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ON THE CAUCHY PROBLEM FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The well-posedness of the Cauchy problem in various function spaces for higher order linear scalar equations is well characterized. On the other hand, the results on the well-posedness for systems are rather poor. The reason is that the principal part of system has not been well caught. In this note, the author proposes the definition of the principal part on the Cauchy problem. In order to understand the structure of an usual matrix, the Jordan normal form and the determinant are very useful. The former includes almost all information on a matrix and the latter is very convenient. Our aims are to establish the corresponding theory for the matrices of differential operators and to give applications —— the necessary and sufficient conditions for the analytic well-posedness and C^∞ well-posedness ——.

Let us consider the following Cauchy problem:

$$\begin{cases} D_t u - \sum_{|\alpha|\le m} A_\alpha(t,x) D_x{}^\alpha u = f(t,x) \quad , \\ u|_{t=t_\circ} = u_\circ(x) \quad , \end{cases} \tag{1.1}$$

where, A_α is a $\mathtt{N}\times\mathtt{N}$ matrix of smooth functions ($|\alpha|\le m$), u, $u_\circ$ and f are vectors of dimension $\mathtt{N}$, $D_t = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$ and $D_x = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}$.

In Section 2, we explain the normal form of systems in the *formal symbol class*. In Section 3, we do the theory of the weighted determinant, so called *p-determinant* and introduce the notion of *p-evolution*. In Section 4, we give the necessary and sufficient condition for the analytic well-posedness (the Cauchy-Kowalevskaya theorem). We give a remark and a conjecture also on the C-K theorem of Nagumo type, relaxation of the regularity of coefficients. In Section 5, we give the necessary and sufficient condition for the C^∞ well-posedness assuming the constant multiplicity of characteristic roots and the real analyticity of coefficients (Levi condition). We give some remarks when the coefficients are not real analytic. The situations on the analytic well-posedness and C^∞ well-posedness in case of the constant multiplicity are very similar if coefficients are real analytic. However, the phenomena are very different when coefficients are non-quasianalytic.

Key words and phrases. normal form of systems, p-determinant of matrix of pseudo-differential operators, p-evolution, the Cauchy-Kowalevskaya theorem for systems, C^∞ well-posedness for systems.

This note is a short course of W. Matsumoto [22]. When the author writes this note, he has corrected the errors in the original [22]. The revised version of [22] can be also available claiming it to the author.

2. Normal form of Systems

We follow the results in W.Matsumoto[20] and [23]. From an arbitrary asymptotic expansion of a symbol of a pseudo-differential operator in an ultradifferentiable class, a true symbol of the same class can be constructed and the ambiguity is of class $S^{-\infty}$. (See L.Boutet de Monvel and P.Krée[7], L.Boutet de Monvel[6] and W.Matsumoto[19].) Therefore, in order to consider many problems on partial differential equations in a ultradifferentiable class, it is sufficient to consider asymptotic expansions, which we call here *formal symbols.* Let $\mathbf{Z}_+$ be $\mathbf{N} \cup \{0\}$. We use the followings for α and β in ${\mathbf{Z}_+}^{1+\ell}$: $|\alpha| = \alpha_0 + \cdots + \alpha_\ell$, $\alpha! = \alpha_0!\alpha_1! \cdots \alpha_\ell!$, $\alpha + \beta = (\alpha_0 + \beta_0, \cdots, \alpha_\ell + \beta_\ell)$ and we denote $\beta \leq \alpha$ when $\beta_i \leq \alpha_i$ for $0 \leq i \leq \ell$. Let us set $a(t,x,\xi)^{(\beta)}_{(\alpha)} = {D_t}^{\alpha_0} {D_{x_1}}^{\alpha_1} \cdots {D_{x_\ell}}^{\alpha_\ell} (\frac{\partial}{\partial \xi})^\beta a(t,x,\xi)$ for $\alpha \in {\mathbf{Z}_+}^{1+\ell}$ and $\beta \in {\mathbf{Z}_+}^{\ell}$.

We introduce a holomorphic formal symbol and a meromorphic one. We say that a set O in $\mathbf{C}_t \times {\mathbf{C}_x}^\ell \times {\mathbf{C}_\xi}^\ell$ is conic when $(t,x,\xi) \in O$ implies $(t,x,\lambda\xi) \in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma \cap \{||\xi|| = 1\}$ is compact in $O \cap \{||\xi|| = 1\}$, where $||\xi|| = \sqrt{\sum_{i=1}^{\ell} |\mathrm{Re}\xi_i|^2 + |\mathrm{Im}\xi_i|^2}$. We say that Σ is a subvariety of O if it is a zero set of a holomorphic function in O.

Definition 1. *(Meromorphic and holomorphic formal symbol,* [20] *)*
I. We say that the formal sum $a(t,x,\xi) = \sum_{i=0}^{\infty} a_i(t,x,\xi)$ is a meromorphic formal symbol ($=$ *m.f.s.*) on O when there exist a conic subvariety Σ in O and a real number κ such that

1) $a_i(t,x,\xi)$ is meromorphic in O, holomorphic in $O \setminus \Sigma$ and positively homogeneous of degree $\kappa - i$ on ξ , ($i \in \mathbf{Z}_+$).
2) For an arbitrary conically compact set Γ in $O \setminus \Sigma$, there are positive constants C, R and R' and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)| \leq C\, R'^i\, R^{|\alpha|+|\beta|}\, i!|\alpha|!|\beta|!|\xi_1|^{\kappa - i} \quad on\ \Gamma, \qquad (i \in \mathbf{Z}_+,\ \alpha \in {\mathbf{Z}_+}^{1+\ell},\ \beta \in {\mathbf{Z}_+}^{\ell}). \tag{2.1}$$

II. The formal sum $\sum_{i=0}^{\infty} a_i$ is called a holomorphic formal symbol ($=$ *h.f.s.*) when it is a meromorphic formal symbol with $\Sigma = \emptyset$.

Remark 2.1. We use ξ_1 as a holomorphic scale of order in case of a complex domain and Σ includes $\{\xi_1 = 0\}$. Of course, ξ_1 can be replaced by another ξ_i and Σ includes $\{\xi_i = 0\}$.

Remark 2.2. It is important that Σ is independent of i.

Now, we define a formal symbol of class $\{M_n\,,\, L_n\}$ on a real domain. Let $\{M_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ be sequences of positive numbers. We assume that $\log M_n = \log L_n = O(n^2)$ (Differentiability condition) and $\{M_n/n!\}_{n=0}^{\infty}$ and $\{L_n/n!\}_{n=0}^{\infty}$ are logarithmically convex and non-decreasing. We say that a set O in $\mathbf{R}_t \times {\mathbf{R}_x}^\ell \times {\mathbf{R}_\xi}^\ell$ is conic when $(t,x,\xi) \in O$ implies $(t,x,\lambda\xi) \in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma \cap \{|\xi| = 1\}$ is compact in $O \cap \{|\xi| = 1\}$, where $|\xi| = \sqrt{\sum_{i=1}^{\ell} {\xi_i}^2}$.

Definition 2. *(Formal symbol of class $\{M_n\,,\, L_n\}$,* [20] *)*
We say that the formal sum $a(t,x,\xi) = \sum_{i=0}^{\infty} a_i(t,x,\xi)$ is a formal symbol of class $\{M_n\,,\, L_n\}$ ($=$ *f.s. of class* $\{M_n\,,\, L_n\}$) on O when there exists a real number κ such that

1) $a_i(t,x,\xi)$ belongs to $C^\infty(O)$ and positively homogeneous of degree $\kappa - i$ on ξ, ($i \in \mathbf{Z}_+$).
2) For an arbitrary conically compact subset Γ in O, there are positive constants C, R and R' and we have

$$|a_{i\,(\alpha)}^{(\beta)}(t,x,\xi)| \le C\, R'^{i}\, R^{|\alpha|+|\beta|}\, M_{i+|\alpha|} L_{i+|\beta|} i!^{-1} \quad |\xi|^{\kappa-i-|\beta|} \quad on\ \Gamma, \quad (i \in \mathbf{Z}_+,\ \alpha \in \mathbf{Z}_+{}^{1+\ell},\ \beta \in \mathbf{Z}_+{}^{\ell}). \tag{2.2}$$

The number κ is called the order of the formal symbol a and denoted by "ord a". When $a_i = 0$ for $0 \le i \le i_\circ - 1$ and $a_{i_\circ} \ne 0$, $\kappa - i_\circ$ is called the true order of a and denoted by "true ord a". The order of 0 is posed $-\infty$. We set $S^\kappa_M(O) = \{$ *the m.f.s.'s on O of order* κ $\}$, $S^\kappa_H(O) = \{$ *the h.f.s.'s on O of order* κ $\}$, $S^\kappa\{M_n, L_n\}(O) = \{$ *the f.s.'s of class* $\{M_n, L_n\}$ *on O of order* κ $\}$, and $S_M(O) = \cup_{\kappa \in \mathbf{R}} S_M(O)$, etc. We denote one of these simply by $S(O)$.

Corresponding to the asymptotic expansion of the symbol of the product of pseudo-differential operators, we introduce the operator product of formal symbols.

Definition 3. *(Operator product)*
Let $a = \sum_{i=0}^\infty a_i$ and $b = \sum_{i=0}^\infty b_i$ be formal symbols. We set

$$a \circ b = \textstyle\sum_{i=0}^\infty c_i \quad , \quad c_i(t,x,\xi) = \sum_{i_1+i_2+|\gamma|=i} \frac{1}{\gamma!} a_{i_1}^{(\gamma)}(t,x,\xi)\, b_{i_2(\gamma)}(t,x,\xi) \tag{2.3}$$

and call it the operator product of a and b.

By the operator product, S_H and $S\{M_n, L_n\}$ become non-commutative rings and S_M does a non-commutative field. S_H is a subring of S_M.

Let us consider a matrix $P = I_N D_t - A(t,x,\xi)$, $A \in M_N(S^m)$, ($m \in \mathbf{N}$). In [20] and [23], we obtained the following theorem.

Theorem 1. (Normal form of system (1), [20])
We assume that every entry of A satisfies (2.2) ((2.1) *in case of m.f.s.*) *with $\kappa = m$ and that the each eigenvalue $\lambda_k(t,x,\xi)$ $(1 \le k \le d)$ of A_0 has the constant multiplicity m_k. Then, there exist finite disjoint open conical sets $\{O_h\}_h$ such that $\cup_h O_h$ is dense in O. On each O_h, there exist natural numbers d_k and $\{n_{kj}\}_{j=1}^{d_k}$ ($\sum_{j=1}^{d_k} n_{kj} = m_k$). For every point $(t_\circ, x_\circ, \xi_\circ)$ in O_h, there exist a conically compact neighborhood Γ, $N(t,x,\xi) = \sum_{i=0}^\infty N_i(t,x,\xi)$ in $GL(N; S(\Gamma))$, and $D_{kj}(t,x,\xi) = \sum_{i=0}^\infty D_{kji}(t,x,\xi)$ in $M_{n_{kj}}(S^m(\Gamma))$, such that*

$$\begin{aligned}
&N^{-1}(t,x,\xi) \circ P(t,x,D_t,\xi) \circ N(t,x,\xi) = \oplus_{1\le k\le d} \oplus_{1\le j\le d_k} P_{kj}\,, \\
&P_{kj}(t,x,D_t,\xi) = I_{n_{kj}}(D_t - \lambda_k(t,x,\xi)) - \textstyle\sum_{i=0}^\infty D_{kji}(t,x,\xi) \\
&D_{kj0} = J(n_{kj})|\xi|^m, \\
&D_{kji} = \begin{pmatrix} & & \\ & 0 & \\ * & \cdots & * \end{pmatrix} \quad \textit{homogeneous of degree } m-i \quad (i \ge 1),
\end{aligned} \tag{2.4}$$

where $J(n) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$: $n\times n$. *We set* $\sum_{i=1}^{\infty} D_{kji} = \begin{pmatrix} & O & \\ d_{kj}(1) & \cdots & d_{kj}(n_{kj}) \end{pmatrix}$.

In the case of meromorphic formal symbol, $\{O_h\}_h$ *is composed by only one element and* $O_1 = O \setminus \Sigma'$ *for a subvariety* Σ'. N *and* D_{kj} *belong to* $GL(\mathtt{N}; S_M(O))$ *and* $M_{n_{kj}}(S_M^m(O))$, *respectively. In* (2.4), *we replace* $|\xi|^m$ *by* ξ_1^m.

One may think that the assumption of the constant multiplicity is too strong. However, if we regard P as an operator of order $m+1$ on D_x, the highest order part is the zero-matrix and has an unique eigenvalue zero of constant multiplicity $\mathtt{N}$. Thus, under no condition on the structure, we can reduce P to the normal form.

Corollary 2. (Normal form of systems (2))
We assume that every entry of A *satisfies* (2.2) ((2.1) *in case of m.f.s.*) *with* $\kappa = m$. *There exist finite disjoint open conical sets* $\{O_h\}_h$ *such that* $\cup_h O_h$ *is dense in* O. *On each* O_h, *there exist natural numbers* d *and* $\{n_k\}_{k=1}^d$ ($\sum_{k=1}^d n_k = \mathtt{N}$). *For every point* $(t_\circ, x_\circ, \xi_\circ)$ *in* O_h, *there exist a conically compact neighborhood* Γ, $N_\circ(t,x,\xi) = \sum_{i=0}^{\infty} N_i(t,x,\xi)$ *in* $\mathrm{GL}(\mathtt{N}; S(\Gamma))$ *and* $B_k(t,x,\xi) = \sum_{i=0}^{\infty} B_{ki}(t,x,\xi)$ *in* $M_{n_k}(S^{m+1}(\Gamma))$ *such that*

$$
(2.5)\quad
\begin{aligned}
&N_\circ^{-1}(t,x,\xi)\circ P(t,x,D_t,\xi)\circ N_\circ(t,x,\xi) = Q = \oplus_{1\le k\le d} Q_k\,,\\
&Q_k(t,x,D_t,\xi) = I_{n_k} D_t - \textstyle\sum_{i=0}^{\infty} B_{ki}(t,x,\xi)\,,\\
&B_{k0} = J(n_k)|\xi|^{m+1}\,,\\
&B_{ki} = \begin{pmatrix} & O & \\ * & \cdots & * \end{pmatrix} : \text{homogeneous of order } m+1-i\,, \quad (i\ge 1)
\end{aligned}
$$

We set $\sum_{i=1}^{\infty} B_{ki} = \begin{pmatrix} & O & \\ b_k(1) & \cdots & b_k(n_k) \end{pmatrix}$.

In the case of meromorphic formal symbol, $\{O_h\}_h$ *is composed by only one element and* $O_1 = O\setminus\Sigma'$ *for a subvariety* Σ'. $N_\circ$ *and* B_k *belong to* $GL(\mathtt{N}; S_M(O))$ *and* $M_{n_k}(S_M^{m+1}(O))$, *respectively. In* (2.5), *we replace* $|\xi|^{m+1}$ *by* ξ_1^{m+1}.

Remark 2.3. $\{O_h\}_h$ and Σ' in Theorems 1 and Corollary 2 are different. In each case, $\{O_h\}_h$ has finite elements but can have countably many connected components in case of non-quasianalytic classes. This causes a difficulty on the Cauchy problem. (See Example 1 in Subsection 4.5.)

In case of non-quasianalytic classes, we stand on the following simple property;

For a continuous function $f(x)$ *on an open set* O, *the set* $\{x \mid f(x)\neq 0\}\cup\{x\mid f(x)=0\}^o$ *is open and dense in* O, *where* A^o *is the open kernel of* A.

By this property, we can also obtain the normal form in case of non-quasianalytic classes on an open dense set.

A higher order scalar equation

$$(\partial t)^m u + \sum_{j=1}^{m} \sum_{|\alpha| \le m(j)} a_{\alpha j}(t,x)(\partial x)^\alpha (\partial t)^{m-j} u = f(t,x) \quad ,$$

is reduced to a first order system on D_t for a suitable positive number p:

$$D_t u - J(\mathtt{N}) D_x{}^p u - B(t,x,D_x)u = f(t,x) \quad ,$$

where the lower order term B has the form $\begin{pmatrix} 0 \\ * \cdots * \end{pmatrix}$. Therefore, Corollary 2 say that a system is reduced to a direct sum of some higher order scalar equations in an open dense set in $\Omega \times \mathbf{C}^\ell \setminus \Sigma$ modulo $S^{-\infty}$. Thus, if we can obtain a result microlocally and modulo $S^{-\infty}$ and if such result on a dense open set implies the global one, we can apply the proof on scalar equations also to systems. In many cases, the necessary condition of the well-posedness has these properties. On the other hand, for the sufficiency, if we assume the real analyticity of coefficients, we can apply the maximum principle. (See, for example, the results in Sections 4 and 5.

In the normal forms in Theorem 1 and Corollary 2, the invariants are not clear. For example, d_k and $\{n_{kj}\}_{j=1}^{d_k}$ in Theorem 1 and d and $\{n_k\}_{k=1}^{d}$ in Corollary 2 are not invariant. Thus, we need the invariant theory and are led to the theory of determinant.

3. p-DETERMINANT OF MATRIX OF DIFFERENTIAL OPERATORS AND p-EVOLUTION

3.1. Definition of p-determinant.

On the matrix of partial differential operators, G.Hufford[10] first introduced the determinant applying the theory of J.Dieudonné[9], which is a determinant theory on a non-commutative field. M.Sato and M.Kashiwara[39] obtained the regularity property of the determinant. The algebraic structure of the determinant on the ring with Ore's property is well characterized by K.Adjamagbo[2] and [3]. The determinant by G.Hufford and M.Sato-M.Kashiwara is homogeneous. However, in order to consider, for example, the parabolic equations and Schrödinger type equations, we encounter inhomogeneous principal parts and need an inhomogeneous determinant. In order to describe the Levi condition for C^∞ well-posedness, we also need an inhomogeneous determinant. Recently, the author has received a preliminary version of a paper by A.D'Agnolo and G.Taglialatela[8], where they define independently the same weighted determinant as mine. Their definition and consideration are more algebraically and systematic than mine.

First we consider $S_M[D_t]$. This is a non-commutative integral domain with Ore's property: for non-zero elements a and b, we can find non-zero c and d such that $ac = bd$. (See, for example, K.Adjamagbo[3].) Ore's property is the necessary and sufficient condition for the existence of the quotient field. (See O.Ore[36].)

We fix a positive rational number p. Let us take $a(t,x,\xi,D_t) = \sum_{j=0}^{m} a^{<j>}(t,x,\xi) D_t{}^{m-j}$, $a^{<j>} = \sum_{i=0}^{\infty} a_i^{<j>} \in S_M$. We reset the order of $a^{<j>}$ to its true order. let us set

$$p\text{-ord}\, a^{<j>}(t,x,\xi) D_t{}^{m-j} = \operatorname{ord} a^{<j>} + p(m-j)$$

$$p\text{-ord}\, a = \max_{0\le j\le m} p\text{-ord}\, a^{<j>}(t,x,\xi){D_t}^{m-j}$$

and call them the p-order. By p-order, $S_M[D_t]$ becomes a filtered ring. We set further

$$R^{(p)}(a) = \{j \,:\, p\text{-ord}\, a^{<j>}{D_t}^{m-j} = p\text{-ord}\, a\}$$

$$a_{p\_pr}(t,x,\xi,\tau) = \sum\nolimits_{j\in R^{(p)}(a)} a_0^{<j>}(t,x,\xi)\tau^{m-j}$$

and call the latter the p-principal symbol of a. The set $\cup_{p>0}\{a_0^{<j>}(t,x,\xi)\tau^{m-j}\}_{j\in R^{(p)}(a)}$ has finite elements and composes the Newton polygon of a.

Let us take $c(t,x,\xi,\tau) = \sum_{j=0}^m c^{<j>}(t,x,\xi)\tau^{m-j}$ a polynomial of τ whose coefficients are homogeneous on ξ respectively. We say that $c(t,x,\xi,\tau)$ is a p-homogeneous polynomial of τ when all $\deg c^{<j>} + p(m-j)$ coincide each other for $0 \le j \le m$. For p-homogeneous c, we call common $\deg c^{<j>} + p(m-j)$ the p-degree of c and denote it by p-$\deg c$. Let us set

$$Y = \{p\text{-}homogeneous\ polynomials\ on\ \tau\}\,.$$

$Y\setminus\{0\}$ is a commutative productive semigroup. The map σ^p from $S_M[D_t]\setminus\{0\}$ to $Y\setminus\{0\}$ defined by $\sigma^p(a) = a_{p\_pr}$ is a homomorphism of the productive semigroup. This is naturally extended to the map from $S_M[D_t]^Q\setminus\{0\}$ to $(Y\setminus\{0\})^Q$ by $\sigma^p(ab^{-1}) = a_{p\_pr}/b_{p\_pr}$ as a homomorphism of the productive group, where $S_M[D_t]^Q$ is the quotient field of $S_M[D_t]$ and $(Y\setminus\{0\})^Q$ is the quotient productive group of $Y\setminus\{0\}$. (By virtue of Ore's property, if $ab^{-1} = a'b'^{-1}$, it holds that $a_{p\_pr}/b_{p\_pr} = a'_{p\_pr}/b'_{p\_pr}$ and the map σ^p is well defined on $S_M[D_t]^Q\setminus\{0\}$.) We put $\sigma^p(0) = 0$. Thus, we can obtain the weighted determinant theory by σ^p following J.Dieudonné[9]. (See also E.Artin[5] and K.Adjamagbo[2], [3].)

In the case of non-quasianalytic classes, we stand on the following simple property mentioned in Section 2;

For a continuous function $f(x)$ on an open set O, the set $\{x \mid f(x)\neq 0\}\cup\{x\mid f(x)=0\}^o$ is open and dense in O, where A^o is the open kernel of A.

By this property, for continuous $\{f_j(x)\}_{1\le j\le d}$, we can find finite disjoint open sets $\{O_h\}_h$ such that the union is dense in O and that $f_j(x)\neq 0$ or else $\equiv 0$ on each O_h. Using this property, we can define p-determinant for matrices with entries in $S\{M_n\,,\,L_n\}[D_t]$ on an open dense set. Of course, we can also take the space of the formal symbols of C^∞-class instead of $S\{M_n\,,\,L_n\}$. The existence of the limit of p-determinant at the boundary of the open dense set is not clear.

Definition 4. *(p-determinant)*
We call the determinant by σ^p of a matrix A with entries in $S[D_t]$ *p-determinant* of A and denote it by p-$\det A$.

Remark 3.1. 1-determinant is just Hufford and Sato-Kashiwara's determinant.

3.2. **properties of p-determinant.**

Following J.Dieudonné[9], we have obtained the elementary properties of p-determinant.

Theorem 3. (Elementary property of p-determinant)

We take $A = (a^{ij})_{1\le i,\, j\le \mathrm{N}}$ and B in $M_{\mathrm{N}}(S[D_t])$.

(1) p-$\det AB = p$-$\det A \cdot p$-$\det B$.

(2) p-$\det A \oplus p$-$\det B = p$-$\det A \cdot p$-$\det B$. (In this case, the sizes of A and B can be different.)

(3) p-determinant is invariant under the similar transformation.

(4)If there are real numbers m_i and n_j such that p-$\operatorname{ord} a^{ij} \le m_i + n_j$ and the ordinary determinant $\det(\sigma^p_{m_i+n_j}(a^{ij}))_{1\le i,\, j\le \mathrm{N}}$ does not vanish, then p-$\det A = \det(\sigma^p_{m_i+n_j}(a^{ij}))$, where $\sigma^p_{m_i+n_j}(a^{ij})$ is $a^{ij}_{p\text{-}pr}$ if p-$\operatorname{ord} a^{ij} = m_i + n_j$, and is 0 if p-$\operatorname{ord} a^{ij} < m_i + n_j$.

Here, on the matrix of the form $P = I_{\mathrm{N}} D_t - A$, $A \in M_{\mathrm{N}}(S^m)$, we give the representation of p-determinant using the element of the normal form in Corollary 2.

Let us set

$$\text{true ord } b_k(h) = r^k_h ,$$

$$M^p_k = \max_{1\le h\le n_k}\{r^k_h + (m+1)(n_k - h) + p(h-1)\}, \tag{3.1}$$

$$R^p_k = \{h : r^k_h + (m+1)(n_k-h) + p(h-1) = M^p_k\}$$

Applying the property (4) in Theorem 3, we have the following.

Proposition 3.1. (Relation between the normal form and p-determinant)

$$p\text{-}\det P \quad = \textstyle\prod_{k=1}^{d} p\text{-}\det Q_k ,$$

$$p\text{-}\det Q_k \quad = \begin{cases} \tau^{n_k} , & (pn_k > M_k) , \\ \tau^{n_k} - \sum_{h\in R^p_k} b_k(h)_0(t,x,\xi)|\xi|^{(m+1)(n_k-h)}\tau^{h-1} , & (pn_k = M_k) , \\ -\sum_{h\in R^p_k} b_k(h)_0(t,x,\xi)|\xi|^{(m+1)(n_k-h)}\tau^{h-1} , & (pn_k < M_k) , \end{cases} \tag{3.2}$$

$$= \textit{the highest p-degree part of the ordinary determinant of } Q_k$$

In case of m.f.s., $|\xi|^{(m+1)(n_k-h)}$ is replaced by ${\xi_1}^{(m+1)(n_k-h)}$.

Thus, p-$\det P$ is a polynomial of τ. On the determinant theory, the regularity property is important. In case of $S = S_H$, as the above P is a polynomial of τ, the meromorphy can occur in (t, x, ξ) space and the proof of Sato-Kashiwara is directly applicable. (We need not transform the pole set to $\xi_1 = 0$.)

Theorem 4. (Regularity of p-determinant)

(1) For $P = I_{\mathrm{N}} D_t - A$, $A \in M_{\mathrm{N}}(S_H)$, p-$\det P$ is a polynomial of τ with holomorphic coefficients on (t, x, ξ).

(2) For a matrix of partial differential operators with holomorphic coefficients on t and x, p-$\det P$ is a polynomial of τ and ξ with holomorphic coefficients on t and x.

A.D'Agnolo and G.Taglialatela[8] algebraically showed the regularity of p-determinant without using the normal form.

3.3. p-evolutive system and Kowalevskian system.

By Proposition 3.1, we have only two cases; 1) there is an unique $p_\circ$ for which $p_\circ$-det P has the term τ^{N} and other terms, 2) p-det P's are always τ^{N} for all $p > 0$. In the former case, we say that P is *$p_\circ$-evolutive* and define the principal part (on the Cauchy problem) of P by $p_\circ$-det P. In the latter case, we say that P is *0-evolutive* and define the principal part by τ^{N}. 0-evolutive operator is essentially an ordinary differential operator. If P is p-evolutive for $p \leq 1$, we say that P is *Kowalevskian.* Our definition of "Kowalevskian system" is different from that in S.Mizohata[32] and M.Miyake[29]. On the other hand, for p-evolutive P ($p > 1$), if every root of p-det $P = 0$ has the positive imaginary part, we say that P is *parabolic* and if every root is real, we do that P is *of Schrödinger type.*

4. Cauchy-Kowalevskaya theorem for system

4.1. Short history.

In 1979, M.Miyake[29] assumed that the coefficients are real analytic and the dimension ℓ of x-space is one and gave the necessary and sufficient condition for the analytic well-posedness on systems introducing the *meromorphic* formal solutions. H.Yamahara and the author[27] and [28] obtained the necessary and sufficient condition for systems in the case of general ℓ. They introduced the formal fundamental solution and estimate it standing on the normal form of systems in the *meromorphic* formal symbol class. M.Miyake[30] further showed that, when $\ell = 1$, one can reduce the analytically well-posed system to a first order one with real analytic coefficients enlarging the size of system.

On the other hand, as the algebraic analysis, M.Kashiwara[11] considered the Cauchy-Kowalevskaya theorem for systems in 1971. He determined the structure of the solution space using the *determinant* of the matrices of pseudo-differential operators introduced by M.Sato and M.Kashiwara[39].

4.2. Complexification and a priori estimate.

We set $A(t,x,D_x) = \sum_{|\alpha|\leq m} A_\alpha(t,x){D_x}^\alpha$ and $P(t,x,D_t,D_x) = I_{N}D_t - A(t,x,D_x)$. The problem (1.1) in the real analytic space is naturally extended to the problem in the holomorphic space in a complex domain. From now on, we consider the problem (1.1) in a complex domain $\Omega \subset \mathbf{C}^{1+\ell}_{t,x}$ and assume that all coefficients of $P(t,x,D_t,D_x)$ are holomorphic there and continuous on its closure. Let $\Omega_{t_\circ}$ be $\{x \in \mathbf{C}^\ell : (t_\circ, x) \in \Omega\}$.

Definition 5. (*The Cauchy-Kowalevskaya theorem = the C-K theorem*)

We say that the Cauchy-Kowalevskaya theorem (= *the C-K theorem*) for $P(t,x,D_t,D_x)$ holds in Ω (or that the Cauchy problem (1.1) is analytically well-posed in Ω) when for each $(t_\circ, x_\circ)$ in Ω, every initial data $u_\circ(x)$ holomorphic in $\Omega_{t_\circ}$ and every right-hand side $f(t,x)$ holomorphic in Ω, there exists a neighborhood ω of $(t_\circ, x_\circ)$ where the Cauchy problem (1.1) has a unique holomorphic solution $u(t,x)$.

We denote the ε-neighborhood of K by K_ε. We say that $v(t,x)$ is holomorphic on a compact set K when v is holomorphic in K^o and continuous on K, where K^o is the open kernel of K. The above proposition implies

Proposition 4.1. (Common existence domain)
For arbitrary compact set K in Ω and arbitrary positive ε, there exists a compact neighborhood K' of K determined by the operator and ε, such that the unique holomorphic solution exists on K' for arbitrary holomorphic initial data on $\bar{K}_{\varepsilon\, t_\circ}$ and arbitrary holomorphic right-hand side on $\bar{K}_\varepsilon$.

When we prove the necessity for the C-K theorem, we need an a priori estimate. For a bounded domain ω in Ω, we set $H(\omega) = \{\ v(t,x) = {}^t(v_1(t,x), \cdots, v_{\mathrm{N}}(t,x))\ :\ v_j$ *is holomorphic in* ω *and continuous on* $\bar{\omega}$, $(1 \le j \le \mathrm{N})\ \}$. It is a Banach space by the norm $||v||_\omega = \max_{1\le j\le \mathrm{N}} \max_{(t,x)\in\bar{\omega}} |v_j(t,x)|$.
The following was essentially given in S.Mizohata.

Proposition 4.2. (A priori estimate, [32])
If the C-K theorem for P holds in Ω, for arbitrary compact set K and arbitrary positive number ε there exist a compact neighborhood K' of K and a positive constant C independent of $u_\circ$ and f such that

$$||u||_{K'} \le C(||u_\circ||_{K_{\varepsilon\, t_\circ}} + ||f||_{K_\varepsilon}) \ , \tag{4.1}$$

where u is the solution of (1.1).

4.3. **Homogeneous problem and the formal fundamental solution.**
Let us consider the homogeneous Cauchy problem:

$$\begin{cases} P(t,x,D_t,D_x)u \equiv D_t u - A(t,x,D_x)u = 0 \quad , \\ \\ u(t_\circ, x) = u_\circ(x) \quad . \end{cases} \tag{4.2}$$

If we can construct the fundamental solution which has an estimate uniform on $t_\circ$, the inhomogeneous problem (1.1) is solved by the Duhamel principle. Therefore, from now on, we consider the problem (4.2).

By the relation $D_t u = A(t,x,D_x)u$, ${D_t}^k u$ is represented by a linear combination of the derivatives on x of u :

$$ {D_t}^k u = A[k](t,x,D_x)u \quad , \quad (k \ge 0)\,. \tag{4.3}$$

$\{A[k]\}_{k=0}^\infty$ satisfies the recurrence formula:

$$\begin{cases} A[0] = I_{\mathrm{N}} \quad , \\ A[k] = A[k-1]\circ A + (A[k-1])_t \quad , \quad (k\ge 1) \quad , \end{cases} \tag{4.4}$$

where $(A)_t$ is obtained by operating D_t to the coefficients of A.

The formal fundamental solution of the problem (4.2) is given by

$$U(t,x,D_x;t_\circ) = \sum_{k=0}^{\infty} \frac{(\sqrt{-1}(t-t_\circ))^k}{k!} A[k](t_\circ,x,D_x) \quad . \tag{4.5}$$

As $A[k]$ is differential operator and $A[k]=\sum_{i\geq 0}A[k]_i$ is a finite sum, when it satisfies (4.8) in Proposition 4.3 below, $\sum_{k=0}^{\infty}\{(\sqrt{-1}(t-t_\circ))^k/k!\}A[k](t_\circ,x,D_x)u_\circ$ converges in a neighborhood ω of $(t_\circ,x_\circ)$ for arbitrary $u_\circ$ in $H(\Omega_{t_\circ})$ and $U(t,x)$ is the true fundamental solution in ω.

4.4. **Cauchy-Kowalevskaya theorem.**

Now we announce our theorem on the Cauchy-Kowalevskaya theorem for systems.

Theorem 5. (Cauchy-Kowalevskaya theorem for systems, [27] and [28])
The following conditions are equivalent.

1) The Cauchy-Kowalevskaya theorem for $P(t,x,D_t,D_x)$ holds in Ω.

2) The lower order terms in the normal form (2.5) *satisfy*

$$\operatorname{ord} b_k(h)\leq 1-m(n_k-h)\,,\quad (1\leq h\leq n_k\,,\,1\leq k\leq d). \tag{4.6}$$

3) $P(t,x,D_t,D_x)$ is reduced to a first order system through a similar transformation by an element in $GL(\mathtt{N};S_M)$.

4) 1-det P *is of degree* $\mathtt{N}$ *: the size of* P .

5) P is Kowalevskian in our sense, that is, p-evolutive for $0\leq p\leq 1$.

6) There exists a natural number $k_\circ$ such that

$$\operatorname{ord} A[k](t,x,D_x)\leq k+k_\circ\,,\quad (k\in\mathbf{Z}_+). \tag{4.7}$$

The equivalences between *2)*, *4)* and *5)* are obvious by virtue of Proposition 3.1. The proof from *1)* to *2)* is the main part of the proof of the necessity for the C-K theorem. The system is microlocally reduced to a backward heat equation of order greater than 1 for a new unknown. We obtain a microlocal energy estimate of this equation in a real domain, which contradicts the a priori estimate (4.1). From *2)* to *3)* is almost trivial. The proof from *3)* to *7)* below: more detailed version of *6)* is the essential part of the proof of the sufficiency. By the estimate (4.8), the formal fundamental solution (4.5) converges and operates on the holomorphic functions. Thus, there exist $\rho_\circ$ and δ ($\rho_\circ>0$, $0<\delta<1$) determined by the operator such that, for an arbitrary $\rho\leq\rho_\circ$, $u_\circ$ in $H(B_\rho(x_\circ))$ and f in $H(B_\rho((t_\circ,x_\circ))$, the unique holomorphic solution u exists in $B_{\delta\rho}((t_\circ,x_\circ))$. This means that *7)* implies *1)*. The proof from *7)* to *6)* is trivial and that from *6)* to *2)* is easy.

Proposition 4.3. (Estimate of $A[k](t,x,\xi)$, [27] and [28])
Condition 3) implies 7):
7) For an arbitrary compact set K in Ω, there exist a positive integer $k_\circ$ and positive constants C, R and $R_\circ$ independent of k, for which the following estimates hold on

$K \times \mathbf{C}^\ell$;

(4.8)
$$|A[k]_{i(\alpha)}^{(\beta)}(t,x,\xi)| \le CR_\circ{}^k \sum_{h=0}^k \ R^{k-h+i+|\alpha|+|\beta|}(k-h)!i!|\alpha|!|\beta|!||\xi||^{k_\circ+h-i-|\beta|}$$
$$(i \in \mathbf{Z}_+\,,\ \alpha \in \mathbf{Z}_+{}^{1+\ell}\,,\ \beta \in \mathbf{Z}_+{}^{\ell}).$$

This Proposition is shown first in $K \times \mathbf{C}^\ell \backslash \Sigma$ for a subvariety Σ. As $A[k]$ is holomorphic, The estimate (4.8) holds in $K \times \mathbf{C}^\ell$ by the maximum principle.

4.5. **Cauchy-Kowalevskaya theorem of Nagumo Type.**

M.Nagumo[34] showed that one can obtain a unique solution, real analytic on x and of C^1-class on t, if $m \le 1$ in (1.1) and the coefficients are real analytic on x and continuous on t. When $m \ge 2$, does the continuity on t of the coefficients and one of 2) to 6) in Theorem 5 assure the existence of a solution? The answer is No.

Example 1. *(announced at ICM'98,* [24] *)*

$$P_6 = I_{\mathrm{N}}\partial t - \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & & & 0 & \mu(t) \\ \nu(t) & & & & 0 \end{pmatrix} (\partial x)^m\,, \qquad \mathrm{N} \times \mathrm{N}\,,$$

where $\mu(t)$ and $\nu(t)$ are non-negative and have the supports in $[0, \infty)$ and $\mu(t)\nu(t) \equiv 0$. More precisely, let us set $t_{2n-1} = t_{2n} = \sum_{j=n}^{\infty} j^{-1}(\log j)^{-2}$ ($\{t_n\}$ is a monotonically decreasing sequence with the limit zero) and take a natural number p ,

$$\mu(t) = \begin{cases} (t_{2n-1} - t)^p (t - t_{2n})^p & t \in (t_{2n},\ t_{2n-1})\,, \\ 0 & otherwise\,, \end{cases}$$

and

$$\nu(t) = \begin{cases} (t_{2n} - t)^p (t - t_{2n+1})^p & t \in (t_{2n+1},\ t_{2n})\,, \\ 0 & otherwise\,. \end{cases}$$
$$(n \in \mathbf{N})$$

$\mu(t)$ and $\nu(t)$ belong to $C^{p-1,\,1}(\mathbf{R})$, that is, the (p-1)-th derivatives are Lipschitzian. As $\mu(t)\nu(t) \equiv 0$, P_6 is 0-evolutive at every point.

For arbitrary small positive ε, we can find $t_{2q} \le \varepsilon$. Let us take $t_\circ = t_{2n+2q}$. We can concretely solve the Cauchy problem for P_6 with the initial data $u_{\circ i} = 0$ $(0 \le i \le \mathrm{N} = 1)$, $u_{\circ\mathrm{N}} = \varphi(x) = \exp(\rho x)$ and the right-hand side $f(t,x) = 0$ from $t_\circ$ to t_{2q} and the solution u has the estimate

$$|u_{\mathrm{N}}(t_{2q}, 0)|$$
$$> \rho^{nm\mathrm{N}} \int_{t_{2n+2q-2k+1}}^{t_{2n+2q-2k}} \int_{t_{2n+2q-2k+2}}^{s_{\mathrm{N}}^{(k)}} \cdots \int_{t_{2n+2q-2k+2}}^{s_2^{(k)}} \prod_{k=1}^{n} \nu(s_{\mathrm{N}}^{(k)})\mu(s_1^{(k)})ds_1^{(k)} \cdots ds_{\mathrm{N}}^{(k)}$$
$$\ge \rho^{m\mathrm{N}n}\{K_\circ \prod_{j=q+1}^{n+q} j(\log j)^2\}^{-(4p+\mathrm{N})}\,.$$

On the other hand, if the Cauchy-Kowalevskaya theorem of Nagumo type holds, we have the same a priori estimate as Proposition 4.2 and the following must hold

$$|u_{\mathtt{N}}(t_{2q},0)| \leq C\exp(K_1\rho)\,.$$

Here, $K_\circ$ and K_1 are positive constants. If $m\mathtt{N} > 4p + \mathtt{N}$, taking $\rho = n$ and making n tend to infinity, the both estimates are not compatible. For the detail, see W.Matsumoto[24].

Thus, in order to assure the Cauchy-Kowalevskaya theorem of Nagumo type for P_6, we need the differentiability on t at least up to $(m-1)\mathtt{N}/4$.

The author propose a conjecture on the Cauchy-Kowalevskaya theorem of Nagumo type for systems. Let us take $\Omega = [T_1, T_2] \times \Omega'$. We denote the space of real analytic functions in Ω' by $A(\Omega')$,

Conjecture (Conjecture on C-K theorem of Nagumo type for systems)

If all coefficients of P belong to $C^\infty([T_1, T_2]; A(\Omega'))$, the assertion in Theorem 5 also holds.

The equivalences between *2)*, $\cdots$, *6)* are rather easily seen. The assertion from *1)* to *2)* is also shown by the same way as the proof in Subsection 4.4, because the analyticity on x is essential but that on t is not required in the proof. Therefore, the sufficiency of *2)* or *3)* or $\cdots$ or *6)* is open. Recently, M.Murai, T.Nagase and the author[26] obtained an affirmative result for the most simple system but non-trivial case: $m = 2$ and $\mathtt{N} = 2$. (The dimension of x space ℓ is free.).

5. Levi condition for the C^∞ well-posedness

We consider the Cauchy problem of a first order system of partial differential equations ((1.1) with $m = 1$) with constantly multiple characteristic roots. If the first order part has only the zero characteristic root, the Levi condition is equivalent to 0-evolution, that is, essentially it is an ordinary differential operator of D_t. When coefficients are real analytic, this is necessary and sufficient for the C^∞ well-posedness. On the other hand, in case of non-quasianalytic coefficients, even if the first order part has constant coefficients, this condition does not rest sufficient. (See W. Matsumoto [20], Remark 4.1. See also W. Matsumoto[16] and [18].)

Through this section, we assume the analyticity of all coefficients.

5.1. p-determinant associated with a characteristic root.

Let $\lambda_k(t,x,\xi)$ be the characteristic roots of constant multiplicity m_k ($1 \leq k \leq d$) of the first order part of P. By virtue of the assumption of the constant multiplicity, every characteristic roots is smooth. In order to describe the Levi condition in an invariant form, we introduce p-determinant associated with $\lambda_k(t,x,\xi)$.

Let p be a rational number such that $0 \leq p < 1$. As $S_M[D_t] = S_M[D_t - \lambda_k(t,x,\xi)]$, every $a(t,x,\xi,D_t)$ is represented as $a(t,x,\xi,D_t) = \sum_{j=0}^m a^{<j>}(t,x,\xi)(D_t-\lambda_k)^{m-j}$, $a^{<j>} = \sum_{i=0}^\infty a_i^{<j>} \in S_M$. We reset the order of $a^{<j>}$ to its true order. Let us set

$$p\text{-ord}_{\lambda_k} a^{<j>}(t,x,\xi)(D_t - \lambda_k)^{m-j} = \operatorname{ord} a^{<j>} + p(m-j)$$

$$p\text{-ord}_{\lambda_k} a = \max_{0\leq j\leq m} p\text{-ord}_{\lambda_k} a^{<j>}(t,x,\xi)(D_t - \lambda_k)^{m-j}$$

and call them the p-order associated with λ_k. By p-order associated with λ_k, $S_M[D_t-\lambda_k]$ becomes a filtered ring. We set further

$$R^{(p)}_{\lambda_k}(a)=\{j\ :\ p\text{-ord}_{\lambda_k}a^{<j>}(D_t-\lambda_k)^{m-j}=p\text{-ord}_{\lambda_k}a\}$$

$$a_{p\_pr\,\lambda_k}(t,x,\xi,\tau)=\sum\nolimits_{j\in R^{(p)}_{\lambda_k}(a)}a_0^{<j>}(t,x,\xi)(\tau-\lambda_k)^{m-j}$$

and call the latter the p-principal symbol of $a(t,x,\xi)$ associated with λ_k. The set $\cup_{p>0}\{a_0^{<j>}(t,x,\xi)(\tau-\lambda_k)^{m-j}\}_{j\in R^{(p)}_{\lambda_k}(a)}$ has finite elements and composes the Newton polygon of a associated with λ_k.

We define the p-homogeneous polynomial on $\tau-\lambda_k$ by the same way in Subsection 3.1. Let us set

$$Y_{\lambda_k}=\{p\textit{-homogeneous polynomials on }\tau-\lambda_k\}\,.$$

Y_{λ_k} is a commutative productive semigroup. The map $\sigma^p_{\lambda_k}$ from $S_M[D_t-\lambda_k]\setminus\{0\}$ to $Y_{\lambda_k}\setminus\{0\}$ defined by $\sigma^p_{\lambda_k}(a)=a_{p\_pr\,\lambda_k}$ is a homomorphism of the productive semigroup. This is naturally extended to the map from $S_M[D_t-\lambda_k]^Q\setminus\{0\}$ to $(Y_{\lambda_k}\setminus\{0\})^Q$ by $\sigma^p_{\lambda_k}(ab^{-1})=a_{p\_pr\,\lambda_k}/b_{p\_pr\,\lambda_k}$ as a homomorphism of the productive group. We put $\sigma^p_{\lambda_k}(0)=0$. Thus, we can obtain the weighted determinant theory by $\sigma^p_{\lambda_k}$ following J.Dieudonné[9]. In case of non-quasianalytic classes, by the same reason as in Subsection 3.1, we can also obtain it on an open dense set.

Definition 6. *(p-determinant associated with λ_k)*
We call the determinant of a matrix A with entries in $S[D_t-\lambda_k]$, by $\sigma^p_{\lambda_k}$ *p-determinant* of A *associated with* λ_k and denote it by p-$\det_{\lambda_k}A$.

We can obtain the corresponding properties in Theorem 3.

On the matrix of the form $P=I_N D_t-A$, $A\in M_N(S^1)$ ($m=1$), we give the representation of p-determinant using the element of the normal form in Theorem 1.

Let us set

$$\text{true order } d_{kj}(h)=r_h^{kj}\,,$$

$$M^p_{kj}=\max\nolimits_{1\le h\le n_{kj}}\{r_h^{kj}+(n_{kj}-h)+p(h-1)\},\tag{5.1}$$

$$R^p_{kj}=\{h: r_h^{kj}+(n_{kj}-h)+p(h-1)=M^p_{kj}\}$$

We have the following.

Proposition 5.1. (Relation between the normal form and p-determinant a.w. λ_k)

(5.2)

$$p\text{-}\det_{\lambda_k} P = \prod_{i=1}^{d} \prod_{j=1}^{d_i} p\text{-}\det_{\lambda_k} P_{ij}\,,$$

$$p\text{-}\det_{\lambda_k} P_{kj}$$

$$= \begin{cases} (\tau-\lambda_k)^{n_{kj}}\,, & (pn_{kj} > M^p_{kj})\,, \\ (\tau-\lambda_k)^{n_{kj}} - \sum_{h\in R^p_{kj}} d_{kj}(h)_0(t,x,\xi)|\xi|^{n_{kj}-h}(\tau-\lambda_k)^{h-1}\,, & (pn_{kj} = M^p_{kj})\,, \\ -\sum_{h\in R^p_{kj}} d_{kj}(h)_0(t,x,\xi)|\xi|^{n_{kj}-h}(\tau-\lambda_k)^{h-1}\,, & (pn_{kj} < M^p_{kj})\,, \end{cases}$$

$= \textit{the highest p-degree part a.w. } \lambda_k \textit{ of the ordinary determinant of } P_{kj}$

$$(1 \le j \le d_k)$$

$$p\text{-}\det_{\lambda_k} P_{ij} = (\lambda_k - \lambda_i)^{n_{ij}} \quad (1 \le i \ne k \le d,\ 1 \le j \le n_{ij})$$

In the case of m.f.s., $|\xi|^{n_{kj}-h}$ *is replaced by* ${\xi_1}^{n_{kj}-h}$.

In case of $S = S_H$, we can obtain the regularity of p-determinant associated with λ_k corresponding to (1) in Theorem 4.

By Proposition 5.1, we have only two cases; 1) There exists a unique $p_\circ$ for which $p_\circ\text{-}\det_{\lambda_k} P/\prod_{1\le i\le d,\, i\ne k}(\lambda_k-\lambda_i)^{m_i}$ has the term τ^{m_k} and other terms,
2) $p_\circ\text{-}\det_{\lambda_k} P/\prod_{1\le i\le d,\, i\ne k}(\lambda_k-\lambda_i)^{m_i}$'s are τ^{m_k} for all $0<p<1$. In the former case, we say that P is $p_\circ$-evolutive with respect to λ_k and define the second principal part (on the Cauchy problem) of P by $p_\circ\text{-}\det_{\lambda_k} P/\prod_{1\le i\le d,\, i\ne k}(\lambda_k-\lambda_i)^{m_i} = \prod_{1\le j\le d_k} p_\circ\text{-}det_{\lambda_k} P_{kj}$ and denote it by $p_\circ\text{-}\det'_{\lambda_k} P$. In the latter case, we say that P is 0-evolutive with respect to λ_k and define the second principal part by τ^{m_k}. 0-evolutive operator with respect to λ_k is essentially an ordinary differential operator along the bicharacteristic strip of λ_k.

5.2. Levi condition.

Let us make clear the definition of C^∞ well-posedness of the Cauchy problem. For the simplicity, we assume that Ω is bounded.

Definition 7. (*C^∞ well-posedness*)

We say that the Cauchy problem is C^∞ well-posed in Ω when for each $(t_\circ, x_\circ)$ in Ω, there exists a neighborhood ω of $(t_\circ, x_\circ)$ where every initial data $u_\circ(x)$ of C^∞-class in $\bar\Omega_{t_\circ}$ and every right-hand side $f(t,x)$ of C^∞-class in $\bar\Omega$, the Cauchy problem (1.1) has a unique solution $u(t,x)$ in $C^\infty(\bar\omega)$.

We give an a priori estimate. For a bounded domain ω in Ω, we set $F(\omega) = \{\, v(t,x) = {}^t(v_1(t,x),\cdots,v_{\mathrm N}(t,x)) : v_j \in C^\infty(\bar\omega) \quad, (1 \le j \le \mathrm{N})\,\}$. It is a Fréchet space by the semi-norms $||v||_{n\,\omega} = \max_{1\le j\le \mathrm N} \sum_{|\alpha|\le n} \max_{(t,x)\in\bar\omega} |D_{tx}{}^\alpha v_j(t,x)|$.

Proposition 5.2. (A priori estimate of C^∞ well-posedness)

If the Cauchy problem for P is C^∞ well-posed in Ω, for arbitrary q in $\mathbf{Z}_+$, there exist r in $\mathbf{Z}_+$ and a positive constant C independent of $u_\circ$ and f such that

(5.3) $$||u||_{q\bar{\omega}} \leq C(||u_{\circ}||_{r\bar{\omega}_{t_{\circ}}} + ||f||_{r\bar{\omega}}) ,$$

where u is the solution of (1.1).

S.Mizohata showed that the following.

Proposition 5.3. (Hyperbolicity, [31])

In order that the Cauchy problem is C^∞ well-posed in Ω, the characteristic roots $\lambda_k(t,x,\xi)$ ($1 \leq k \leq d$) must be real.

Now we announce our theorem on the C^∞ well-posedness for systems.

Theorem 6. (C^∞ well-posedness for systems, [20] Section 4 and [25])

We assume that every characteristic root $\lambda_k(t,x,\xi)$ is real and has the constant multiplicity m_k ($1 \leq k \leq d$). The following conditions are equivalent.

i) The Cauchy problem for P is C^∞ well-posed in Ω.

ii) The lower order terms in the normal form (2.4) *with $m = 1$ satisfy*

(5.4) $$\operatorname{ord} d_{kj}(h) \leq -(n_k - h) , \quad (1 \leq h \leq n_{kj} , 1 \leq j \leq d_k , 1 \leq k \leq d).$$

iii) P is reduced to a first order system with a diagonal first order part through a similar transformation by an element in $GL(\mathtt{N}; S_M)$.

iv) P is 0-evolutive with respect to λ_k ($1 \leq k \leq d$).

Remark 5.1. The conditions in Theorems 5 and 6 are similar each other and the proofs also similar in the case of real analytic coefficients. In the case of non-quasianalytic coefficients, the proofs on the necessity also hold. However, not only the proofs of the sufficiency loose the validity but also the phenomena themselves become different.

Remark 5.2. In the case of non-quasianalytic coefficients, under the equivalent condition *ii*), *iii*) or *iv*), the greatest space for the well-posedness of the Cauchy problem was studied by W.Matsumoto[18] for 2×2 systems. It depends on the regularity of coefficients. For example, when coefficients belong to a Gevrey class, it is much bigger than the union of the Gevrey classes.

Remark 5.3. A.D'Agnolo and G.Taglialatela[8] also discussed another representation of the Levi condition using their determinant theory.

The equivalences between *ii*) and *iv*) is obvious by virtue of Proposition 5.1. The proofs from *ii*) to *iii*) is evident. The proofs from *i*) to *ii*) is rather easy and we need not assume the real analyticity of the coefficients. From *iii*) to *i*), we take four steps. First, we separate each eigenvalue and consider each block independently (W.Matsumoto[23] and T.Nishitani[35]). Second, we reduce the eigenvalue in each block to zero by a Fourier integral operator (H.Kumano-go[14]). Thus, each block becomes 0-evolutive, that is, essentially an ordinary differential system and we can obtain a similar estimate as 7) in Proposition 4.3 replacing $||\xi||^{k_\circ+h-i-|\beta|}$ by $|\xi|^{k_\circ-i-|\beta|}$. Third, we construct a true symbol from

$A[k]$ (L.Boutet de Monvel and P.Krée[7], L.Boutet de Monvel[6] and W.Matsumoto[19]). This gives a parametrix acting on C^∞-functions. Finally, we obtain the fundamental solution from the parametrix. (See H. Kumano-go[13].)

References

[1] K.Adjamagbo; *Problème de Cauchy non caractéristique pour le système général d'équations différentielles linéaires*, C. R. Acad. Sc. Paris, **294**, Série I, (1982), 159-162.

[2] ——; *Panorama de la théorie des déterminants sur un anneau non commutatif*, Bull.Sc.Math., 2e Série, **117**, (1993), 401-420.

[3] ——; *Les fondements de la théorie des déterminants sur un domaine de Ore*, Thèse Doc. État, Univ. Paris VI (1991).

[4] V.I.Arnold; *Matrices depending on parameters*, Uspehi Mat Nauk, **26-2 (158)** (1971), 101-114, (English translation) Russ. Math. Surveys, **26-2**(1971), 29-43.

[5] E.Artin; *Geometric Algebra*, Chap. IV, 1, Interscience Publishers (1957).

[6] L.Boutet de Monvel; *Opérateurs pseudo-différentiels analytiques et opérateurs d'ordre infini* , Ann.Inst.Fourier, Grenoble, **22** (1972), 229-268.

[7] L.Boutet de Monvel and P.Krée; *Pseudo-differential operators and Gevrey classes* , Ann.Inst.Fourier, Grenoble, **17** (1967), 295-323.

[8] A.D'Agnolo and G.Taglialatela; *Sato-Kashiwara determinant and Levi conditions for systems*, (Preliminary version).

[9] J.Dieudonné; *Les déterminants sur un corps non commutatif*, Bull.Soc.Math.France, **71** (1943), 27-45.

[10] G.Hufford; *On the characteristic matrix of a matrix of differential operators*, Jour. Diff. Eq. **1** (1965), 27-38.

[11] M.Kashiwara; *Algebraic study of systems of partial differential equations*, Mém.Soc.Math.France, **63** (1995), (Translation of M.Kashiwara's thesis in 1971 written in Japanese to English by A.D'Agnolo and J.-P.Schneiders).

[12] S.von Kowalevsky; *Zur Theorie der partiellen Differentialgleichungen*, Jour.reine angew.Math., **80** (1875), 1-32.

[13] H.Kumano-go; *A calculus of Fourier integral operators on* $\mathbf{R}^n$ *and the fundamental solution for an operator of hyperbolic type*, Comm.Part.Diff.Eq. **1** (1976), 1-44.

[14] ——; *Pseudo-Differential Operators* , Chap.10, MIT Press (1981).

[15] E.E.Levi; *Caracteristiche multiple e problema di Cauchy*, Ann.Math.Pura Appl., Ser.III, **16** (1909), 161-201.

[16] W.Matsumoto; *On the conditions for the hyperbolicity of systems with double characteristics, I*, Jour.Math.Kyoto Univ. **21** (1981), 47-84.

[17] ——; *On the conditions for the hyperbolicity of systems with double characteristics, II*, Jour.Math.Kyoto Univ. **21** (1981), 251-271.

[18] ——; *Sur l'espace de données admissibles dans le problème de Cauchy*, C.R.Acad.Sc.Paris, Série I **292** (1981), 621-623.

[19] ——; *Theory of pseudo-differential operators of ultradifferentiable class*, Jour.Math.Kyoto Univ., **27** (1987), 453-500.

[20] ——; *Normal form of systems of partial differential and pseudo-differential operators in formal symbol classes* , Jour.Math.Kyoto Univ. **34** (1994), 15-40.

[21] ——; *Levi condition for general systems*, Physics on Manifolds, Proc.Intern.Colloq., in honour of Y.Choquet-Bruhat, 1992, Ed. M.Flato et al. Kluwer Academic Publishers (1994), 303-307.

[22] ______; *The Cauchy problem for systems – through the normal form of systems and the theory of the weighted determinant –*, Séminaire de E.D.P., Exposé XVIII, Centre de Mathématiaue, Ecole Polytechnique, (1998-99).
[23] ______; *Direct proof of the perfect block diagonalization of systems of pseudo-differential operators in the ultradifferentiable classes*, (To appear in Jour.Math.Kyoto Univ.).
[24] ______; *On the Cauchy Kowalevskaya theorem of Nagumo type for systems* , (to appear).
[25] ______; *Levi condition for systems with characteristics of constant multiplicity*, (to appear).
[26] W.Matsumoto, M.Murai and T.Nagase; *On the necessary and sufficient condition for the Cauchy-Kowalevskaya theorem of Nagumo type - one of the simplest cases -*, (to appear).
[27] W.Matsumoto and H.Yamahara; *On the Cauchy-Kowalevskaya theorem for systems*, Proc.Japan Acad., **67**, Ser.A (1991), 181-185.
[28] ______; *The Cauchy-Kowalevskaya theorem for systems*, (to appear).
[29] M.Miyake; *On Cauchy-Kowalevski's theorem for general systems*, Publ.RIMS, Kyoto Univ. **15** (1979), 315-337.
[30] ______; *Reduction to the first order systems of the Kowalevskian systems in the sense of Volevič*, Publ.RIMS, Kyoto Univ. **15** (1979), 339-355.
[31] S.Mizohata; *Some remarks on the Cauchy problem*, Jour.Math.Kyoto Univ. **1** (1961), 109-127.
[32] ______; *On Kowalevskian systems*, Uspehi Mat.Nauk. **29** (1974), translated in English: Russ.Math.Survey **29** (1974), 223-235.
[33] ______; *On the hyperbolicity in the domain of real analytic functions and Gevrey classes*, Hokkaido Math.Jour. **12** (1983), 298-310.
[34] M.Nagumo; *Über des Anfangswertproblem partieller Differentialgleichungen* Japan Jour.Math. **18** (1941-43), 41-47.
[35] T.Nishitani; *On the Lax-Mizohata theorem in the analytic and Gevrey classes*, Jour.Math.Kyoto Univ. **18** (1978), 509-521.
[36] O.Ore; *Linear equations in non-commutative fields*, Ann.Math. **32** (1931), 463-477.
[37] V.M.Petkov; *On the Cauchy problem for first-order hyperbolic systems with multiple characteristics* Dokl.Acad.Nauk.SSSR, **209** (1973), 795-797, (English translation) Soviet Math.Dokl. **14** (1973), 1-13.
[38] ______; *Microlocal forms for hyperbolic systems*, Math.Nachr. **93** (1979), 117-131.
[39] M.Sato and M.Kashiwara; *The determinant of matrices of pseudo-differential operators*, Proc.Japan Acad. **51**, Ser.A, (1975), 17-19.
[40] G.Taglialatela and J.Vaillant; *Conditions invariantes d'hyperbolicité des systèmes et réduction des systèmes*, Bull.Sci.Math., **120** (1996), 19-97.
[41] J.Vaillant; *Caractéristiques multiples et bicaractéristiques des systèmes d'équations aux dérivées partielles linéaires et à coefficients constantes*, Ann.Inst.Fourier, Grenoble, **51** (1965), 225-311.
[42] ______; *Conditions d'hyperbolicité des systèmes*, C.R.Acad.Sci.Paris, **313** (1991), 227-230.
[43] ______; *Conditions invariantes pour un système du type conditions de Levi*, Physics on Manifolds, Proc.Intern.Colloq., in honour of Y.Choquet-Bruhat, 1992, Ed. M.Flato et al. Kluwer Academic Publishers (1994),

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