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THE CONTINUATION OF HOLOMORPHIC SOLUTIONS TO CONVOLUTION EQUATIONS IN COMPLEX DOMAINS

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1 Introduction

First of all, The problem of analytic continuation of the solutions to a homogeneous linear partial differential equation with constant coefficients was considered by Kiselman [7]. He proved that the directions to whom every solution is analytically continued are determined by *its characteristic set.* (See also Zerner [12].) After that, under an additional hypothesis, Sébbar [11] extended the method of [7] to the case of local differential operators of infinite order with constant coefficients. Motivated by [11], Aoki [1] proved a local continuation theorem for the general differential operators of infinite order with variable coefficients, using his theory of exponential calculus for pseudo-differential operators. In the case of convolution equation with a hyperfunction kernel defined in tube domains invariants by any real translations, Ishimura and Y. Okada [2] proved that the directions to whom not every solution can be continued at once were contained to *the characteristic set* of the operator, by using the method developped by [7] and [11].

In this talk, we consider the homogeneous convolution equation $S*f=0$ with an analytic functional S and study the analytic continuation of the solution f.

We refer to [5] for the details and the proof.

2 The characteristic set and the condition (S)

In this section, we shall introduce the characteristic set and the condition $(S)_{\zeta_0}$. For any open set $\omega\subset\mathbb{C}^n$, we denote by $\mathcal{O}(\omega)$ the space of holomorphic functions defined on ω. Let S be an analytic functional on $\mathbb{C}^n$ and we suppose that S is supported by a compact convex set $K \subset \mathbb{C}^n$. $\hat{S}$ denote its Fourier-Borel-transform

$$\hat{S}(\zeta) =< S, \exp(z \cdot \zeta) >_z, \tag{2.1}$$

which is an entire function of exponential type satisfying the following estimate (the theorem of Polyà-Ehrenpreis-Martineau). For every ε, we can take a constant $C_\varepsilon > 0$ such that

$$|\hat{S}(\zeta)| \leqq C_\varepsilon \exp\left(H_K(\zeta) + \varepsilon|\zeta|\right), \tag{2.2}$$

where $H_K(\zeta) = \sup_{z \in K} \mathrm{Re} < z, \zeta >$ is the supporting function of K.

For a set $A \subset \mathbb{C}^n$, we set $A^a = -A$. we define the convolution operator $S*$ by

$$(S * f)(z) =< S, f(z-w) >_w \qquad \text{for } f \in \mathcal{O}(\omega + K^a), \tag{2.3}$$

and consider the homogeneous convolution equation

$$S * f = 0. \tag{2.4}$$

We define the sphere at infinity

$$S^{2n-1}_\infty = (\mathbb{C}^n \setminus \{0\})/\mathbb{R}_+$$

and denote by $\zeta\infty$ the equivalent class of $\zeta \in \mathbb{C}^n \setminus \{0\}$. We consider the compactification with directions

$$\mathbb{D}^{2n} = \mathbb{C}^n \sqcup S^{2n-1}_\infty$$

of $\mathbb{C}^n$.

Let $f(\zeta)$ be an entire function of exponential type. In accordance with Lelong and Gruman [9], we define the growth indicator of f by

$$h_f(\zeta) = \limsup_{r \to \infty} \frac{\log|f(r\zeta)|}{r}, \tag{2.5}$$

and the regularized growth indicator of f by

$$h_f^*(\zeta) = \limsup_{\zeta' \to \zeta} h_f(\zeta). \tag{2.6}$$

As in [2], and generalizing to the present case, we define the characteristic set of $S*$:

Definition 2.1. We set

$$\begin{aligned}\mathrm{Char}_\infty(S*) =& \text{the complement of } \{\tau\infty \in S_\infty^{2n-1}\ ;\\ & \text{for every } \varepsilon > 0, \text{ there exist } N > 0 \text{ and } \delta > 0 \text{ such that}\\ & \text{for any } r > N \text{ and } \zeta \in \mathbb{C}^n \text{ satisfying } \left|\zeta - \frac{\tau}{|\tau|}\right| < \delta,\\ & \text{we have } |\hat{S}(r\zeta)| \geqq \exp(h^*_{\hat{S}}(\zeta) - \varepsilon)r\}\end{aligned}$$

and call it the characteristic set of the operator $S*$.

Now we recall the definition of the condition (S), originally due to T. Kawai [6] and was defined in a direction in [4].

Definition 2.2. We say that an entire function f of exponential type satisfies the condition (S) at direction $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$, if it satisfies the following:

$$(S)_{\zeta_0} \qquad \begin{cases}\text{For every } \varepsilon > 0, \text{ there exisits } N > 0 \text{ such that}\\ \text{for any } r > N, \text{ we have } \zeta \in \mathbb{C}^n \text{ satisfying}\\ |\zeta - \zeta_0| < \varepsilon, |f(r\zeta)| \geqq \exp\left(h^*_f(\zeta_0) - \varepsilon\right) r.\end{cases}$$

Remark . This condition is equivalent to the condition of regular growth which is the classialc notion in the theory of entire functions (see [4]).

Remark . By (2.2) and (2.6), we have in general $h^*_{\hat{S}}(\zeta) \leqq H_K(\zeta)$. Hereafter we shall make assumption $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$. For open convex domains, this condition and the condition (S) are , in a sense, necessary and sufficient conditions for the solvability of inhomogeneous convolution equation $S * f = g$. See Krivosheev [8] for the more precise statement.

3 Main theorem and example

For the characteristic set $\mathrm{Char}_\infty(S*)$ and an open convex set $\omega \subset \mathbb{C}^n$, we set

$$\Omega = \text{the interior of } \left(\bigcap_{\zeta\infty \in \mathrm{Char}_\infty(S*)^a} \{z \in \mathbb{C}^n\ ;\ \mathrm{Re} < z, \zeta > \leqq H_\omega(\zeta)\}\right). \quad (3.1)$$

Our main theorem is the following:

Theorem 3.1. *Let $K \subset \mathbb{C}^n$ be a compact convex set and S an analytic functional supported by K. We suppose that S satisfies the condition $(S)_{\zeta_0}$ in any directions in $\mathbb{C}^n$ and $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$. For an open convex set $\omega \subset \mathbb{C}^n$, we define the open set Ω by (3.1). Then every holomorphic solution f to $S * f = 0$ defined on $\omega + K^a$ extends analytically to $\Omega + K^a$.*

Example . Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_l\}$ be a finite set in $\mathbb{C}^n$, K its convex-hull and $p_j(\zeta)$ an entire function of minimal type for $1 \leqq j \leqq l$. For the analytic functional S, we suppose its Fourier-Borel transform $\hat{S} = \sum_{j=1}^{l} p_j(\zeta) \exp <\zeta, \lambda>$. Then S is supported by K and by Ronkin [10] and by [4], we also know $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$ and that $\hat{S}$ satisfies the condition $(S)_{\zeta_0}$ in any directions in $\mathbb{C}^n$. Therefore this analytic functional S satisfies all hypothesis of the theorem above.

In particular, in case where p_j's are elliptic, that is to say, its characteristic set is empty, we can prove that the characteristic set $\mathrm{Char}_\infty(S*)$ coincides with the following:

$$\left\{\zeta\infty \in S^{2n-1}_\infty \ ; \ \#\{j \ ; \ \mathrm{Re} <\zeta, \lambda_j> = H_K(\zeta)\} \geqq 2\right\}.$$

See [3] for more detailed results. In the case of $n = 1$, $l = 4$ and $K =$ the convex-hull of Λ, the figures are the following:

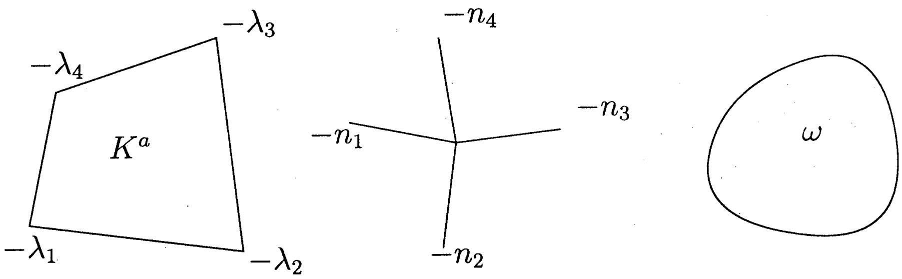

Figure 1: K^a, $\mathrm{Char}(S*)^a$ and ω

In this case, we remark

$$\mathrm{Char}_\infty(S*) = \text{the exterior normal directions } \{n_1\infty, n_2\infty, n_3\infty, n_4\infty\}.$$

In Figure 2, every solution $f \in \mathcal{O}(\omega + K^a)$ of $S * f = 0$ can be analytically continued to four corners.

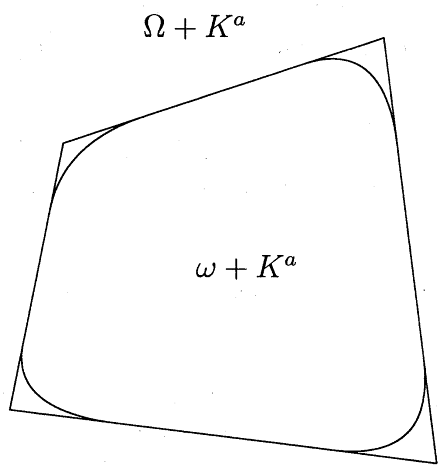

Figure 2: $\omega + K^a$ and $\Omega + K^a$

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