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Microlocalization of the Topological Boundary Value Morphism for Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) $\mathcal{D}$-Module, here in this article, we shall write *Module* with a capital letter, instead of *sheaf of modules*). If the system is *regular-specializable*, we can define the *nearby-cycle* of the system in the theory of $\mathcal{D}$-Modules. The definitions of regular-specializable $\mathcal{D}$-Module and its nearby-cycle are initiated by Kashiwara [Kas], Kashiwara and Kawai [K-K 1] and Malgrange [Mal] for regular-holonomic cases. These definitions extended to the *specializable* $\mathcal{D}$-Module (see Laurent [L], Laurent and Malgrange [L-Ma] and Mebkhout [Me]). After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 3], [Sc 4], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism (called the *topological boundary value morphism*) which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the *induced* system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the *near-hyperbolicity*). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

In this section, we shall fix the notation used in later sections.

We denote the set of integers, of real numbers and of complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\ n \geqslant 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In this article, all the manifolds are assumed to be paracompact. Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Moreover in this paper, we assume the existence of a partial complexification of M in X; that is, there exists a $(2n+1)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to $(\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R})$ by a local coordinate system $(z, \tau) = (x + \sqrt{-1}\, y, t + \sqrt{-1}\, s)$ of X around each point of N. We say such a coordinate system *admissible*. We shall mainly follow the notation in Kashiwara-Schapira [K-S 2]; we denote the normal deformations of N and Y in M and L by $\widetilde{M}_N$ and $\widetilde{L}_Y$ respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:

$$
\begin{tikzcd}
T_N M \arrow[rr, hook, "s_M"] \arrow[dr, "\tau_N"] \arrow[dd, hook, "i'"'] & & \widetilde{M}_N \arrow[dr, "p_M"] \arrow[dd, "\widetilde{i}'"] & & \Omega_M \arrow[ll, hook', "j_M"'] \arrow[dl, "\widetilde{p}_M"'] \arrow[dd, "\widetilde{i}"] & \\
& N \arrow[rr, hook] \arrow[dd, hook, "i|_N"] & & M \arrow[rr, hook, "i_M"] \arrow[dd, hook, "i"] & & X \arrow[dd, equal] \\
T_Y L \arrow[rr, "s_L"] \arrow[dr, "\tau_Y"] & & \widetilde{L}_Y \arrow[dr, "p_L"] & & \Omega_L \arrow[ll, "j_L"'] \arrow[dl, "\widetilde{p}_L"'] & \\
& Y \arrow[rr, hook, "i_Y"] & & L \arrow[rr, hook, "i_L"] & & X,
\end{tikzcd}
$$

and by admissible coordinates we have locally the following relation:

$$
\begin{tikzcd}
N = \mathbb{R}^n_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & M = \mathbb{R}^n_x \times \mathbb{R}_t \arrow[d, hook, "i"'] \arrow[dr, "i_M"] & \\
Y = \mathbb{C}^n_z \times \{0\} \arrow[r, hook, "i_Y"] & L = \mathbb{C}^n_z \times \mathbb{R}_t \arrow[r, hook, "i_L"] & X = \mathbb{C}^n_z \times \mathbb{C}_\tau .
\end{tikzcd}
$$

With these coordinates, we often identify $T_Y X$ and $T_Y L$ with X and L respectively.

The projection $\tau_Y \colon T_Y L \longrightarrow Y$ induces natural mappings:

$$
T^*_N Y \underset{\tau_{Y\pi}}{\longleftarrow} T_N M \underset{N}{\times} T^*_N Y \underset{{}^t\tau'_Y}{\xrightarrow{\ \sim\ }} T^*_{T_N M} T_Y L,
$$

and by ${}^t\tau'_Y$ we identify $T^*_{T_N M} T_Y L$ with $T_N M \underset{N}{\times} T^*_N Y$. Similarly by natural mappings

$$
T^*_{\widetilde{M}_N} \widetilde{L}_Y \underset{s_{L\pi}}{\longleftarrow} T_N M \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N} \widetilde{L}_Y \underset{{}^t s'_L}{\xrightarrow{\ \sim\ }} T^*_{T_N M} T_Y L,
$$

we identify $T_N M \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N} \widetilde{L}_Y$ with $T^*_{T_N M} T_Y L$.

$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z,t) \in T_Y L;\, t > 0\}.$$

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Note that to define $T_Y L^+$ (or $T_N M^+$) by means of admissible coordinates is equivalent to determining a local isomorphism $or_{Y/L} \simeq \mathbb{Z}_Y$ (or equivalently $or_{N/M} \simeq \mathbb{Z}_N$). Here $or_{Y/L}$ denotes the relative orientation sheaf.

Define open embeddings f and f_N by:

$$\begin{tikzcd} T_Y L^+ \arrow[r, hook, "f"] \arrow[d, phantom, "\cup"] \arrow[dr, phantom, "\circlearrowleft"] & T_Y L \arrow[d, phantom, "\cup"] \\ T_N M^+ \arrow[r, hook, "f_N"] & T_N M. \end{tikzcd}$$

Thus we regard $T_N M^+ \underset{N}{\times} T^*_N Y$ as an open set of $T^*_{T_N M} T_Y L$. Moreover f induces mappings:

$$\begin{tikzcd} T^*_{T_N M^+} T_Y L^+ & T_N M^+ \underset{T_N M}{\times} T^*_{T_N M} T_Y L \arrow[l, "\sim"', "{}^t f'"] \arrow[r, "f_\pi"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & T^*_{T_N M} T_Y L \arrow[d, "\wr"] \\ & T_N M^+ \underset{N}{\times} T^*_N Y \arrow[r, "f_N \times \mathrm{id}"] & T_N M \underset{N}{\times} T^*_N Y. \end{tikzcd}$$

Hence we identify $T^*_{T_N M^+} T_Y L^+$ with $T_N M^+ \underset{N}{\times} T^*_N Y$, and f_π with $f_N \times \mathrm{id}$.

2 Several Sheaves Attached to the Boundary

In this section, we recall several sheaves attached to the boundary due to Oaku [Oa 2]. These sheaves will play essential roles for our boundary value problem. We remark that in Oaku [Oa 2] these sheaves are defined on cosphere bundles. So we shall present equivalent but slightly different definitions on cotangent bundles along the line of Oaku-Yamazaki [O-Y]. We refer to Oaku [Oa 2] or Oaku-Yamazaki [O-Y] for the proofs. Note that although the higher-codimensional case is treated in Oaku-Yamazaki [O-Y], the same proofs also work as in the one-codimensional case.

As usual, we denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of *holomorphic functions* on X, of *hyperfunctions* on M and of *microfunctions* on $T^*_M X$ respectively. Further, we denote by $\mathcal{B}\mathcal{O}_L$ the sheaf of *hyperfunctions with holomorphic parameters* on L; that is,

$$\mathcal{B}\mathcal{O}_L := \mathcal{H}^1_L(\mathcal{O}_X) \otimes or_{L/X} \simeq i_L^! \,\mathcal{O}_X \otimes or_{L/X}[1]\,.$$

We denote as usual by ν and μ the Sato specialization and microlocalization functors respectively.

2.1 Definition. We set:

$$\mathcal{C}_{N|M} := s_{L\pi}^{-1}\,\mathcal{H}^n\big(\mu_{\widetilde{M}_N}(j_{L*}\widetilde{p}_L^{-1}\,\mathcal{B}\mathcal{O}_L)\big)\otimes or_{M/L}\,,$$
$$\mathcal{B}_{N|M} := \mathcal{C}_{N|M}\big|_{T_NM}\,.$$

We denote by $\pi_{N|M}$ the natural projection from $T^*_{T_NM}T_YL$ to T_NM. Let $\dot{\pi}_{N|M}$ be the restriction of $\pi_{N|M}$ to $T^*_{T_NM}T_YL\setminus T^*_{T_NM}T_NM$ as usual. By virtue of the following proposition, we can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$:

2.2 Proposition. *There exists the following exact sequence on T_NM:*

$$0\longrightarrow \nu_Y(\mathcal{B}\mathcal{O}_L)\big|_{T_NM}\longrightarrow \mathcal{B}_{N|M}\longrightarrow \dot{\pi}_{N|M*}\,\mathcal{C}_{N|M}\longrightarrow 0.$$

Moreover, an isomorphism $\nu_N(\mathcal{B}_M)\simeq\mathcal{B}_{N|M}$ holds.

2.3 Definition. We set:

$$\widetilde{\mathcal{C}}_{N|M} := \mathcal{H}^n\big(\mu_{T_NM}(\nu_Y(\mathcal{B}\mathcal{O}_L))\big)\otimes or_{N/Y}\,,$$
$$\widetilde{\mathcal{B}}_{N|M} := \widetilde{\mathcal{C}}_{N|M}\big|_{T_NM}\simeq \mathcal{H}^n_{T_NM}(\nu_Y(\mathcal{B}\mathcal{O}_L))\otimes or_{N/Y}\,.$$

By the following fact, we can regard $\mathcal{C}_{N|M}$ as a subsheaf of $\widetilde{\mathcal{C}}_{N|M}$:

2.4 Proposition. *There exists a natural monomorphism $\mathcal{C}_{N|M}\longrightarrow\widetilde{\mathcal{C}}_{N|M}$.*

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable $\mathcal{D}$-Module and its nearby-cycle.

As usual, we denote by $\mathcal{D}_X$ the sheaf on X of holomorphic differential operators, and by $\{\mathcal{D}_X^{(m)}\}_{m\in\mathbb{N}_0}$ the usual order filtration on $\mathcal{D}_X$. First, let us recall the definition of the V-filtration:

3.1 Definition. Denote by $\mathcal{I}_Y$ the defining Ideal of Y in $\mathcal{O}_X$ with a convention that $\mathcal{I}_Y^j=\mathcal{O}_X$ for $j\leqslant 0$. The *V-filtration* $\{V_Y^k(\mathcal{D}_X)\}_{k\in\mathbb{Z}}$ *(along Y)* is a filtration on $\mathcal{D}_X|_Y$ defined by

$$V_Y^k(\mathcal{D}_X) := \bigcap_{j\in\mathbb{Z}}\{P\in\mathcal{D}_X|_Y;\ P\,\mathcal{I}_Y^j\subset\mathcal{I}_Y^{j-k}\}.$$

It is easy to see that by admissible coordinates, this filtration written as

$$V_Y^k(\mathcal{D}_X)=\Big\{\sum_{j-i\leqslant k}P_{ij}(z;\partial_z)\,\tau^i\,\partial_\tau^j\in\mathcal{D}_X|_Y\Big\}.$$

For the fundamental properties of this filtration, we refer to Björk [Bj], Sabbah [Sab] and Schapira [Sc 2]).

Let us denote by ϑ the Euler operator. Note that $\vartheta\in V_Y^0(\mathcal{D}_X)\setminus V_Y^{-1}(\mathcal{D}_X)$ and that ϑ can be represented by $\tau\,\partial_\tau$ by admissible coordinates.

3.2 Definition. A coherent $\mathcal{D}_X$-Module $\mathcal{M}$ defined on a neighborhood of Y is said to be *regular-specializable* (*along* Y) if there exist locally a coherent $\mathcal{O}_X$-sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

(1) $\mathcal{M}_0$ generates $\mathcal{M}$ over $\mathcal{D}_X$; that is, $\mathcal{M} = \mathcal{D}_X \, \mathcal{M}_0$;

(2) $b(\vartheta)\, \mathcal{M}_0 \subset (\mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X))\, \mathcal{M}_0$, where m is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase "along Y" since Y is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module for which Y is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K 1] proved that every regular-holonomic $\mathcal{D}_X|_Y$-Module is regular-specializable.

3.4 Proposition. *If* $\mathcal{M}$ *is a regular-specializable* $\mathcal{D}_X$*-Module, then each cohomology of* $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_Y(\mathcal{O}_X))$ *and* $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_Y(\mathcal{O}_X))$ *is a locally* $\mathbb{C}^\times$*-conic sheaf.*

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module. Recall that a V-filtration $\{F^k \mathcal{M}\}_{k\in\mathbb{Z}}$ is said to be *good* if there exist locally a system of generators $\{u_j\}_{j=1}^m$ and $k_j \in \mathbb{Z}$ such that for any $k \in \mathbb{Z}$

$$F^k \mathcal{M} = \sum_{j=1}^{m} V_Y^{k-k_j}(\mathcal{D}_X)\, u_j$$

holds. The following theorem is proved by Kashiwara [Kas] (cf. also Björk [Bj]):

3.5 Theorem. *Set* $G := \{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha < 1\}$. *Then, for any regular-specializable* $\mathcal{D}_X$*-Module* $\mathcal{M}$, *there exist a unique good* V*-filtration* $\{V_G^k(\mathcal{M})\}_{k\in\mathbb{Z}}$ *on* $\mathcal{M}$ *and a non-zero polynomial* $b_G(\alpha) \in \mathbb{C}[\alpha]$ *such that* $b_G^{-1}(0) \subset G$ *and that for any* $k \in \mathbb{Z}$ *the following holds*:

$$b_G(\vartheta + k)\, V_G^k(\mathcal{M}) \subset V_G^{k-1}(\mathcal{M}).$$

3.6 Definition. Under the notation of Theorem 3.5, we set:

$$\Psi_Y(\mathcal{M}) := V_G^0(\mathcal{M})/V_G^{-1}(\mathcal{M}),$$
$$\Phi_Y(\mathcal{M}) := V_G^1(\mathcal{M})/V_G^0(\mathcal{M}),$$

and call $\Psi_Y(\mathcal{M})$ the *nearby-cycle* of $\mathcal{M}$ and $\Phi_Y(\mathcal{M})$ the *vanishing-cycle* of $\mathcal{M}$ respectively.

3.7 Remark. Laurent [L] extended the definitions of nearby and vanishing cycles to the derived category of bounded complexes with (regular) specializable cohomology by using the theory of second microlocalization.

Let $\iota\colon Y \longrightarrow X$ be the natural inclusion. Then the *induced system,* or the *inverse image* in the sense of $\mathcal{D}$-Modules is defined by

$$\boldsymbol{D}\iota^* \mathcal{M} := \mathcal{O}_Y \overset{\boldsymbol{L}}{\underset{\iota^{-1}\mathcal{O}_X}{\otimes}} \iota^{-1}\mathcal{M}.$$

Then we have (cf. Laurent [L], Mebkhout [Me] or Sabbah [Sab]):

3.8 Proposition. *If $\mathcal{D}_X$-Module $\mathcal{M}$ is regular-specializable, then $\Psi_Y(\mathcal{M})$, $\Phi_Y(\mathcal{M})$ and each cohomology of $\boldsymbol{D}\iota^*\mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle*:

$$\Phi_Y(\mathcal{M}) \xrightarrow{\mathrm{var}} \Psi_Y(\mathcal{M}) \longrightarrow \boldsymbol{D}\iota^*\mathcal{M} \xrightarrow{+1} .$$

As usual, we denote by $\mathcal{C}^{\mathbb{R}}_{Y|X} := \mu_Y(\mathcal{O}_X)[1]$ the sheaf of *real holomorphic microfunctions* on T^*_YX. Set $\dot{T}_YX := T_YX \setminus T_YY$ as usual (the definition of $\dot{T}^*_YX$ is similar). Using an admissible coordinate system we define a continuous section $\sigma\colon Y \longrightarrow \dot{T}_YX$ by $z \longmapsto (z,1)$. Similarly we define ${}^t\sigma\colon Y \longrightarrow \dot{T}^*_YX$ by $z \longmapsto (z,1)$. Denote by $\mathcal{N}_{X|Y}$ the sheaf of Nilsson class functions on X along Y and regard as a sheaf on Y. Then the following theorem is proved by Laurent [L] (cf. also Kashiwara-Kawai [K-K 2]):

3.9 Theorem. *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$-Module. Then, there exists the following isomorphism of distinguished triangles*:

$$\begin{array}{ccccccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)|_Y & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\sigma^{-1}\nu_Y(\mathcal{O}_X)) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},{}^t\sigma^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X}) & \xrightarrow{+1} & \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}\iota^*\mathcal{M},\mathcal{O}_Y) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}),\mathcal{O}_Y) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}),\mathcal{O}_Y) & \xrightarrow{+1} & .
\end{array}$$

Moreover, a natural morphism $\mathcal{N}_{X|Y} \longrightarrow \sigma^{-1}\nu_Y(\mathcal{O}_X)$ induces an isomorphism:

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}),\mathcal{N}_{X|Y}) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\sigma^{-1}\nu_Y(\mathcal{O}_X)).$$

3.10 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}\iota^*\mathcal{M},\mathcal{O}_Y) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Value Morphism

In this section, we shall define our injective boundary value morphism. Recall the mappings f_π and $\tau_{Y\pi}$ defined in Section 1.

4.1 Theorem. *For any regular-specializable $\mathcal{D}_X$-Module $\mathcal{M}$, there exists the following isomorphism:*

$$f_\pi^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M}) \xrightarrow{\sim} f_\pi^{-1}\,\tau_{Y\pi}^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

The proof is based on Proposition 3.4 and Theorem 3.9 .

4.2 Definition. For any regular-specializable $\mathcal{D}_X$-Module $\mathcal{M}$, we define by virtue of Proposition 2.4 and Theorem 4.1:

$$\begin{aligned}\beta\colon f_\pi^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) &\longrightarrow f_\pi^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M})\\ &\xrightarrow{\sim} f_\pi^{-1}\,\tau_{Y\pi}^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).\end{aligned}$$

By the construction, we can obtain the following Holmgren type theorem:

4.3 Theorem. (1) *The morphism β gives a monomorphism*

$$\beta^0\colon f_\pi^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrowtail f_\pi^{-1}\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

(2) *The restriction of β^0 to the zero-section $T_N M^+$ coincides with the topological boundary value morphism in the sense of Monteiro Fernandes* [MF 1].

4.4 Remark. (1) For a general Fuchsian system in the sense of Tahara [T], Oaku [Oa 2] defined an injective boundary value morphism under additional conditions of characteristic exponents by using a detailed study due to Tahara [T].

(2) Let $\mathcal{C}^F_{N|M} \subset \mathcal{C}_{N|M}$ be the subsheaf consisting of F-mild microfunctions, and $\widetilde{\mathcal{C}}^A_{N|M} := \mu_N(\mathcal{O}_X|_Y) \otimes or_{N/Y}[n]$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$-Module and set $\mathcal{M}_Y := \mathcal{H}^0(\boldsymbol{D}\iota^*\,\mathcal{M}) = \mathcal{O}_Y \underset{\iota^{-1}\mathcal{O}_X}{\otimes} \iota^{-1}\,\mathcal{M}$. Since $\mathcal{M}$ is a Fuchsian system, by the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$\begin{array}{ccccc} f_\pi^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^F_{N|M}) & \rightarrowtail & f_\pi^{-1}\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}^A_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ f_\pi^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \rightarrowtail & f_\pi^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N), \end{array}$$

that is, the boundary value morphism

$$\gamma^F\colon f_\pi^{-1}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^F_{N|M}) \rightarrowtail f_\pi^{-1}\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

for F-mild microfunctions and β^0 are compatible.

5 Solvability

In this section, we shall state the solvability theorem under a kind of hyperbolicity condition. First, let us recall the following (Laurent-Monteiro Fernandes [L-MF 2]):

5.1 Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module on a neighborhood of Y. Then we say $\mathcal{M}$ is *near-hyperbolic* at $x_0 \in N$ (in dt-codirection) if there exist positive constants C and ε_1 such that

$$\begin{aligned}\operatorname{char}(\mathcal{M}) \cap \big\{(z,\tau;z^*,\tau^*) \in T^*X;\ |z-x_0|,\ |\tau| < \varepsilon_1,\ \operatorname{Re}\tau > 0\big\}\\ \subset \big\{(z,\tau;z^*,\tau^*) \in T^*X;\ |\operatorname{Re}\tau^*| < C\big(|\operatorname{Im} z^*|(|\operatorname{Im} z| + |\operatorname{Im}\tau|) + |\operatorname{Re} z^*|\big)\big\}\end{aligned}$$

holds by an admissible coordinate system.

5.2 Remark. As is shown by Laurent-Monteiro Fernandes [L-MF 2, Lemma 1.3.2], the near-hyperbolicity condition is weaker than the hyperbolicity condition (see also Bony-Schapira [B-S]).

5.3 Theorem. *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$-Module. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* = (x_0, t_0; \sqrt{-1}\,\langle \xi_0, dx\rangle) \in T^*_{T_N M^+} T_Y L^+$*

$$\beta\colon \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})_{p^*} \longrightarrow \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism.

5.4 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module for which Y is non-characteristic. Then, it is known that $\Psi_Y(\mathcal{M}) \xrightarrow{\sim} \boldsymbol{D}\iota^*\,\mathcal{M} \simeq \mathcal{M}_Y$. Moreover by virtue of the commutative diagram in Remark 4.4, we see that β^0 is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of β^0 to the zero-section $T_N M^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Moreover, if each $\pm dt \in T^*_N M$ is hyperbolic for $\mathcal{M}$, then the nearly-hyperbolic condition is satisfied (cf. Kashiwara-Schapira [K-S 1]).

(2) Assume that $X = \mathbb{C}^{n+1}$ an so on by taking an admissible coordinate system. Let $b(\alpha)$ be a non-zero polynomial with degree m, and $Q \in \mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)$ and set $\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X\,(b(\vartheta) + Q)$. Then $\mathcal{M}$ is regular-specializable. For simplicity, assume that

$$b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j} \quad (\alpha_i - \alpha_j \notin \mathbb{Z} \text{ for } 1 \leqslant i \neq j \leqslant \mu)$$

(note that $\sum_{j=1}^{\mu} \nu_j = m$). Then a direct calculation shows that $\Psi_Y(\mathcal{M}) \simeq \mathcal{D}_Y^{\oplus m}$, and β^0 is equivalent to γ in Oaku [Oa 2]: Let $p^* = (x_0, t_0; \sqrt{-1}\,\langle \xi_0, dx\rangle)$ be a point of $T^*_{T_N M^+} T_Y L^+$, and $f(x,t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^*. Then, since $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}_{X|Y}) \simeq$

$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1}\nu_Y(\mathcal{O}_X))$ by virtue of Theorem 3.9, we can see that $f(x,t)$ has a defining function

$$F(z,\tau) = \sum_{j=1}^{\mu}\sum_{k=1}^{\nu_j} F_{jk}(z,\tau)\,\tau^{\alpha_j}\,(\log\tau)^{k-1}$$

as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M})$ at p^*. Here each $F_{jk}(z,\tau)$ is holomorphic on a neighborhood of $\{(z,0) \in X;\ |x_0 - z| < \varepsilon,\ \operatorname{Im} z \in \Gamma\}$ with a positive constant ε and an open convex cone Γ such that $\xi_0 \in \operatorname{Int}(\Gamma^\circ)$ (the interior of the dual cone Γ° of Γ). Then, $\beta^0(f)$ is equivalent to $\{\mathrm{sp}_N(F_{jk}(x + \sqrt{-1}\,\Gamma\,0, 0));\ 1 \leqslant k \leqslant \nu_j,\ 1 \leqslant j \leqslant \mu\}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ written as $\tau^m P(z,\tau;z^*,\tau^*)$ for a hyperbolic polynomial P at dt-codirection, then the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

5.5 Example. Assume that $X = \mathbb{C}^{n+1}$. Take an operator $A(z;\partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 :=$ id and $A^{(j)} := \dfrac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geqslant 1$. Let $p^* = (0,1;\sqrt{-1}\,\langle\xi, dx\rangle)$ be a point of $T^*_{T_N M^+} T_Y L^+$ and set $p_0 := (0;\sqrt{-1}\,\langle\xi, dx\rangle) \in T^*_N Y$. Consider the following differential equations:

$$\begin{aligned}
\mathcal{M}_1 &:= \mathcal{D}_X / \mathcal{D}_X\big(\vartheta(\vartheta-1) - \tau A(z;\partial_z)\vartheta\big),\\
\mathcal{M}_2 &:= \mathcal{D}_X / \mathcal{D}_X\big((\vartheta-1)^2 - \tau A(z;\partial_z)\vartheta\big),\\
\mathcal{M}_3 &:= \mathcal{D}_X / \mathcal{D}_X\big((\vartheta-1)(\vartheta-2) - \tau A(z;\partial_z)\vartheta\big).
\end{aligned}$$

Let $f_i(x,t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \mathcal{C}_{N|M})$ at p^*. Then:

(1) $f_1(x,t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M})$ at p^*:

$$F_1(z,\tau) = U_0(z) + \sum_{j=0}^{\infty} \frac{A^{(j)}U_1(z)}{j+1}\,\tau^{j+1}.$$

In this case, $f_1(x,t)$ is always F-mild. Hence $\beta^0(f_1(x,t))$ is given by $\gamma^F(f_1(x,t)) = \{(\partial_t{}^l f_1)(x,+0)\}_{l=0,1} = \{\mathrm{sp}_N(U_l)(x)\}_{l=0,1}$ at p_0. Indeed if $\tau \neq 0$, $\mathcal{M}_1$ is isomorphic to $\mathcal{D}_X / \mathcal{D}_X(\partial_\tau^2 - \partial_\tau A(z;\partial_z))$ for which Y is non-characteristic.

(2) $f_2(x,t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M})$ at p^*:

$$F_2(z,\tau) = \sum_{j=0}^{\infty} A^{(j)}U_1(z)\,\tau^{j+1} - \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{A^{(j)}U_0(z)}{k}\,\tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)}U_0(z)\,\tau^{j+1}\log\tau\,,$$

and $\beta^0(f_1(x,t))$ is given by $\{\mathrm{sp}_N(U_l)(x)\}_{l=0,1}$ at p_0. Further if $f_1(x,t)$ is F-mild, then $U_0(z) = 0$ and $\gamma^F(f_2(x,t)) = \{(\partial_t{}^l f_2)(x,+0)\}_{l=0,1} = \{\mathrm{sp}_N(U_1)(x)\}$ at p_0.

(3) $f_3(x,t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M})$ at p^*:

$$F_3(z,\tau) = \sum_{j=0}^{\infty} A^{(j)}U_2(z)\,\tau^{j+2} + U_1(z)\,\tau - \sum_{j=2}^{\infty}\sum_{k=1}^{j-1} \frac{jA^{(j)}U_1(z)}{k}\,\tau^{j+1}$$
$$+ \Big(AU_1(z)\,\tau^2 + \sum_{j=2}^{\infty} jA^{(j)}U_1(z)\,\tau^{j+1}\Big)\log\tau,$$

and $\beta^0\big(f_3(x,t)\big)$ is given by $\{\mathrm{sp}_N(U_l)(x)\}_{l=1,2}$ at p_0. In the case where $f_3(x,t)$ is F-mild, we must impose the condition $AU_1(z) = 0$. Under this condition, $\gamma^F\big(f_3(x,t)\big)$ is given by $\gamma^F\big(f_3(x,t)\big) = \{(\partial_t{}^l f_3)(x,+0)\}_{0\leqslant l\leqslant 2} = \{0, \mathrm{sp}_N(U_1)(x), 2\,\mathrm{sp}_N(U_2)(x)\}$ at p_0 with $A(\partial_t f_3)(x,+0) = A\,\mathrm{sp}_N(U_1)(x) = 0$.

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