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Existence of weakly singular solutions to linear partial differential equations with holomorphic coefficients

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§0 Introduction

Let $P(z,\partial)$ be a linear partial differential operator with holomorphic coefficients in a neighborhood Ω of $z=0$ in $\mathbb{C}^{d+1}$. Consider the equation

$$(0.1) \qquad P(z,\partial)u(z)=f(z),$$

where $f(z)$ is holomorphic except on the surface $K=\{z_0=0\}$, but $f(z)$ is *weakly singular* on K. In the present paper *"weakly singular"* means that $f(z)$ has an asymptotic expansion $f(z)\sim\sum_{n=0}^{\infty}f_n(z')z_0^n$ with respect to z_0 as $z_0\to 0$ in some sectorial domain. We study the existence of a solution $u(z)$ with an asymptotic expansion such as $f(z)$. Firstly we remark that if we do not restrict the behaviors of solutions near K, there exists a solution $u(z)$ with singularities on K under some conditions on the principal symbol of $P(z,\partial)$. But the singularities of $u(z)$ may be much stronger than $f(z)$ (see [1], [2], [4] and [8]).

The behaviors and growth properties of solutions near singularities are studied in [5], [6] and [7] and they are characterized by the lower order terms of operators. The lower order terms of operators are important to know the behaviors of solutions. However the existence of solutions with asymptotic expansions is not studied in those papers. So we study it in this paper.

§1 Notations and Results

In order to state the problem we concern and results more precisely we give simply notations and definitions. The coordinate of $\mathbb{C}^{d+1}$ is denoted by $z=(z_0,z_1,\cdots,z_d)=(z_0,z')\in\mathbb{C}\times\mathbb{C}^d$. $|z|=\max\{|z_i|;\ 0\le i\le d\}$ and $|z'|=\max\{|z_i|;\ 1\le i\le d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\cdots,\xi_d)$. $\mathbb{N}$ is the set of all nonnegative integers, $\mathbb{N}=\{0,1,2,\cdots\}$. The partial differentiation is denoted by $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial_1,\cdots,\partial_d)=$

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(∂_0, ∂'). For a multi-index $\alpha = (\alpha_0, \alpha') \in \mathbb{N} \times \mathbb{N}^d$, $|\alpha| = \alpha_0 + |\alpha'| = \sum_{i=0}^{d} \alpha_i$. We use the notations $\partial^\alpha = \prod_{i=0}^{d} \partial_i^{\alpha_i}$ and $\xi^{\alpha'} = \prod_{i=1}^{d} \xi_i^{\alpha_i}$.

Next let us define spaces of holomorphic functions in some regions. Let $\Omega = \Omega_0 \times \Omega'$ be a polydisc centered at $z = 0$, where $\Omega_0 = \{z_0 \in \mathbb{C}^1;\ |z_0| < R_0\}$ and $\Omega' = \{z' \in \mathbb{C}^d; |z'| < R\}$ for some positive constants R_0 and R. Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\};\ |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$, $\Omega(\theta)$ is sectorial with respect to z_0. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega'),\ \mathcal{O}(\Omega(\theta)))$ is the set of all holomorphic functions on Ω (*resp.* Ω', $\Omega(\theta)$).

Definition 1.1. (i) $Asy_{\{\kappa\}}(\Omega(\theta))$ $(0 < \kappa \leq +\infty)$ *is the set of all* $u(z) \in \mathcal{O}(\Omega(\theta))$ *such that for any* θ' *with* $0 < \theta' < \theta$

$$(1.1) \qquad |u(z) - \sum_{n=0}^{N-1} u_n(z') z_0^n| \leq AB^N |z_0|^N \Gamma(\frac{N}{\kappa} + 1) \quad z \in \Omega(\theta'),$$

where $u_n(z') \in \mathcal{O}(\Omega')$ $(n \in \mathbb{N})$, *holds for constants* $A = A(\theta')$ *and* $B = B(\theta')$.
(ii) $Asy_{\{0\}}(\Omega(\theta))$ *is the set of all* $u(z) \in \mathcal{O}(\Omega(\theta))$ *such that for any* θ' *with* $0 < \theta' < \theta$

$$(1.2) \qquad |u(z) - \sum_{n=0}^{N-1} u_n(z') z_0^n| \leq A_N |z_0|^N \quad z \in \Omega(\theta'),$$

where $u_n(z') \in \mathcal{O}(\Omega')$ $(n \in \mathbb{N})$, *holds for constants* $A_N = A(N, \theta')$ *depending on* N *and* θ'

We say that $u(z) \in Asy_{\{\kappa\}}(\Omega(\theta))$ has an asymptotic expansion with Gevrey exponent (or index) κ. $u(z) \in Asy_{\{+\infty\}}(\Omega(\theta))$ means that $u(z)$ is holomorphic at $z = 0$. $u(z) \in Asy_{\{0\}}(\Omega(\theta))$ means that it has merely an asymptotic asymptotic expansion. So put $Asy(\Omega(\theta)) := Asy_{\{0\}}(\Omega(\theta))$.

Now let $P(z, \partial)$ be an m-th order linear partial differential operator with holomorphic coefficients in a neighborhood of $z = 0$,

$$(1.3) \qquad P(z, \partial) = \sum_{|\alpha| \leq m} a_\alpha(z) \partial^\alpha.$$

We introduce the characteristic polygon of $P(z, \partial)$ with respect to K, which is important to study the behaviors of solutions of (0.1) near K. Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0. Hence if $a_\alpha(z) \not\equiv 0$, $a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z)$ with $b_\alpha(0, z') \not\equiv 0$ and for $a_\alpha(z) \equiv 0$ put $j_\alpha = \infty$. Put

$$(1.4) \qquad e_\alpha = j_\alpha - \alpha_0,$$

where $e_\alpha = +\infty$ if $a_\alpha(z) \equiv 0$. We denote by $\Pi(a,b)$ the set $\{(x,y) \in \mathbb{R}^2; x \leq a, y \geq b\}$. The characteristic polygon of Σ is defined by

$$\Sigma := \textit{the convex hull of } \bigcup_\alpha \Pi(|\alpha|, e_\alpha).$$

The boundary of Σ consists of a vertical half line $\Sigma(0)$ and a horizontal half line $\Sigma(p^*)$ and p^*-1 segments $\Sigma(i)$ $(1 \leq i \leq p^*-1)$ with slope γ_i, $0 = \gamma_{p^*} < \gamma_{p^*-1} < \cdots < \gamma_1 < \gamma_0 = +\infty$.

Let $\{(k_i, e(i)) \in \boldsymbol{R}^2; 0 \leq i \leq p^*-1\}$ be vertices of Σ, where $0 \leq k_{p^*-1} < \cdots < k_i < k_{i-1} < \cdots < k_0 = m$. So the endpoints of $\Sigma(i)$ $(1 \leq i \leq p^*-1)$ are $(k_{i-1}, e(i-1))$ and $(k_i, e(i))$.

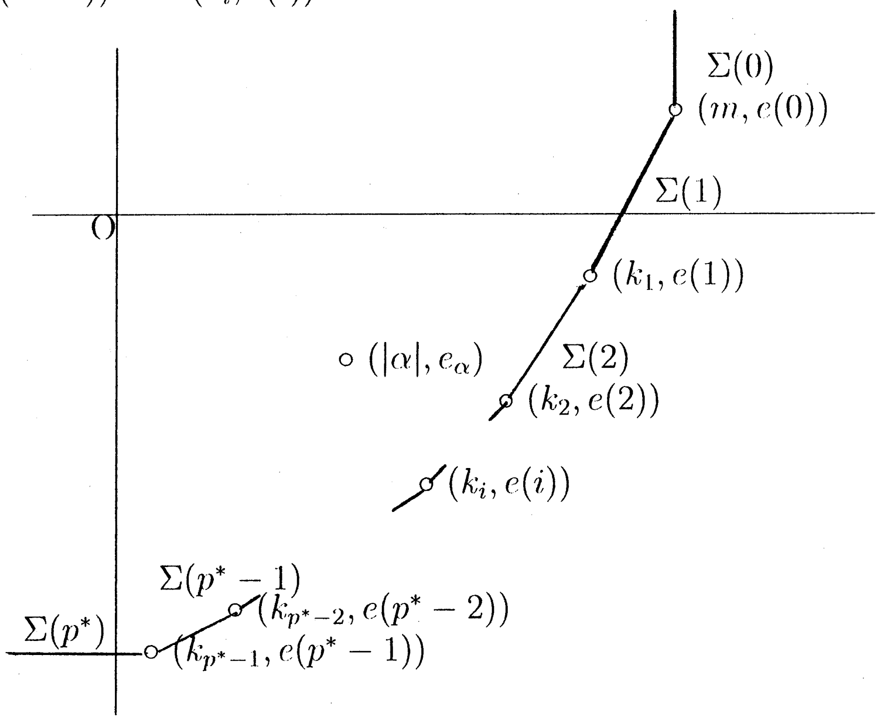

Figure 1: Characteristic polygon

We call the slope γ_i of $\Sigma(i)$ the *i-th characterisitc index* of $P(z,\partial)$ with respect to $K = \{z_0 = 0\}$. Now we notice the vertices of the polygons Σ. So put subsets $\Delta(i)$ of multi-indices and quantities $l_i \in \mathbb{N}$ $(0 \leq i \leq p^*-1)$ as follows:

$$(1.5) \qquad \begin{cases} \Delta(i) := \{\alpha \in \mathbb{N}^{d+1}; |\alpha| = k_i,\ j_\alpha - \alpha_0 = e(i)\} \\ \quad l_i := \max\{|\alpha'| :\ \alpha \in \Delta(i)\} \end{cases}$$

Define the subset $\Delta_0(i)$ of $\Delta(i)$ and a polynomial $\chi_{P,i}(z', \xi')$ in ξ' $(0 \leq i \leq$

$p^* - 1)$ by

$$\tag{1.6} \begin{cases} \Delta_0(i) = \{\alpha \in \Delta(i); |\alpha'| = l_i\} \\ \chi_{P,i}(z', \xi') = \sum_{\alpha \in \Delta_0(i)} b_\alpha(0, z')\xi^{\alpha'}. \end{cases}$$

$\chi_{P,i}(z', \xi')$ is homogeneous in ξ' with degree l_i.

Let us return to the equation (0.1). Our problem is precisely the following. *Does the equation*

$$P(z, \partial)u(z) = f(z) \in Asy_{\{\kappa\}}(\Omega(\theta))$$

have a solution $u(z) \in Asy_{\{\kappa\}}(U(\theta'))$ *for a polydisc* $U \subset \Omega$ *and a constant* θ' $(0 < \theta' < \theta)$ *?*

In order to answer it we give conditions (C_i). For fixed i $(0 \leq i \leq p^* - 1)$

(C_i) $\quad j_\alpha = 0$ *for* $\alpha \in \Delta_0(i)$ *and* $\chi_{P,i}(0, \xi') \not\equiv 0.$

Firstly we have

Theorem 1.2. *Suppose that* $P(z, \partial)$ *satisfies* (C_i) *and* $f(z) \in Asy_{\{\gamma\}}(\Omega(\theta))$ *with* $\gamma_{i+1} \leq \gamma < \gamma_i$. *Let* θ' *be a constant such that if* $i \neq 0$, $0 < \theta' < \min\{\theta, \pi/2\gamma_i\}$ *and if* $i = 0$, $0 < \theta' < \theta$.

Then there is $g(z) \in Asy_{\{\gamma\}}(U(\theta'))$ *for some polydisc* U *centered at* $z = 0$ *such that* $(Rf)(z) := P(z, \partial)g(z) - f(z) \sim 0$ *in* $Asy_{\{\gamma_i\}}(U(\theta'))$.

If $i = 0$, then $(Rf)(z) = 0$, that is, $P(z, \partial)g(z) = f(z)$. We show Theorem 1.2 by constructing a parametrix. As for the existence of a solution $u(z)$ whose asymptotic expansion is the same type as $f(z)$, we have

Theorem 1.3. *Suppose that* $P(z, \partial)$ *satisfies* (C_i) *for* $i = 0, 1, \cdots s$, *and let* $f(z) \in Asy_{\{\gamma\}}(\Omega(\theta))$ *with* $\gamma_{s+1} \leq \gamma < \gamma_s$. *Then for any* $0 < \theta' < \min\{\theta, \pi/2\gamma_1\}$ *there is* $u(z) \in Asy_{\{\gamma\}}(U(\theta'))$ *satisfying* $P(z, \partial)u(z) = f(z)$ *in* $U(\theta')$ *for some polydisc* U *centered at* $z = 0$.

The problem of the existence of solutions with asymptotic expansion was studied in [3], where the characteristic Cauchy problem was treated. The characteristic Cauchy problem has a formal power series solution. The purpose of [3] was to study the relation between genuine solutions and formal power series solutions. We studied in [3] the existence of a genuine solution with the same asymptotic expansion as the formal power series solution. The main result in [3] follows from Theorem 1.3.

We give an example. Let us consider

$$P(z,\partial) = \partial_1^5 + \partial_1^3\partial_0 + \partial_0^2, \quad z = (z_0, z_1) \in \mathbb{C}^2. \tag{1.7}$$

We have

$$\begin{cases} \gamma_0 = +\infty, \quad \gamma_1 = 1, \quad \gamma_2 = 1/2, \quad \gamma_3 = 0, \\ \chi_{P,0}(z',\xi_1) = \xi_1^5, \quad \chi_{P,1}(z',\xi_1) = \xi_1^3, \quad \chi_{P,0}(z',\xi_1) = 1. \end{cases}$$

Obviously $P(z,\partial)$ satisfies (C_i) and $\chi_{P,i}(z',1) \neq 0$ for $i = 0, 1, 2$.

Suppose $f(z) \in Asy(\Omega(\theta))$, that is, it has merely an asymptotic expansion, and consider

$$(\partial_1^5 + \partial_1^3\partial_0 + \partial_0^2)u(z) = f(z). \tag{1.8}$$

Let $0 < \theta' < \min\{\theta, \pi/2\}$. Then we have a solution $u(z) \in Asy(U(\theta'))$ for some polydisc U centered at $z = 0$ by Theorem 1.3.

§2 Construction of Parametrix

In order to find $g(z)$ in Theorem 1.2 we construct a parametrix of $P(z, \partial_z)$. For this purpose we use some auxiliary functions. Let $0 < \delta \leq 1$ and put

$$\hat{g}_p(\lambda) = \begin{cases} \dfrac{\lambda^\delta}{\Gamma(\frac{p}{\delta}+1)} \displaystyle\int_0^d \exp(-\lambda^\delta\zeta)\zeta^{\frac{p}{\delta}}d\zeta \quad \mathit{for} \quad p > 0 \\ \\ \lambda^{-p} \quad \mathit{for} \quad p \leq 0, \end{cases} \tag{2.1}$$

where $d > 0$ is a small constant. If $p > 0$, $\hat{g}_p(\lambda)$ depends on δ and d but if $p \leq 0$, $\hat{g}_p(\lambda)$ does not. Define

$$K_p(\delta; t) = \frac{1}{2\pi i}\int_1^\infty \exp(-\lambda t)\hat{g}_p(\lambda)d\lambda \tag{2.2}$$

and

$$K_{p,q}(\delta; w_0 - z_0, w_0) = w_0^q\left(-\frac{\partial}{\partial w_0}\right)^q K_p(\delta; w_0 - z_0). \tag{2.3}$$

If $0 < \delta < 1$, $K_p(\delta; t)$ is multi-valued holomorphic on $t \neq 0$.

We construct the paremetrix G as follows: Let $w = (w_0, w') \in \mathbb{C} \times \mathbb{C}^d$. Let C_0 be a path in w_0-space which starts at $w_0 = 0$, encloses $w_0 = z_0$ once anticlockwise and ends at $w_0 = 0$ and C' be the d-dimensional product of circles defined by $\prod_{h=1}^{d}\{|w_i| = r_1\}$ in $\mathbb{C}^d$. Suppose $f(w) \in Asy_{\{\gamma\}}(\Omega(\theta))$. Then the parametix is of the form

$$(Gf)(z) = \int_{C'} dw' \int_{C_0} G(z,w)f(w)dw_0,$$

which is an integral operator with kernel

$$\text{(2.4)} \quad G(z,w) = \sum_{p=-\infty}^{+\infty}\sum_{q=0}^{+\infty} k_{p,q}(z,w')K_{p,q}(\delta_i; w_0 - z_0, w_0), \ \delta_i = \gamma_i/(\gamma_i+1).$$

The coefficients $k_{p,q}(z,w')$'s are determined so as to satisfy

$$P(z,\partial_z)G(z,w) = \frac{1}{(2\pi i)^d}\frac{K_{0,0}(\delta_i; w_0 - z_0, w_0)}{\prod_{h=1}^{d}(w_h - z_h)} + R(z,w),$$

where $R(z,w)$ satisfies $(Rf)(z) = \int_{C_0\times C'} R(z,w)f(w)dw \sim 0$ in $Asy_{\{\gamma_i\}}(U(\theta'))$. So put $g(z) := (Gf)(z)$.

The details, that is, the existence of $k_{p,q}(z,w')$, its estimate and the properties of $G(z,w)$ and $R(z,w)$ *etc.* will be published elsewhere.

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