpha\left(\mu_0;\nu\right)\right), \tag{3.80}$$

*with (3.78)*

$$\Psi_{\rm ww} \notin \mathcal{F}_0 \cup \mathcal{F}_1 \tag{3.81}$$

*for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\rm ww}(\nu)$.*

**(c)** *Let $\nu = 0$ (massless bosons). Assume (A.4), $\nabla\omega \in L^\infty(\mathbf{R}^d)$ and (2.9) in addition. Then, there exists a ground state $\Psi_{\mathrm{WW}} \in \mathcal{F}$ of $H_\alpha(\mu_0;\nu)$ such that (3.80), (3.78) and (3.81) hold for every $\alpha \in \mathbf{R}$ with $|\alpha| > \alpha_{\mathrm{WW}}(0)$.*

**Open Problem 3.1** *We knew in Theorem 3.5 that there exists a non-perturbative ground state state $\Psi_{\mathrm{WW}}$ in $\mathcal{F}$, and $\Psi_{\mathrm{WW}}$ does not belong to the 0-sector or 1-sector. As we remark in Remark 3.8 below, this fact plays a important role to show the phenomena for WW model, which cannot be derived from the regular perturbation theory (see Remark 3.8). But we have not yet known which sector $\Psi_{\mathrm{WW}}$ belongs to. This is an open problem.*

**Open Problem 3.2** *Concerning Open Problem 3.1, in Theorem 3.5 we assumed the infrared regularity condition, $\lambda/\omega \in L^2(\mathbf{R}^d)$. The next open problem is whether the ground state $\Psi_{\mathrm{WW}}$ appears in the standard state space $\mathcal{F}$ under the infrared singularity condition, $\lambda/\omega \notin L^2(\mathbf{R}^d)$, or not.*

**Remark 3.7** *When the case of massive bosons ($\nu > 0$), we can apply the regular perturbation theory to WW model for sufficiently small absolute value of the coupling constant $\alpha$, and then Theorem 2.1 says that we get either $E^\alpha_{\mu_0,\nu}$ or 0 as the ground state energy. Theorem 3.5 means that, for sufficiently large absolute value of the coupling constant, a non-perturbative ground state appears as an influence of the scalar Bose field with its ground state energy so low that we cannot conjecture it by the regular perturbation theory for sufficiently small absolute value of the coupling constant. For other models, the similar phenomenon were investigated by Hiroshima and Spohn [HfS]. So, Theorem 3.5 may make a statement on the existence of a superradiant ground state in physics (see, for instance, [Pr1, Pr2, En]) for WW model. Namely, we can say that, even for WW model which is simple and familiar in physics, we may be able to show a phenomena of superradiant ground state influenced by the scalar Bose field.*

**Remark 3.8** *By applying the existence of such a non-perturbative ground state in Theorem 3.5 (b) & (c) to our new result on stability of ground states [AH3], we shall show in [AH3, Theorem 2.3] that there exists a value of coupling constants at which WW model has degenerate ground states, and the following fact: We denote the ground state (resp. first excited) state energy by $E^p_0(\alpha)$ (resp. $E^p_1(\alpha)$) iff the ground (resp. first excited) state exists for $\alpha \in \mathbf{R}$, i.e.,*

$$E^p_0(\alpha) := \inf \sigma_{pp}(H_\alpha(\mu_0;\nu)) = E_0(H_\alpha(\mu_0;\nu)) \tag{3.82}$$

$$\textit{(resp.}\quad E^p_1(\alpha) := \inf\{\sigma_{pp}(H_\alpha(\mu_0;\nu)) \setminus \{E^p_0(\alpha)\}\} \qquad ). \tag{3.83}$$

*Then, we obtain that for $\nu > 0$ there exists $\alpha_1$ in a region such that*

$$E^p_0(\alpha_1) < E^p_1(\alpha_1) < \inf \sigma_{ess}(H_{\alpha_1}(\mu_0;\nu)). \tag{3.84}$$

*even if we assume that* $\nu > 0$ *is so small that*

$$E_0^p(0) < \inf \sigma_{ess}\left(H_0\left(\mu_0; \nu\right)\right) < E_1^p(0) \tag{3.85}$$

*holds [AH3, Theorem 2.3]. Although we can find many papers stating the possibility of the existence of such the first excited state in quantum field theory, there is few papers pointing out the definite existence despite under (3.85). These phenomena cannot be obtained by the regular perturbation theory. Namely, they occur in the region* $\{\alpha \in \mathbf{R} \,|\, d_\nu^\alpha(\mu_0) < 0\}$ *(see Remark 3.3), not in the region of the coupling constants treated by Hübner and Spohn in [HüS, §6] and ourselves in [AH2, Theorem 6.14(i)]. So, the existence of the non-perturbative ground state derives very interesting phenomena.*

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