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1159 巻 2000 年 10-25

On the boundedness and invertibility of boundary potentials in Lipschitz domains

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The method of boundary layer potentials is at the same time a theoretical and a practical tool to solve boundary value problems on an open subset of $\mathbb{R}^{n}$. Boundedness and index properties of these boundary layers have been studied in general Lipschitz domains, [3], [6], [15], [22]. In this paper, we present some improvements of these results in three special situations that are motivated by numerical applications. We emphasize on the strong ellipticity of the generated boundary operators and on their mapping properties in Sobolev spaces of high order.

The first one is the Dirichlet boundary value problem in a plane domain with cuts. We present some results obtained in [13]. The main difficulty is that the double layer potential presents degeneracy and has a large kernel. To obtain a continuous and bijective operator, we add a new term to the potential along each cut. It essentially acts as a Hilbert transform along the cuts. Unfortunately, mixed with the double layer potential, it does not lead to a coercive operator. We modify it with an explicit inverse of the Hilbert transform in an interval. This choice has also the advantage that the singular functions generated by the cuts are immediately taken into account by the form of the potential. In this way, the boundary unknown on the cuts has the same regularity as the data. Since we consider non-connected boundaries, we also have to add an operator with a finite dimensional range in order to obtain a bijective operator.

Next, we consider the mixed Dirichlet-Neumann problem for the laplacian and present some results of [14]. The difficulty here comes essentially from the transition of the spaces where the problem is well posed, $H^{1/2}$ on the Dirichlet part of the boundary and $H^{-1/2}$ on the Neumann part. This leads us to the use of a mixed single and double layer potential which is quite different from the ansatz used in the direct method, [2]. The index of the boundary operator depends on the value of a generalized capacity associated to the polygon and to the decomposition of the boundary in a Dirichlet and a Neumann part.

The third problem is the Lamé system in a bounded Lipschitz open subset of $\mathbb{R}^{n}$. We discuss the properties of the double layer potential in this framework. Here, this potential depends on the choice of the boundary operator generalizing the normal derivative. It turns out that in general it involves a singular operator of Cauchy's type. This problem is avoided using a special value of a free parameter in the generalized stress boundary operator.

1 The Dirichlet problem in domains with cuts

Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$ and denote by ν the unit outward normal to $\partial\Omega$. If $f \in L^2(\partial\Omega)$ then the double layer potential

$$(\mathcal{K}f)(x) = \frac{1}{2\pi} \int_\Gamma \frac{(x-y).\nu_y}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \Omega,$$

is a harmonic function in Ω and has a non tangential limit on $\partial\Omega$ almost everywhere, see [17] and [22]. If, in the sense of [22], $\Gamma_i(x)$ is a regular family of cones of Ω, we have

$$\lim_{y \to x,\, y \in \Gamma_i(x)} \mathcal{K}f(y) = -\frac{1}{2}(I-K)f(x)$$

almost everywhere with

$$(Kf)(x) = \frac{1}{\pi} \int_\Gamma \frac{(x-y).\nu_y}{|x-y|^n} f(y)\, d\sigma(y).$$

Moreover, if we define the non-tangential maximal function by

$$u_i^*(x) = \sup_{y \in \Gamma_i(x)} |u(y)|,$$

we have

$$\|(\mathcal{K}f)^*\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$

In this way, the Dirichlet problem for the laplacian in Ω

$$\begin{cases} -\Delta u = 0 \\ u_{|\partial\Omega} = g \end{cases}$$

can be solved by inverting the operator $I - K$ on the boundary. If the boundary data f belongs to $H^{1/2}(\partial\Omega)$ then $\mathcal{K}f \in H^1(\Omega)$. The equivalence between the integral equation and the boundary value problem is discussed in [2]. The invertibility of $I-K$ in $L^2(\partial\Omega)$ and in $H^1(\partial\Omega)$ are proved in [22] when the boundary is connected.

Let us consider the case of a polygonal open subset of $\mathbb{R}^2$ with cuts. Let U be a bounded open subset of $\mathbb{R}^2$ with a connected polygonal boundary ∂U and corners $P_0, P_1, P_2, \ldots, P_N = P_0$. The boundary is the union of closed straight lines L_j joining P_j to P_{j+1}. Denote by $\omega_j \in]0, 2\pi[\setminus\{\pi\}$ the measure of the interior angle at P_j and by γ_j a parametrization of L_j by the arc length from $[0, t_j]$ onto L_j. We consider the Dirichlet problem

$$\begin{cases} -\Delta u = 0 \quad \text{in } \Omega \\ u_{|\partial U} = f, \quad u_{|\Gamma_\pm} = f_\pm. \end{cases} \tag{1}$$

where $\Omega = U \setminus (\Gamma_0 \cup \ldots \cup \Gamma_{M-1})$ and $\Gamma_0, \ldots, \Gamma_{M-1}$ are separate closed line segments included in U. Let $\Gamma = \cup_{j=0}^{M-1}\Gamma_j$. Since we allow different boundary values on both sides of the cuts, we choose an orientation for each Γ_j and make a distinction between the two sides $\Gamma_{j,\pm}$ of Γ_j.

If cuts are present the boundary of Ω is not connected. This is a first obstruction to the use of the double layer potential in this situation. Moreover along the two sides of a cut, the two boundary values would be opposite modulo a smooth contribution from the other parts of the boundary. It follows that the boundary operator would have a large kernel. Hence the double layer potential cannot be used alone.

To avoid these problems, we consider a more general potential. We look for the solution in the form

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y).\nu_y}{|x-y|^2}\,g(y)\,d\sigma(y) \\ &\quad+\frac{1}{2\pi}\int_{\Gamma}\frac{(x-y).t_y}{|x-y|^2}\,T_\Gamma h(y)\,d\sigma(y) \\ &\quad+\sum_{j=1}^{p}u_j(x)\int_{\Gamma_j}s_j(y)h(y)\,d\sigma(y). \end{aligned}$$

where t_y is the unit tangent vector to $\partial\Omega$ choosen in such a way that (ν_y, t_y) is a positive basis.

Here again the non tangential limits are known away from the endpoints if $T_\Gamma h \in L^2(\Gamma)$, see for example [20] p.186. We have

$$\lim_{x\to\sigma(s),\, x\in C_i(\sigma(s))}\frac{1}{2\pi}\int_{\Gamma_j}\frac{(x-y).t_y}{|x-y|^2}\,g(y)\,d\sigma(y)=\frac{1}{2\pi}pv\int_a^b\frac{g(\sigma(t))}{\sigma(s)-\sigma(t)}\,dt$$

almost everywhere on $\Gamma_{j,+}$ and $\Gamma_{j,-}$ if $g \in L^2(\Gamma_j)$. Up to a constant, this is the Hilbert transform restricted to the cut.

The suitable functional spaces for g and h are described below. The insertion of the operator T_Γ in the second term is important. Without it, we loose the coercivity of the boundary operator since the operator on the boundary would then reduce to a Hilbert transform.

We use the following notations. For a fixed j, let $\sigma : [a, b] \to \Gamma_j$ be a parametrization of the cut Γ_j by arc length. The operator T_Γ is defined by

$$(T_\Gamma h)(\sigma(s)) = -\frac{1}{\pi}pv\int_a^b\frac{\sqrt{(b-s)(s-a)}}{\sqrt{(b-t)(t-a)}}\,\frac{h(\sigma(t))}{s-t}\,dt.$$

The function s_j is given by

$$s_j(\sigma(s)) = \frac{1}{2\pi}\frac{1}{\sqrt{(b-s)(s-a)}}.$$

The choice of the operator T_Γ comes from the following result. Let $I = [a, b]$ be a compact interval of $\mathbb{R}$. If $f \in C^\infty(I)$, let

$$H_I f(x) = \frac{1}{\pi}pv\int_a^b\frac{f(y)}{x-y}\,dy$$

and

$$U_I f(x) = -\frac{1}{\pi}pv\int_a^b\frac{\sqrt{(b-x)(x-a)}}{\sqrt{(b-y)(y-a)}}\,\frac{f(y)}{x-y}\,dy.$$

The operator H_I is injective in $L^2(I)$ but has a dense non closed range on this space. However, we can avoid this problem. If $|\alpha| < \frac{1}{2}$, denote by $L_\alpha(\mathbb{R}_+)$ the set of functions f on $\mathbb{R}_+$ such that $x^{-\alpha}f(x) \in L^2(\mathbb{R}_+)$. With the norm

$$\|f\|^2_{L_\alpha} = \int_0^{+\infty} x^{-2\alpha}|f(x)|^2\,dx$$

this is a Hilbert space. We have the following result [13].

Proposition 1.1 *If* $|\alpha| < \frac{1}{2}$ *(resp.* $0 < \alpha < \frac{1}{2}$*) then* H_I *(resp.* U_I*) can be extended as a continuous operator from* $L_\alpha(]a, b[)$ *into itself.*

If $0 < \alpha < \frac{1}{2}$ *and* $f \in L_\alpha(]a, b[)$*, we have* $U_I H_I f = f$ *and*

$$H_I U_I f(x) = f(x) - \frac{1}{\pi} \int_a^b \frac{f(y)}{\sqrt{(b-y)(y-a)}}\, dy.$$

On these spaces, we have

$$\ker H_I = \{0\}, \quad imH_I = \{f \in L_\alpha(]0,1[) : \int_a^b \frac{f(y)}{\sqrt{(b-y)(y-a)}}\, dy = 0\},$$

$$\ker U_I = \rangle 1 \langle, \quad imU_I = L_\alpha(]a, b[).$$

Using the lemma 1.2, we obtain that the contribution of $h_{|\Gamma_j}$ in (2) can also be written

$$\frac{1}{2\pi} \int_a^b (\Re\,(\frac{\sqrt{(z-\sigma(a))(z-\sigma(b))}}{z-\sigma(t)}) - 1) \frac{h(\sigma(t))}{\sqrt{(b-t)(t-a)}}\, dt$$

where $z = x_1 + ix_2$ and $\sqrt{(z-\sigma(a))(z-\sigma(b))}$ is chosen analytic in $\mathbb{C} \setminus \Gamma_j$ and such that

$$\lim_{z\to\infty} \frac{1}{z} \sqrt{(z-\sigma(a))(z-\sigma(b))} = 1.$$

The two forms of the new term are convenient for different purposes.

Lemma 1.2 *If* $z \in \mathbb{C} \setminus [0,1]$ *and* $f \in L_\alpha(]0,1[)$ *with* $0 < \alpha < \frac{1}{2}$ *then*

$$\begin{aligned} &\frac{1}{\pi} \int_0^1 \frac{r\cos\theta - t}{r^2 - 2tr\cos\theta + t^2} U_{[0,1]} f(t)\, dt \\ = &\ \frac{1}{\pi} \int_0^1 (\Re(\frac{\sqrt{z(z-1)}}{z-s}) - 1)\, \frac{f(s)}{\sqrt{s(1-s)}}\, ds \end{aligned}$$

where $z = re^{i\theta}$ *and* $\sqrt{z(z-1)}$ *is chosen analytic in* $\mathbb{C} \setminus [0,1]$ *and positive in* $]1, +\infty[$.

The functions u_j are necessary to obtain an injective boundary operator. This essentially occurs because the boundary of Ω is not connected. One has to choose them in such a way that their linear hull has a trivial intersection with the image of the two other terms in (2). With the previous notations, a good choice is

$$u_j(z) = \log \big| z - \frac{\sigma(a)+\sigma(b)}{2} + \sqrt{(z-\sigma(a))(z-\sigma(b))}\,\big| - \log \frac{|\sigma(a)-\sigma(b)|}{2}$$

for $z \in \mathbb{C} \setminus \Gamma_j$. This is an harmonic function and it belongs to $H^1(\Omega \setminus \Gamma_j)$ for every bounded open subset Ω of $\mathbb{C}$. Extended by 0 on Γ_j, it becomes continuous and subharmonic in $\mathbb{C}$.

Note that if the cut is the positive real axis, the singular function $\Im(z \log z)$ is generated by the first term in (2) whereas $\Im\sqrt{z}$ is generated by the second term.

We denote by T the operator mapping $(g_{|\partial U}, g_{|\Gamma}, h)$ to the boundary values of (2) and (2) on ∂U, Γ_+ and Γ_-.

On the boundary of U, we consider the usual Sobolev spaces

$$H^s(\partial U) = C_0(\partial U) \cap \{f \in L^2(\partial U) : f_{|L_j} \circ \gamma_j \in H^s(]0, t_j[),\ j = 0, \dots, N-1\}$$

with the norm

$$\|f\|^2_{H^s(\partial U)} = \sum_{j=0}^{N-1} \|f_{|L_j} \circ \gamma_j\|^2_{H^s(]0,t_j[)}.$$

We also define

$$H^s_0(L_j) = \{f \in L^2(L_j) : f \circ \gamma_j \in H^s_0(]0, t_j[)\},$$

$$H^s_0(\partial U) = \{f \in H^s(\partial U) : f_{|L_j} \in H^s_0(L_j),\ j = 0, \dots, N-1\}$$

with the induced norms.

We have to add the singular functions on the boundary which are generated by the inversion of the double layer potential on ∂U. They are defined in the following way. If $0 < \omega < 2\pi$, let

$$\begin{aligned} e^+_\omega &= \{\frac{(2k+1)\pi}{\omega} : k \in \mathbb{N}\} \cup \{\frac{2k\pi}{2\pi - \omega} : k \in \mathbb{N}\} \\ e^-_\omega &= \{\frac{2k\pi}{\omega} : k \in \mathbb{N}\} \cup \{\frac{(2k+1)\pi}{2\pi - \omega} : k \in \mathbb{N}\} \end{aligned}$$

be the set of singular exponents for the angle ω. Let us describe the functions associated to a fixed corner P_j. Denote by x the distance to P_j. We use the notation (f, g) to specify the function equal to f on Γ_j and to g on Γ_{j-1}. The function is equal to zero on the other sides. If $s - \frac{1}{2} \notin e^\pm_{\omega_j}$, we denote by $\mathcal{L}_{j,s}$ the linear hull in $L^2(\partial U)$ of the functions

- (x^α, x^α) if $\alpha \in e^+_{\omega_j}$, $\alpha < s - \frac{1}{2}$,
- $(x^\alpha \log(x), x^\alpha \log(x))$ if $\alpha \in e^+_{\omega_j}$, $\alpha < s - \frac{1}{2}$ and α is of the form $p + \frac{1}{2}$ with $p \in \mathbb{N}$,
- $(x^\alpha, -x^\alpha)$ if $\alpha \in e^-_{\omega_j} \setminus \{0\}$, $\alpha < s - \frac{1}{2}$,
- $(x^\alpha \log(x), -x^\alpha \log(x))$ if $\alpha \in e^-_{\omega_j} \setminus \{0\}$, $\alpha < s - \frac{1}{2}$ and α is of the form $p + \frac{1}{2}$ with $p \in \mathbb{N}$.

For each j, choose a function χ_j on ∂U that is the restriction of a function of $C^\infty_0(\mathbb{R}^2)$ and equal to δ_{jk} near P_k for every k. Consider the space

$$\mathcal{H}^s(\partial U) = H^s_0(\partial U) + \sum_{j=0}^{N-1} \chi_j \mathcal{L}_{j,s}.$$

Of course, it does not depend on the choice of the χ_j's. For every j, let u_{jk}, $0 \le k < K_j$, be a basis of $\mathcal{L}_{j,s}$. If

$$f = g + \sum_{j=0}^{N-1} \chi_j \sum_{k=0}^{K_j - 1} c_{jk} u_{jk}$$

with $g \in H_0^s(\partial U)$, define

$$\|f\|^2_{\mathcal{H}^s(\partial U)} = \|g\|^2_{H_0^s(\partial U)} + \sum_{j=0}^{N-1}\sum_{k=0}^{K_j-1} |c_{jk}|^2.$$

With this norm, $\mathcal{H}^s(\partial U)$ is a Hilbert space.

In the same way, we consider

$$H^s(\Gamma) = \{f \in L^2(\Gamma) : f_{|\Gamma_j} \in H^s(\Gamma_j), \, j = 0, \dots, N-1\}$$

and denote by $H^{s,0}(\Gamma)$ the subset of $H^s(\Gamma)$ containing the elements vanishing at the endpoints of each Γ_j. These spaces are endowed with the usual H^s norm as above.

If $s > 1/2$,, define $H^s(\Gamma_+, \Gamma_-)$ as the subset of $H^s(\Gamma) \times H^s(\Gamma)$ formed by the pairs (f, g) where f and g take the same value at the endpoints of each Γ_j.

It is known, see for example [7], that if $s > 0$ and s is not an integer then the trace operator maps $H^{s+1/2}(\Omega)$ on $H^s(\partial U) \times H^s(\Gamma_+, \Gamma_-)$. With these notations, we have the following result, [13].

Theorem 1.3 *If $s > \frac{1}{2}$ and $s - \frac{1}{2} \notin e^{\pm}_{\omega_j}$ for every j then the operator T is continuous and bijective from $\mathcal{H}^s(\partial U) \times H^{s,0}(\Gamma) \times H^s(\Gamma)$ onto $H^s(\partial U) \times H^s(\Gamma_+, \Gamma_-)$.*

2 The mixed problem

As above, let us consider a bounded polygonal open subset Ω of $\mathbb{R}^2$ with a connected boundary $\Gamma = \partial\Omega = \bigcup_{j=0}^{M-1} \Gamma_j$. Here Γ_j, $0 \le j < M$, is a closed straight line segment. We denote by P_j the corner point where Γ_{j-1} and Γ_j meet. The interior angle at P_j is denoted by ω_j. It is assumed that this angle belongs to $]0, 2\pi[\setminus\{\pi\}$. As above, denote by ν the unit inward normal vector and by t the unit tangent vector on the boundary. In the definitions of the singular exponents and of the associated spaces and operators, we use the index $+$ for the interior domain and $-$ for the exterior one.

Assume that we have a decomposition $\Gamma = \Gamma_D \cup \Gamma_N$ where $\Gamma_D = \cup_{j \in e_D} \Gamma_j$, $\Gamma_N = \cup_{j \in e_N} \Gamma_j$, $e_D \cap e_N = \emptyset$ and $e_D \cup e_N = \{0, 1, \dots, M-1\}$. We assume that $e_N \neq \emptyset$ since this simplifies the exposition.

We denote by e_{ND} the set of indices $j \in e_D$ such that $j - 1$ ($M - 1$ if $j = 0$) belongs to e_N. Let p be the number of elements of e_{ND}. It follows that Γ_D has p connected components $\Gamma_{D,1}, \dots, \Gamma_{D,p}$.

We consider the interior mixed Dirichlet-Neumann problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u_{|\Gamma_D} = u_0, & \partial_\nu u_{|\Gamma_N} = u_1, \end{cases} \tag{2}$$

and also the exterior problem

$$\begin{cases} -\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ u_{|\Gamma_D} = u_0, \quad \partial_\nu u_{|\Gamma_N} = u_1, \\ u(x) = a \log|x| + \mathcal{O}(1), \quad x \to \infty, \end{cases} \tag{3}$$

with a singularity at infinity. Here u_0, u_1 and a are given. This formulation of the exterior problem with $a \neq 0$ contains the Green's function with pole at infinity. It is useful for the presentation of our results below.

If Γ_D is not empty, these problems have a unique variational solution u for any data $u_0 \in H^{1/2}(\Gamma_D)$, $u_1 \in H^{-1/2}(\Gamma_N)$ and $a \in \mathbb{C}$. If $\Gamma_D = \emptyset$, then the solution exists and is unique modulo a constant if $u_1 \in H^{-1/2}(\Gamma_N)$ has mean value 0 in the interior case and satisfies

$$a + \frac{1}{2\pi} \int_\Gamma u_1 \, d\sigma = 0$$

in the exterior case.

2.1 The spaces

The space $H^{1/2}$ is of particular use in the following constructions and its special nature requires some care. See for example [7] for basic properties and some characterizations of $H^{1/2}(\mathbb{R})$.

Let $C = \Gamma, \Gamma_D$ or Γ_N. Following [2], we define $\widetilde{H}^{1/2}(C)$ as the set of elements of $L^2(C)$ whose extension by 0 outside C belongs to $H^{1/2}(\Gamma)$. Of course $\widetilde{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$. We also need the dual spaces

$$\widetilde{H}^{-1/2}(C) = H^{1/2}(C)'$$

and

$$H^{-1/2}(C) = \widetilde{H}^{1/2}(C)'.$$

We define $\widetilde{H}_v^{-1/2}(C)$ as the subspace of $\widetilde{H}^{-1/2}(C)$ formed by the elements whose integrals on each connected component of C vanish.

As in the previous section, the spaces have to be refined to take into account the singularities generated by the corners. The singular exponents can be defined in the following way.

For a mixed corner P with interior angle ω, let

$$e_{\omega,e}^{(m)} = \{(2k - \frac{1}{2})\frac{\pi}{\omega} - 1 : k \in \mathbb{N} \setminus \{0\}\} \cup \{(2k - \frac{1}{2})\frac{\pi}{2\pi - \omega} - 1 : k \in \mathbb{N} \setminus \{0\}\},$$

$$e_{\omega,o}^{(m)} = \{(2k + \frac{1}{2})\frac{\pi}{\omega} - 1 : k \in \mathbb{N}\} \cup \{(2k + \frac{1}{2})\frac{\pi}{2\pi - \omega} - 1 : k \in \mathbb{N}\}$$

and $e_\omega^{(m)} = e_{\omega,e}^{(m)} \cup e_{\omega,o}^{(m)}$. If $s - \frac{1}{2} \notin e_\omega^{(m)}$, denote by $\mathcal{L}_{\omega,s,\pm}^{(m)}$ the linear hull in $L^2(\Gamma)$ of the functions

- $\begin{pmatrix} x^\alpha \\ \pm x^\alpha \end{pmatrix}$ if $\alpha \in e_{\omega,e}^{(m)}$, $-1 < \alpha < s - \frac{1}{2}$,
- $\begin{pmatrix} x^\alpha \\ \pm x^\alpha \end{pmatrix} \ln(x)$ if $\alpha \in e_{\omega,e}^{(m)}$, $-1 < \alpha < s - \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{N}$,
- $\begin{pmatrix} x^\alpha \\ \mp x^\alpha \end{pmatrix}$ if $\alpha \in e_{\omega,o}^{(m)}$, $-1 < \alpha < s - \frac{1}{2}$,
- $\begin{pmatrix} x^\alpha \\ \mp x^\alpha \end{pmatrix} \ln(x)$ if $\alpha \in e_{\omega,o}^{(m)}$, $-1 < \alpha < s - \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{N}$,

where x is the distance to P. Here, the vector notation means that the first component is the value of the function on the segment preceding P in the direction of the tangent vector t and the second one is the value on the segment following P. The function is equal to zero on the other sides.

Note also that if n is an integer and $n - 1/2 \notin e_\omega^{(m)}$ then $\dim \mathcal{L}_{\omega,n,\pm}^{(m)} = 2n+1$.

If P_j is a mixed corner, the set of singular exponents is $e_{\omega_j} = e_{\omega_j}^{(m)}$. The associated singular functions space is $\mathcal{L}_{j,s}^{\pm} = \mathcal{L}_{\omega_j,s,\pm}^{(m)}$ if $j-1 \in e_N$ and $j \in e_D$. In the other case, $j-1 \in e_D$ and $j \in e_N$, we have to use $\mathcal{L}_{j,s}^{\pm} = \mathcal{L}_{\omega_j,s,\mp}^{(m)}$. We remind the reader that the $+$ (resp. $-$) corresponds to the interior (resp. exterior) problem.

We proceed in the same way for the pure Dirichlet and Neumann corners. Let

$$e_{\omega,e}^{(p)} = \{\frac{2k\pi}{\omega} - 1 : k \in \mathbb{N}\} \cup \{\frac{(2k+1)\pi}{2\pi - \omega} - 1 : k \in \mathbb{N}\},$$

$$e_{\omega,o}^{(p)} = \{\frac{(2k+1)\pi}{\omega} - 1 : k \in \mathbb{N}\} \cup \{\frac{2k\pi}{2\pi - \omega} - 1 : k \in \mathbb{N}\}$$

and $e_\omega^{(p)} = e_{\omega,e}^{(p)} \cup e_{\omega,o}^{(p)}$. If $s - \frac{1}{2} \notin e_\omega^{(p)}$, denote by $\mathcal{L}_{\omega,s,\pm}^{(p)}$ the linear hull in $L^2(\Gamma)$ of the functions

- $\begin{pmatrix} x^\alpha \\ \pm x^\alpha \end{pmatrix}$ if $\alpha \in e_{\omega,e}^{(p)}$, $-\frac{1}{2} < \alpha < s - \frac{1}{2}$,
- $\begin{pmatrix} x^\alpha \\ \pm x^\alpha \end{pmatrix} \ln(x)$ if $\alpha \in e_{\omega,e}^{(p)}$, $0 < \alpha < s - \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{N}$,
- $\begin{pmatrix} x^\alpha \\ \mp x^\alpha \end{pmatrix}$ if $\alpha \in e_{\omega,o}^{(p)}$, $-\frac{1}{2} < \alpha < s - \frac{1}{2}$,
- $\begin{pmatrix} x^\alpha \\ \mp x^\alpha \end{pmatrix} \ln(x)$ if $\alpha \in e_{\omega,o}^{(p)}$, $0 < \alpha < s - \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{N}$

with the same notations as above.

Clearly $-1/2 \notin e_\omega^{(m)}$ if $\omega \in]0, 2\pi[\backslash\{\pi\}$. Hence $\dim \mathcal{L}_{\omega,0,\pm}^{(m)} = 1$. This one dimensional space is not included in $L^2(\Gamma)$. However it is included in $H^{-1/2}(\Gamma)$ since $x^\alpha \in H^{-1/2}(\mathbb{R}_+)$ if $-1 < \alpha < -1/2$.

Note that if n is an integer and $n - \frac{1}{2} \notin e_\omega^{(p)}$ then $\dim \mathcal{L}_{\omega,n,\pm}^{(p)} = 2n$. Hence there is a shift in the dimension of the spaces of singular functions between the two types of corners.

If P_j is a pure Dirichlet or Neumann corner, the set of singular exponents is $e_{\omega_j} = e_{\omega_j}^{(p)}$ and the associated singular functions space is $\mathcal{L}_{j,s}^{\pm} = \mathcal{L}_{\omega_j,s,\pm}^{(p)}$.

For each j, choose a function χ_j on Γ which is the restriction of an element of $C_0^\infty(\mathbb{R}^2)$ and equal to δ_{jk} near P_k for every k. Denote by $\mathcal{H}_\pm^s(\Gamma_D, \Gamma_N)$ the subspace of

$$H_0^s(\Gamma_D) \times H_0^s(\Gamma_N) + \sum_{j=0}^{M-1} \chi_j \mathcal{L}_{j,s}^{\pm}$$

formed by the elements (f, g) such that the integral of f on each connected component of Γ_D is 0. This space does not depend on the choice of the functions χ_j.

Note that $\mathcal{H}^s_\pm(\Gamma_D, \Gamma_N)$ is not a subspace $L^2(\Gamma_D) \times L^2(\Gamma_N)$ if there is at least one mixed corner P_j. Indeed $-1/2 \notin e^{(m)}_\omega$ if $\omega \in]0, 2\pi[\setminus\{\pi\}$. Hence $\dim \mathcal{L}^\pm_{j,0} = 1$. The one dimensional space $\chi_j \mathcal{L}^\pm_{j,0} = 1$ is not included in $L^2(\Gamma)$. However it is included in $H^{-1/2}(\Gamma)$ since x^α belongs to $H^{-1/2}(\mathbb{R}_+)$ if $-1 < \alpha < -1/2$. This additional degree of freedom in the unknown function at the mixed corners is balanced by the requirement that the integral of f vanishes on each connected component of Γ_D.

We can of course see the elements of $\mathcal{H}^s_\pm(\Gamma_D, \Gamma_N)$ as a pair of functions, one on Γ_D and one on Γ_N, or as a single function on Γ. For every j, let u_{jk}, $0 \leq k < K_j$, be a basis of $\mathcal{L}^\pm_{j,s}$. If

$$u = v + \sum_{j=0}^{M-1} \chi_j \sum_{k=0}^{K_j-1} c_{jk} u_{jk}$$

with $v \in H^s_0(\Gamma_D) \times H^s_0(\Gamma_N)$ then $\mathcal{H}^s_\pm(\Gamma_D, \Gamma_N)$ is a Hilbert space for the norm

$$\|u\|^2_{\mathcal{H}^s_\pm(\Gamma_D,\Gamma_N)} = \sum_{j=0}^{M-1} \sum_{k=0}^{K_j-1} |c_{jk}|^2 + \|v\|^2_{H^s_0(\Gamma_D)\times H^s_0(\Gamma_N)}.$$

The choice of the basis of singular functions does not matter for the asymptotic estimates since the space is finite dimensional. However, a well designed basis close to orthogonality is important for the condition number in practical computations.

2.2 The boundary operators

To solve (2), we use the following ansatz

$$\begin{aligned} K(g,h)(x) &= \frac{1}{2\pi}\int_{\Gamma_N} g(y) \log|x-y|\, d\sigma(y) && (4)\\ &\quad + \frac{1}{2\pi}\int_{\Gamma_D} \frac{(x-y).\nu_y}{|x-y|^2}\, h(y)\, d\sigma(y) && (5)\end{aligned}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. It is quite different from the one used in the direct method. In this case the two previous integrals contain the known boundary data u_1 and u_0 and the boundary unknowns appear in the single layer potential on Γ_D and the double layer potential on Γ_N.

To solve (3), we use the similar ansatz

$$\begin{aligned} K(g,h,c)(x) &= c + \frac{1}{2\pi}\int_{\Gamma_N} g(y) \log|x-y|\, d\sigma(y) && (6)\\ &\quad + \frac{1}{2\pi}\int_{\Gamma_D} \frac{(x-y).\nu_y}{|x-y|^2}\, h(y)\, d\sigma(y) && (7)\end{aligned}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. The constant c is required to get surjectivity of the boundary operators defined below.

Let $(f,g) \in \widetilde{H}_v^{-1/2}(\Gamma_D) \times \widetilde{H}^{-1/2}(\Gamma_N)$. There is a unique $h \in \widetilde{H}^{1/2}(\Gamma_D)$ such that $\partial_t h = f$. By the Theorem 1 of [1], the function (4) defined by g and h belongs to $H^1(\Omega)$. In the same way, the function (6) is H^1 in any bounded open subset of $\mathbb{R}^2 \setminus \overline{\Omega}$ and has the asymptotic behavior required in (3).

We consider the operators

$$\begin{aligned} T_\pm : \widetilde{H}_v^{-1/2}(\Gamma_D) \times \widetilde{H}^{-1/2}(\Gamma_N) &\to H^{-1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_N) \\ (f,g) &\to (\partial_t u_{|\Gamma_D}, \partial_\nu u_{|\Gamma_N}) \end{aligned}$$

where u is defined by (4) for the interior problem, by (6) for the exterior one and $h \in \widetilde{H}^{1/2}(\Gamma_D)$ satisfies $\partial_t h = f$. The subscript + (resp. −) means that we consider the interior (resp. exterior) problem and that the boundary values are taken from inside (resp. outside).

The existence of the boundary values and of the normal derivative in the space $H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_N)$ is a consequence of the Lemmas 3.2 and 3.6 of [1] since u is in the maximal domain of Δ. This shows that the operators $T_\pm$ make sense since the tangential derivative maps $H^{1/2}(\Gamma_D)$ into $H^{-1/2}(\Gamma_D)$.

If $(f,g) \in L^2(\Gamma_D) \times L^2(\Gamma_N)$, it follows from the results of [9] and [10] that the function u defined by (4) belongs to $H^{3/2}(\Omega)$ and that the one defined by (6) is $H^{3/2}$ in any bounded open subset of $\mathbb{R}^2 \setminus \overline{\Omega}$. Hence the traces are continuous on the closure of each side Γ_j. This remains true if $(f,g) \in \mathcal{H}^0_\pm(\Gamma_D, \Gamma_N)$.

For each $j = 1, \ldots, p$, we fix a point Q_j on $\Gamma_{D,j}$. We can for example take the points P_j with $j \in e_{ND}$. We use the finite rank operators

$$S_+ : \mathcal{H}^0_+(\Gamma_D, \Gamma_N) \to \mathbb{C}^p : (f,g) \mapsto (u(Q_1), \ldots, u(Q_p))$$

$$S_- : \mathcal{H}^0_-(\Gamma_D, \Gamma_N) \times \mathbb{C} \to \mathbb{C}^p : (f,g) \mapsto (u(Q_1), \ldots, u(Q_p))$$

where u is defined as above by (4) and (6) respectively.

2.3 The mapping properties

If $\Gamma_D = \emptyset$, our ansatz (4) is the single layer potential. It defines a one to one boundary operator if and only if the capacity of Ω is not one. In the general case, denote by G the solution of the mixed problem

$$\begin{cases} \Delta G = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{\Omega} \\ G_{|\Gamma_N} = 0, \quad \partial_\nu G_{|\Gamma_D} = 0, \\ G(x) = \log|x| + \mathcal{O}(1), \quad x \to +\infty. \end{cases} \tag{8}$$

We define the mixed capacity γ of Ω with respect to the decomposition (Γ_D, Γ_N) of the boundary Γ by the limit

$$\log\gamma = \lim_{x\to\infty} G(x) - \log|x|.$$

The central mapping properties of the boundary operator defined by (4) are summed up in the following result.

Theorem 2.1 *If $s \geq 0$, then the operator $T_\pm$ maps $\mathcal{H}^s_\pm(\Gamma_D, \Gamma_N)$ into $H^s_c(\Gamma_D) \times H^s_c(\Gamma_N)$.*

Assume that $s \geq 0$ and $s - \frac{1}{2} \notin e_{\omega_j}$ for every j. It follows that

- *if $\Gamma_D \neq \emptyset$ and the mixed capacity of Ω with respect to the decomposition (Γ_D, Γ_N) is not 1, then the operator*

  $$\mathcal{T}_+ : \mathcal{H}^s_+(\Gamma_D, \Gamma_N) \to H^s_c(\Gamma_D) \times H^s_c(\Gamma_N) \times \mathbb{C}^p : (f,g) \mapsto (T_+(f,g), S_+(f,g))$$

  is bijective,

- *if* $\Gamma_D \neq \emptyset$ *then the operator*

$$\mathcal{T}_- : \mathcal{H}^s_-(\Gamma_D, \Gamma_N) \times \mathbb{C} \to H^s_c(\Gamma_D) \times H^s_c(\Gamma_N) \times \mathbb{C}^{p+1} :$$
$$(f, g, c) \mapsto (T_-(f, g), S_-(f, g), \int_{\Gamma_N} g \, d\sigma)$$

is bijective,

- *if* $\Gamma_D = \emptyset$ *then* $\mathcal{T}_\pm$ *are Fredholm operators with index* 0 *and one dimensional kernel;* $\mathcal{T}_+$ *has the same property if* $\Gamma_D \neq \emptyset$ *and the mixed capacity is* 1.

The condition on the capacity is analogous to the one met in the use of the single layer potential for the pure Dirichlet problem and should not be a real problem. Any dilation by a factor r multiplies the capacity by r. Hence, for any open subset Ω of $\mathbb{R}^2$, there is one and only one $r > 0$ such that $r\Omega$ has the capacity 1. Moreover, in this case, the addition of a good operator of rank one gives a bijective boundary operator. For example, if the mixed capacity is 1, we can replace (4) by

$$\begin{aligned} K(g,h)(x) &= \frac{1}{2\pi} \int_{\Gamma_N} g(y) \log|x-y| \, d\sigma(y) \\ &\quad + \frac{1}{2\pi} \int_{\Gamma_D} \frac{(x-y).\nu_y}{|x-y|^2} h(y) \, d\sigma(y) + \frac{1}{2\pi} \int_{\Gamma_N} g \, d\sigma. \end{aligned}$$

3 Boundary potentials for the Lamé system

3.1 The problem

Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$ with a connected boundary. We consider the system of linearized elastostatics

$$Pu = -\mu \Delta u - (\lambda + \mu) graddivu$$

in Ω. Here μ and λ are the Lamé moduli. We assume that $\mu > 0$ and $\lambda + 2\mu > 0$. If λ converges to infinity, this system can be interpreted as the Stokes system of hydrostatics.

If $n = 2$, a fundamental solution for P is given by

$$E_2(x) = -\frac{1}{4\pi\mu(\lambda + 2\mu)} \big((\lambda + 3\mu) \log|x| \, I - (\lambda + \mu) \frac{\rangle x, x \langle}{|x|^2} \big) .$$

If $n > 2$, we can take

$$E_n(x) = \frac{1}{2\omega_{n-1}\mu(\lambda + 2\mu)} \big(\frac{\lambda + 3\mu}{(n-2)|x|^{n-2}} I + (\lambda + \mu) \frac{\rangle x, x \langle}{|x|^n} \big) .$$

Here $\rangle x, x \langle$ denotes the n by n matrix $(x_j x_k)_{1 \leq j,k \leq n}$ and I is the identity matrix.

For a given $g \in H^{1/2}(\partial\Omega; \mathbb{C}^n)$, we consider the problem of finding $u \in H^1(\Omega; \mathbb{C}^n)$ satisfying

$$\begin{cases} Pu = 0 \\ u_{|\partial\Omega} = g. \end{cases} \tag{9}$$

Let $\kappa > 0$. The generalized stress boundary operator $\mathcal{T}_\kappa = \mathcal{T}_\kappa(\nu)$ is defined by

$$(\mathcal{T}_\kappa(\nu)u)_i = \mu\partial_\nu u_i + (\lambda+\mu)\nu_i divu + \kappa\sum_{j=1}^{n}(\nu_j\partial_i - \nu_i\partial_j)u_j$$

where ν is the almost everywhere defined outward unit normal to $\partial\Omega$. If U is any Lipschitz open subset of $\mathbb{R}^n$, we consider the maximal domain of P in U

$$H^1(U,P) = \{u \in H^1(U;\mathbb{C}^n) : Pu \in L^2(U;\mathbb{C}^n)\}.$$

It follows from the results of [1], see also [7], that the space $\overline{C_0^\infty}(\Omega;\mathbb{C}^n)$ of restrictions to Ω of the elements of $C_0^\infty(\mathbb{R}^n;\mathbb{C}^n)$ is dense in $H^1(\Omega,P)$ and that the operator $\mathcal{T}_\kappa$ first defined on $\overline{C_0^\infty}(\Omega;\mathbb{C}^n)$ extends as a continuous linear operator

$$\gamma_1^{(\kappa)} : H^1(\Omega,P) \to H^{-1/2}(\Omega;\mathbb{C}^n)$$

for any κ. Moreover, we have the Green formula

$$\begin{aligned}&\int_\Omega Pu.v\,d\lambda + \int_{\partial\Omega}\mathcal{T}_\kappa u.v\,d\sigma\\ &= \int_\Omega(\mu\sum_j \partial_j u.\partial_j v + \kappa\sum_{j,k}\partial_j u_k.\partial_k v_j + (\lambda+\mu-\kappa)divu\,.divv)\,d\lambda\end{aligned}$$

for any $u \in H^1(\Omega,P)$, $v \in H^1(\Omega;\mathbb{C}^n)$ and $\kappa > 0$. A similar formula holds in $\mathbb{R}^n \setminus \overline{\Omega}$ for functions with bounded supports.

3.2 Boundary potentials

We first follow some ideas of [1] to obtain the basic properties of the boundary potentials. Let $u \in L^2_{comp}(\mathbb{R}^n;\mathbb{C}^n)$ such that $u_{|\mathbb{R}^n\setminus\partial\Omega} \in H^1(\mathbb{R}^n\setminus\partial\Omega,P)$ and define

$$f = P(u_{|\mathbb{R}^n\setminus\partial\Omega}) \in L^2_{comp}(\mathbb{R}^n).$$

Denote by γ_0 the trace map from $H^1(\Omega;\mathbb{C}^n) \cup H^1(\mathbb{R}^2\setminus\overline{\Omega};\mathbb{C}^n)$ onto $H^{1/2}(\partial\Omega;\mathbb{C}^n)$. Consider the jumps

$$[\gamma_0 u] = \gamma_0(u_{|\mathbb{R}^n\setminus\overline{\Omega}}) - \gamma_0(u_{|\Omega}),$$

$$[\gamma_1^{(\kappa)}u] = \gamma_1^{(\kappa)}(u_{|\mathbb{R}^n\setminus\overline{\Omega}}) - \gamma_1^{(\kappa)}(u_{|\Omega}).$$

Using the Green formula, we obtain

$$\begin{aligned}(Pu)(\varphi) &= u({}^tP\varphi) = \int_\Omega u.\,{}^tP\varphi\,d\lambda + \int_{\mathbb{R}^n\setminus\overline{\Omega}} u.\,{}^tP\varphi\,d\lambda\\ &= \int_{\mathbb{R}^n} f.\varphi\,d\lambda + \left\langle[\gamma_1^{(\kappa)}u],\gamma_0\varphi\right\rangle_{L^2(\partial\Omega)} - \left\langle\gamma_1^{(\kappa)}\varphi,[\gamma_0 u]\right\rangle_{L^2(\partial\Omega)}\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^N)$, Hence, denoting by G the convolution with the fundamental solution, we get

$$\begin{aligned}u(\varphi) &= (GPu)(\varphi) = (Pu)({}^tG\varphi)\\ &= \int_{\mathbb{R}^n} f.{}^tG\varphi\,d\lambda + \left\langle[\gamma_1^{(\kappa)}u],\gamma_0{}^tG\varphi\right\rangle_{L^2(\partial\Omega)} - \left\langle\gamma_1^{(\kappa)t}G\varphi,[\gamma_0 u]\right\rangle_{L^2(\partial\Omega)}.\end{aligned}$$

This gives the representation formula

$$\begin{aligned} u(x) &= (Gf)(x) + \left\langle [\gamma_1^{(\kappa)} u], \gamma_0 {}^t G(x,.) \right\rangle_{L^2(\partial\Omega)} \\ &\qquad - \left\langle \gamma_1 {}^t G(x,.), [\gamma_0 u] \right\rangle_{L^2(\partial\Omega)} \\ &= (Gf)(x) + \int_{\partial\Omega} G(x,y).[\gamma_1^{(\kappa)} u](y)\, d\sigma(y) \\ &\qquad - \int_{\partial\Omega} {}^t(\gamma_1^{(\kappa)}(\nu_y) {}^t G(x,y)).[\gamma_0 u](y)\, d\sigma(y) \end{aligned}$$

for every $x \in \mathbb{R}^n \setminus \partial\Omega$.

If $g \in L^1_{loc}(\partial\Omega; \mathbb{C}^n)$, the simple layer potential Vg and the double layer potential $W_\kappa g$ generated by g are defined by

$$Vg(x) = \int_{\partial\Omega} G(x,y).g(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega$$

and

$$W_\kappa g(x) = \int_{\partial\Omega} (\gamma_1^{(\kappa)}(\nu_y) G(x,y))^t.g(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega.$$

These functions always satisfy $Pu = 0$ in $\mathbb{R}^n \setminus \partial\Omega$.

For any $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \partial\Omega)$, we get

$$\begin{aligned} \int_{\mathbb{R}^n} Vg.\varphi\, d\lambda &= \int_{\mathbb{R}^n} \int_{\partial\Omega} G(x,y) g(y)\, d\sigma(y).\varphi(x)\, dx \\ &= \int_{\partial\Omega} g(y). \int_{\mathbb{R}^n} G(x,y)\varphi(y)\, dx\, d\sigma(y) \\ &= \langle g, \gamma_0 G\varphi \rangle = (G \circ \gamma_0^t g)(\varphi). \end{aligned}$$

Hence $V = G \circ \gamma_0^t$.

The first properties of these operators can be obtained as in [1]. Since the trace map γ_0 maps $H^1(\mathbb{R}^n; \mathbb{C}^n)$ onto $H^{1/2}(\partial\Omega; \mathbb{C}^n)$, the transpose map γ_0^t maps $H^{-1/2}(\partial\Omega; \mathbb{C}^n)$ into $H^{-1}_{comp}(\mathbb{R}^2; \mathbb{C}^n)$. Since G is a pseudodifferential operator of order -2, it follows that $V = G \circ \gamma_0^t$ extends as a continuous operator

$$V : H^{-1/2}(\partial\Omega; \mathbb{C}^n) \to H^1_{loc}(\mathbb{R}^n; \mathbb{C}^n).$$

Let $S : H^{1/2}(\partial\Omega; \mathbb{C}^n) \to H^1(\Omega, P)$ be the operator solving the Dirichlet problem for P. It follows from the representation formula that for any $g \in H^{1/2}(\partial\Omega; \mathbb{C}^n)$ we have

$$Sg = V(\mathcal{T}_\kappa Sg) - W_\kappa g.$$

Hence $W_\kappa = (V\mathcal{T}_\kappa - I)S$ maps $H^{1/2}(\partial\Omega; \mathbb{C}^n)$ on $H^1(\Omega, P)$.

¿From the results of [3], we know that the non-tangential limits of $W_\kappa f$ exist almost everywhere on $\partial\Omega$ for any $f \in L^2(\partial\Omega; \mathbb{C}^n)$. If Γ_i is a regular family of interior cones for Ω, the boundary value of $W_\kappa f$ is

$$\begin{aligned} \frac{1}{2}(-I + K_\kappa) f(x) &:= \lim_{y \to x,\, y \in \Gamma_i(x)} W_\kappa f(y) \\ &= -\frac{1}{2} f(x) + pv \int_{\partial\Omega} (\mathcal{T}_\kappa(\nu_y) G(x-y))^t.f(y)\, d\sigma(y). \end{aligned}$$

We have to change the sign of the first term if the exterior boundary value is considered. Since $W_\kappa f \in H^1(\Omega, P)$ for any $f \in H^{1/2}(\partial\Omega; \mathbb{C}^n)$, the operator K_κ maps $H^{1/2}(\partial\Omega; \mathbb{C}^n)$ on itself.

The following jump relations can be obtained as in [1]

$$[\gamma_0 V f] = 0, \quad [\mathcal{T}_\kappa V f] = -f,$$

$$[\gamma_0 W_\kappa g] = g, \quad [\mathcal{T}_\kappa W_\kappa g] = 0$$

for any $f \in H^{-1/2}(\partial\Omega; \mathbb{C}^n)$ and $g \in H^{1/2}(\partial\Omega; \mathbb{C}^n)$.

3.3 Mapping properties

Let

$$\tilde{\kappa} = \frac{\mu(\lambda + \mu)}{\lambda + 3\mu}.$$

The next result is proved in [3], lemma 3.3, for an open subset of $\mathbb{R}^n$ with $n \geq 3$. We can follow almost verbatim the proof in this paper to obtain the same result in $\mathbb{R}^2$. Another technique is used in [15] to study the Fredholm property and the index of $I - K_\kappa$ in the spaces $L^p(\partial\Omega)$ for a curved polygon, $n = 2$ and $\kappa = 0$, see also [5]. This method can also be used.

Proposition 3.1 *If Ω is a bounded Lipschitz domain of $\mathbb{R}^n$ with a connected boundary then the operator $I - K_{\tilde{\kappa}} : L^2(\partial\Omega; \mathbb{C}^n) \to L^2(\partial\Omega; \mathbb{C}^n)$ is invertible.*

Let us explain why the special value $\tilde{\kappa}$ appears. For simplicity assume that $n = 2$. An elementary computation shows that

$$\begin{aligned} \mathcal{T}_\kappa(\nu_y) G(x-y) &= \frac{1}{2\pi}\Big((1-a)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + 2a \frac{\rangle x-y, x-y\langle}{|x-y|^2}\Big)\frac{(x-y).\nu_y}{|x-y|^2}\Big) \\ &\qquad + \frac{c}{2\pi}\frac{(x-y).t_y}{|x-y|^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

where $t_y = (-\nu_{y,2}, \nu_{y,1})$ is the unit tangent vector and

$$a = \frac{(\lambda+\mu)(\kappa+\mu)}{2\mu(\lambda+2\mu)}, \quad c = \frac{\kappa(\lambda+3\mu) - \mu(\lambda+\mu)}{2\mu(\lambda+2\mu)}.$$

The last term generates a singular operator of Cauchy's type on the boundary. It disappears exactly for $\tilde{\kappa}$. In this case the operator $K_{\tilde{\kappa}}$ is compact in L^2 for smooth domains since it has a bounded kernel. The corresponding value of a is

$$\tilde{a} = \frac{\lambda + \mu}{\lambda + 3\mu}.$$

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