数理解析研究所講究録
1159巻 2000年 26-44

SOME MICRODIFFERENTIAL EQUATIONS FOR MICROFUNCTIONS WITH A HOLOMORPHIC PARAMETER AND THEIR FORMAL SYMBOL TYPE SOLUTIONS

KIYOOMI KATAOKA (UNIVERSITY OF TOKYO) 片岡清臣
AND
YOSHIAKI SATOH (FUJITSU LTD.) 佐藤芳光

§1. Introduction

Let X be a complex manifold $\mathbb{C}_z \times \mathbb{C}_x^n$ and M be its submanifold

$$M = \{(z, x) \in X; \mathrm{Im} x = 0\} \simeq M^{\mathbb{R}},$$

where $M^{\mathbb{R}}$ is the underlying real structure of M. We denote by $(z, x; \zeta, \xi)$ the coordinates of T^*X. We use the notation $D_z = \dfrac{\partial}{\partial z}$ and $D_x = \dfrac{\partial}{\partial x}$.

Around a point $(0, x^0; 0, i\eta^0) \in T^*(\mathbb{C} \times \mathbb{C}^n)$ with real x^0 and $\eta^0 \neq 0$, we construct a microfunction solution $v(z, x)$ with a holomorphic parameter z of

$$P(z, x, D_z, D_x)v(z, x) := \left(\sum_{k=0}^{m} A_k(z, x, D_z, D_x)D_z^{m-k}\right)v(z, x) = 0 \tag{1.1}$$

with ramified singularities along $\{z - \varphi(x, \xi) = 0\}$. Here $\varphi(x, \xi)$ is a holomorphic function of homogenous degree 0 with respect to ξ defined in a neighbourhood of $(0, x^0; 0, i\eta^0)$ with

$$\varphi(x^0, i\eta^0) = 0.$$

We suppose that $P(z, x, D_z, D_x)$ has Fuchsian singularities along $\{z = \varphi(x, \xi)\}$; that is each $A_k(z, x, D_z, D_x)$ is a microdifferential operator with $ord(A_k) \leq 0$ and satisfies

$$\sigma_0(A_0)(z, x, 0, \xi) = z - \varphi(x, \xi) \quad \text{and} \quad \sigma_0(A_1)(0, x^0, 0, i\eta^0) \notin \{0, -1, -2, \dots\}. \tag{1.2}$$

Definition 1.1

A $Q(z, x, D_z, D_x)$ is called an m-th order microdifferential operator if there exists a formal symbol $\{Q_j(z, x, \zeta, \xi)\}_{j=-\infty}^{m}$ such that

$$Q(z, x, D_z, D_x) = \sum_{j=-\infty}^{m} Q_j(z, x, D_z, D_x). \tag{1.3}$$

Here, there exists a neighbourhood W of $(z^0, x^0; \zeta^0, \xi^0)$ in T^*X and a positive constant C such that each $Q_j(z, x, \zeta, \xi)$ is holomorphic on W, and homogeous of degree j with respect to $(\zeta, \xi) \in \mathbb{C} \times \mathbb{C}^n$, and that we have

$$\sup_{(z,x;\zeta,\xi)\in W} |Q_j(z, x, \zeta, \xi)| \leq (-j)! C^{-j} \quad (-j \gg 1). \tag{1.4}$$

We denote by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators above.

Definition 1.2

We denote by $\mathcal{CO}_M$ a subsheaf of $\mathcal{C}_{M^{\mathbb{R}}}$:

$$\mathcal{CO}_M = \{v(z, x) \in \mathcal{C}_{M^{\mathbb{R}}}; \bar{\partial}_z v(z, x) = 0\}. \tag{1.5}$$

We call a section of $\mathcal{CO}_M$ a microfunction in (z, x) with a holomorphic parameter z.

Before constructing the solutions of (1.1) we reduce P to a simpler microdifferential operator by using some quantized contact transformation preserving sheaf $\mathcal{CO}_M$. By the implicit function theorem $\sigma_0(A_0)(z, x, \zeta, \xi)$ is written as follows:

$$\sigma_0(A_0)(z, x, \zeta, \xi) = \alpha(z, x, \zeta, \xi)(z - \Phi(x, \zeta, \xi)),$$

where α and Φ are homogenous of degree 0 with respect to (ζ, ξ) and satisfying

$$\alpha(0, x^0, 0, i\eta^0) = 1 \quad \text{and} \quad \Phi(x, 0, \xi) = \varphi(x, \xi).$$

Therefore by applying $\alpha(z, x, D_z, D_x)^{-1}$ to both sides of (1.1) we can reduce P to the case that

$$\sigma_0(A_0)(z, x, \zeta, \xi) = z - \Phi(x, \zeta, \xi)$$

with the same condition (1.2).

Proposition 1.3.

There exists a holomorphic contact transformation

$$S : \begin{cases} z^* &= z - \Phi(x, \zeta, \xi) \\ x^* &= x^*(z, x, \zeta, \xi) \\ \zeta^* &= \zeta \\ \xi^* &= \xi^*(z, x, \zeta, \xi) \end{cases}$$

satisfying

$$x^*(z, x, 0, \xi) = x, \quad \xi^*(z, x, 0, \xi) = \xi.$$

[Proof]

Solve the following Cauchy problem for $\psi = \psi(x, \zeta^*, \xi^*)$

$$\begin{cases} \frac{\partial \psi}{\partial \zeta^*} + \Phi(x, \zeta^*, \xi^* + \frac{\partial \psi}{\partial x}) = 0, \\ \psi|_{\zeta^*=0} = 0. \end{cases}$$

Then a function

$$\chi(z, x, \zeta^*, \xi^*) = z\zeta^* + x \cdot \xi^* + \psi(x, \zeta^*, \xi^*)$$

generates the desired contact transformation S. □

We note here that S preserves

$$T_M^* X = \{(z, x; \zeta, \xi) | \zeta = 0, \mathrm{Im} x = 0, \mathrm{Re} \xi = 0\}.$$

Hence there exists a quantized contact transformation

$$\mathcal{S} : S^{-1}\mathcal{C}\mathcal{O}_M \xrightarrow{\sim} \mathcal{C}\mathcal{O}_M$$

such that

$$\mathcal{S} \circ D_{z^*} \circ \mathcal{S}^{-1} = D_z,$$
$$\mathcal{S} \circ z^* \circ \mathcal{S}^{-1} = z - \Phi(x, D_z, D_x).$$

Therefore $\mathcal{S}^{-1} \circ P \circ \mathcal{S}$ gives a desired reduction of P. That is, we have

$$A_0(z, x, D_z, D_x) = z \tag{1.6}$$

under the same condition (1.2) with $\varphi = 0$. Hereafter we suppose this form (1.6) of A_0.

We construct a solution $v(z, x) \in \mathcal{C}\mathcal{O}_M$ around $\{z = 0\}$ of

$$P(z, x, D_z, D_x)v(z, x) = 0 \tag{1.7}$$

of the form

$$v(z, x) = U(z, x, D_x)f(x). \tag{1.8}$$

Here, $f(x)$ is any microfunction in x , and

$$U(z, x, D_x) = \sum_{j=-\infty}^{0} u_j(z, x, D_x) \tag{1.9}$$

is a microdifferential operator commuting with z with ramified singularities along $\{z = 0\}$ and satisfying the following equation as a microdifferential operator :

$$P(z, x, D_z, D_x)U(z, x, D_x) = 0 \mod \mathcal{E}_X \cdot D_z. \tag{1.10}$$

Indeed, (1.10) is equivalent to some system of equations for formal symbols. However, here we use the method of successive approximation.

Let us introduce a fundamental Fuchsian ordinary differential operator by

$$L := \sum_{k=0}^{m} a_k(z, x, \xi)\partial_z^{m-k}, \tag{1.11}$$

where $a_k(z,x,\xi) = a_{k,0}(z,x,0,\xi)$ for the homogeous expansion

$$A_k(z,x,D_z,D_x) = \sum_{j=-\infty}^{0} a_{k,j}(z,x,D_z,D_x) \tag{1.12}$$

of microdifferential operator $A_k(z,x,D_z,D_x)$ in (1.1). Further we define an operation L and $\mathcal{L}$ on formal symbols

$$U(z,x,\xi) = \sum_{j=-\infty}^{0} u_j(z,x,\xi) \tag{1.13}$$

by

$$LU(z,x,\xi) = \sum_{j=-\infty}^{0} (Lu_j)(z,x,\xi) \tag{1.14}$$

and

$$\mathcal{L}U(z,x,\xi) = \sum_{j=-\infty}^{0} \Bigg(\sum_{0\le k\le m,\ |r|+q=-j} \frac{1}{r!} \partial_\xi^r a_k(z,x,\xi) \partial_z^{m-k} \partial_x^r u_{-q}(z,x,\xi) \Bigg). \tag{1.15}$$

Then, our successive approximation process is formulated as follows:

$$\begin{cases} LU_0 = 0 \\ LU_{k+1} = \{(L-\mathcal{L}) - R\circ\}U_k \quad (k=0,1,2,\dots). \end{cases} \tag{1.16}$$

Here each U_k is a formal symbol of the form

$$U_k = \sum_{j=-\infty}^{0} u_j^k(z,x,\xi), \tag{1.17}$$

($u_j^k(z,x,\xi)$ is the j-th degree homogeous part of U_k) and R is a microdifferential operator given by

$$R = \sum_{k=1}^{m} A_k'(z,x,D_z,D_x) D_z^{m-k}, \tag{1.18}$$

where

$$A_k'(z,x,D_z,D_x) \equiv \sum_{j=-\infty}^{0} a_{k,j}'(z,x,D_z,D_x)$$

with

$$a_{k,j}'(z,x,\zeta,\xi) = a_{k,j}(z,x,\zeta,\xi) - \delta_{j0} \cdot a_{k,0}(z,x,0,\xi).$$

Further $R\circ$ denotes the usual operator product mod $\mathcal{E}_x \cdot D_z$; that is,

$$R \circ U \equiv S(z,x,0,D_x) \quad \text{when} \quad R(z,x,D_z,D_x)U(z,x,D_x) = S(z,x,D_z,D_x).$$

It is easy to see that the sum

$$U(z,x,D_x) = \sum_{k=0}^{\infty} U_k(z,x,D_x) \tag{1.19}$$

formally satisfies (1.9).

Therefore our problem is reduced to the following:

(1) Can we find formal symbols U_k around $\{z=0\}$ successively?
(2) Does $\sum_{k=0}^{\infty} U_k(z,x,D_x)$ converge around $\{z=0\}$ as a series of microdifferential operators?

In §2, we get suitable estimations along $\{z = 0\}$ for regular and ramified solutions of L, which are important for the successive construction of formal symbols $\{U_k\}$.

In §3, we introduce some formal norms with weight around $\{z=0\}$, and obtain some a' pri·o'ri estimations for these formal norms.

In §4, we solve our reduced problems (1), (2) above. Therefore we succeed in constructing one ramified and $m-1$ regular independent solutions around $\{z=0\}$.

§2. Preliminaries

Let L be an m-th order ordinary differential operator of the form

$$L = \sum_{k=0}^{m} a_k(z)\partial_z^{m-k},$$

where $a_0(z) = z$ and each $a_k(z)$ is holomorphic in a neighbourhood of

$$D = \{z \in \mathbb{C}; |z| \leq 1\}.$$

For an $\varepsilon > 0$ we set

$$\Omega = \{z \in \mathbb{C}; 0 < |z| \leq 1, |\arg z| \leq \pi - \varepsilon\}.$$

We obtain estimations for solutions of

$$Lu = f \tag{2.1}$$

for two cases: Holomorphic functions $f(z)$ on D and also on Ω.

Notation

For a holomorphic function u in a neighbourhood of D, we define two norms as follows:

$$\|u\| = \sup_{|z|\leq 1} |u(z)|$$

$$\|u\|' = \sup_{|z|\le 1, j=0,\dots,m} |u^{(j)}(z)|$$

and define another two norms with weight $\mu \in \mathbb{R}$

$$\|u\|_\mu = \sup_{z\in\Omega} |z|^\mu |u(z)|$$

$$\|u\|'_\mu = \sup_{z\in\Omega, j=0,\dots,m} |z|^{\mu-m+1+j}|u^{(j)}(z)|$$

for a holomorphic function $u(z)$ defined in a neighbourhood of Ω .

Theorem 2.1.

We suppose that $a_1(0) \neq 0, -1, -2, \dots$. Set

$$M = \max\{1, \sup_{z\in D} \sum_{k=1}^{m} |a_k(z)|\} < +\infty \tag{2.2}$$

and

$$\delta = \min\{|p + a_1(0)|; p = 0, 1, 2, \dots\} > 0. \tag{2.3}$$

Then we have a positive constant C depending only on M and δ, which satisfies the following estimations:
(1) Regular case: For a holomorphic function $f(z)$ in a neighbourhood of D any holomorphic solution $u(z)$ in a neighbourhood of D of (2.1) satisfies

$$\|u\|' \le C\{\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)|\}. \tag{2.4}$$

(2)Non-regular case : For a holomorphic function $f(z)$ in a neighbourhood of Ω any holomorphic solution $u(z)$ in a neighbourhood of Ω of (2.1) satisfies

$$\|u\|'_\mu \le C\{\|f\|_\mu + |u(1)| + \dots + |u^{(m-1)}(1)|\} \tag{2.5}$$

with $\forall \mu \ge M + m + 1$.

Remark. It is well known by the theory of Fuchsian differential equations that under the assumption $a_1(0) \neq 0, -1, -2, \dots$ there exists a unique solution for any given $(u(0), \dots, u^{(m-2)}(0))$ or $(u(1), \dots, u^{(m-1)}(1))$ for both cases.

[Proof]

Put an $m \times m$-matrix

$$A(z) = \begin{pmatrix} 0 & z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & z \\ -a_m(z) & \cdots & \cdots & \cdots & -a_1(z) \end{pmatrix},$$

and two m-dimensional vectors

$$\overrightarrow{x(z)} = \begin{pmatrix} u(z) \\ u'(z) \\ \vdots \\ u^{(m-1)}(z) \end{pmatrix}, \quad \overrightarrow{b(z)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(z) \end{pmatrix}.$$

Then, equation (2.1) reduces to

$$\frac{\overrightarrow{dx(z)}}{dz} = \frac{1}{z}A(z)\overrightarrow{x(z)} + \frac{1}{z}\overrightarrow{b(z)}. \tag{2.6}$$

Hence,

$$\overrightarrow{x(z)} = \overrightarrow{x(z_0)} + \int_{z_0}^{z} \frac{1}{s}A(s)\overrightarrow{x(s)}ds + \int_{z_0}^{z} \frac{1}{s}\overrightarrow{b(s)}ds. \tag{2.7}$$

Here we introduce the following norms for $m \times m$ matrix $X = (x_{i,j})_{i,j=1}^{m}$ and m-vector $\overrightarrow{x} = (x_i)_{i=1}^{m}$:

$$|X| \equiv \max_{i=1,\dots,m} \Big(\sum_{j=1}^{m} |x_{i,j}|\Big), \quad |\overrightarrow{x}| \equiv \max_{i=1,\dots,m} |x_i|.$$

Then we have an estimation

$$|A(z)| \leq M \quad \text{on } D.$$

We shall prove (1) after [Proof of (2)].

[Proof of (2)]

Firstly we put $z = e^{i\theta}$ and $z_0 = 1$ in (2.7) and we get the following integral inequality for $\theta \in [0, \pi - \varepsilon]$:

$$\begin{aligned}
|\overrightarrow{x(e^{i\theta})}| &\leq |\overrightarrow{x(1)}| + \left|\int_1^{e^{i\theta}} \frac{1}{s}A(s)\overrightarrow{x(s)}ds\right| + \left|\int_1^{e^{i\theta}} \frac{1}{s}\overrightarrow{b(s)}ds\right| \\
&= |\overrightarrow{x(1)}| + \left|\int_0^{\theta} \frac{1}{e^{i\varphi}}A(e^{i\varphi})\overrightarrow{x(e^{i\varphi})}ie^{i\varphi}d\varphi\right| + \left|\int_0^{\theta} \frac{1}{e^{i\varphi}}\overrightarrow{b(e^{i\varphi})}ie^{i\varphi}d\varphi\right| \\
&\leq |\overrightarrow{x(1)}| + \int_0^{\theta} |A(e^{i\varphi})||\overrightarrow{x(e^{i\varphi})}|d\varphi + \int_0^{\theta} |\overrightarrow{b(e^{i\varphi})}|d\varphi \\
&\leq |\overrightarrow{x(1)}| + \int_0^{\theta} M|\overrightarrow{x(e^{i\varphi})}|d\varphi + \int_0^{\theta} |f(e^{i\varphi})|d\varphi \\
&\leq |\overrightarrow{x(1)}| + \pi\|f\|_{\mu} + \int_0^{\theta} M|\overrightarrow{x(e^{i\varphi})}|d\varphi.
\end{aligned} \tag{2.8}$$

Secondly we put $z = re^{i\theta}$ and $z_0 = e^{i\theta}$ and we get the following integral inequality for $|\theta| \leq \pi - \varepsilon$ and $r \in (0,1]$:

$$\begin{aligned}
|\overrightarrow{x(re^{i\theta})}| &\leq |\overrightarrow{x(e^{i\theta})}| + \left| \int_{e^{i\theta}}^{re^{i\theta}} \frac{1}{s} A(s)\overrightarrow{x(s)}ds \right| + \left| \int_{e^{i\theta}}^{re^{i\theta}} \frac{1}{s}\overrightarrow{b(s)}ds \right| \\
&= |\overrightarrow{x(e^{i\theta})}| + \left| \int_{1}^{r} \frac{1}{se^{i\theta}} A(se^{i\theta})\overrightarrow{x(se^{i\theta})}e^{i\theta}ds \right| + \left| \int_{1}^{r} \frac{1}{se^{i\theta}}\overrightarrow{b(se^{i\theta})}e^{i\theta}ds \right| \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_{r}^{1} \frac{1}{s}|A(se^{i\theta})||\overrightarrow{x(se^{i\theta})}|ds + \int_{r}^{1} \frac{1}{s}|\overrightarrow{b(se^{i\theta})}|ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_{r}^{1} \frac{M}{s}|\overrightarrow{x(se^{i\theta})}|ds + \int_{r}^{1} \frac{|f(se^{i\theta})|}{s}ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_{r}^{1} \frac{M}{s}|\overrightarrow{x(se^{i\theta})}|ds + \int_{r}^{1} \frac{\|f\|_\mu}{s^{\mu+1}}ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}-1}{\mu}\|f\|_\mu + \int_{r}^{1} \frac{M}{s}|\overrightarrow{x(se^{i\theta})}|ds. \qquad (2.9)
\end{aligned}$$

We prepare next Lemma:

Lemma 2.2 (Gronwall).

Let $f(t), g(t), h(t)$ be non-negative valued continuous functions defined on $[a,b]$. If they satisfy

$$f(t) \leq g(t) + \int_a^t h(s)f(s)ds \quad for \ \forall t \in [a,b],$$

then we have

$$f(t) \leq g(t) + \int_a^t g(s)h(s)exp\Big(\int_s^t h(r)dr\Big)ds \quad for \ \forall t \in [a,b].$$

[Proof of Lemma]

We put

$$H(t) = \int_a^t h(s)f(s)ds,$$

then we get

$$\frac{dH(t)}{dt} = h(t)f(t) \leq h(t)\{g(t) + H(t)\} = h(t)g(t) + h(t)H(t).$$

That is,

$$\frac{dH(t)}{dt} - h(t)H(t) \leq h(t)g(t).$$

Multiplying both sides by $\exp\big(-\int_a^t h(s)ds\big)$, we obtain

$$\frac{d}{dt}\Big[H(t)\exp\Big(-\int_a^t h(s)ds\Big)\Big] \leq h(t)g(t)\exp\Big(-\int_a^t h(s)ds\Big).$$

Integrating both sides from a to t, we have

$$H(t)\exp\Big(-\int_a^t h(s)ds\Big) \le \int_a^t h(s)g(s)\exp\Big(-\int_a^s h(r)dr\Big)ds.$$

Therefore,

$$H(t) \le \int_a^t h(s)g(s)\exp\Big(\int_s^t h(r)dr\Big)ds.$$

Combining these inequalities, we get

$$f(t) \le g(t) + \int_a^t h(s)g(s)\exp\Big(\int_s^t h(r)dr\Big)ds. \quad \Box$$

Applying Lemma 2.2 to (2.8), we obtain

$$\begin{aligned}
|\overrightarrow{x(e^{i\theta})}| &\le |\overrightarrow{x(1)}| + \pi\|f\|_\mu + \int_0^\theta \{|\overrightarrow{x(1)}| + \pi\|f\|_\mu\} M\exp\Big(\int_\varphi^\theta M dr\Big)d\varphi \\
&= |\overrightarrow{x(1)}| + \pi\|f\|_\mu + \{|\overrightarrow{x(1)}| + \pi\|f\|_\mu\}\int_0^\theta Me^{M(\theta-\varphi)}d\varphi \\
&= |\overrightarrow{x(1)}| + \pi\|f\|_\mu + \{|\overrightarrow{x(1)}| + \pi\|f\|_\mu\}e^{M\theta}(-e^{-M\theta}+1) \\
&\le e^{M\pi}\{|\overrightarrow{x(1)}| + \pi\|f\|_\mu\}. \qquad (2.10)
\end{aligned}$$

It is easy to see that the conclusion of (2.10) is valid also for $\theta \in [-\pi+\varepsilon, 0]$.

Applying Lemma 2.2 to (2.9) for $\mu \ge M+m+1$, we obtain

$$\begin{aligned}
|\overrightarrow{x(re^{i\theta})}| &\le |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}-1}{\mu}\|f\|_\mu + \int_r^1 \{|\overrightarrow{x(e^{i\theta})}| + \frac{t^{-\mu}-1}{\mu}\|f\|_\mu\}\frac{M}{t}\exp\Big(\int_r^t \frac{M}{s}ds\Big)dt \\
&\le |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}}{\mu}\|f\|_\mu + \int_r^1 \frac{M}{r^M}\{|\overrightarrow{x(e^{i\theta})}|t^{M-1} + \frac{\|f\|_\mu}{\mu}t^{M-\mu-1}\}dt \\
&\le r^{-M}|\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}}{\mu-M}\|f\|_\mu \le r^{-M}|\overrightarrow{x(e^{i\theta})}| + r^{-\mu}\|f\|_\mu. \qquad (2.11)
\end{aligned}$$

Combining (2.11) with (2.10), we have

$$\begin{aligned}
|u^{(m)}(z)| &= |z|^{-1}| - a_1(z)u^{(m-1)}(z) - \cdots - a_m(z)u(z) + f(z)| \\
&\le |z|^{-\mu-1}M(1+\pi e^{M\pi})\big(\|f\|_\mu + |\overrightarrow{x(1)}|\big).
\end{aligned}$$

Further

$$|u^{(m-1)}(z)| \le |\overrightarrow{x(z)}| \le |z|^{-\mu}(1+\pi e^{M\pi})\big(\|f\|_\mu + |\overrightarrow{x(1)}|\big),$$

and so

$$\begin{aligned}|u^{(m-2)}(z)| &\le |u^{(m-1)}(e^{i\theta})| + \int_r^1 |u^{(m-1)}(se^{i\theta})|ds \\ &\le \bigl(1+\pi e^{M\pi}\bigr)\bigl(\|f\|_\mu + |\overrightarrow{x(1)}|\bigr)\left(1+\frac{r^{1-\mu}-1}{\mu-1}\right) \\ &\le r^{1-\mu}\bigl(1+\pi e^{M\pi}\bigr)\bigl(\|f\|_\mu + |\overrightarrow{x(1)}|\bigr).\end{aligned}$$

Since $\mu \ge m+1$, we can repeat this process $m-1$ times. Therefore we have

$$|u^{(j)}(z)| \le |z|^{-\mu+m-j-1}M\bigl(1+\pi e^{M\pi}\bigr)\bigl(\|f\|_\mu + |\overrightarrow{x(1)}|\bigr) \quad \text{for } j=0,\dots,m.$$

Hence the inequality (2.5) holds for $C = M\bigl(1+\pi e^{M\pi}\bigr)$.

[Proof of (1)]

In equation (2.6), we expand all the functions into power series with center 0:

$$A(z) = \sum_{p=0}^{\infty} A_p z^p, \quad \overrightarrow{b(z)} = \sum_{p=0}^{\infty} \overrightarrow{b_p} z^p, \quad \overrightarrow{x(z)} = \sum_{p=0}^{\infty} \overrightarrow{x_p} z^p.$$

Hence we have the following equations for the coefficients:

$$(p-A_0)\overrightarrow{x_p} = \sum_{q=1}^{p} A_q \overrightarrow{x_{p-q}} + \overrightarrow{b_p} \tag{2.12}$$

for $\forall p = 0,1,2,\dots$. Here we note that

$$\det(p-A_0) = p^{m-1}(p+a_1(0)) \ne 0$$

for $\forall p \ge 1$. Therefore we get for $\forall p \ge 1$ that

$$|\overrightarrow{x_p}| = \left|(p-A_0)^{-1}\left(\sum_{q=1}^{p} A_q\overrightarrow{x_{p-q}} + \overrightarrow{b_p}\right)\right| \le |(p-A_0)^{-1}|\left(\sum_{q=1}^{p} |A_q||\overrightarrow{x_{p-q}}| + |\overrightarrow{b_p}|\right).$$

Since A_q is written as integration of $z^{-q-1}A(z)$ on the unit circle, we have estimations

$$|A_q| \le \sup_{z\in D}|A(z)| \le M, \quad |\overrightarrow{b_q}| \le \|f\|$$

for every q. Therefore we obtain

$$|\overrightarrow{x_p}| \le |(p-A_0)^{-1}|\left(\sum_{q=1}^{p} M|\overrightarrow{x_{p-q}}| + \|f\|\right) \quad (\forall p \ge 1). \tag{2.13}$$

On the other hand $(p - A_0)^{-1}$ is given by

$$(p - A_0)^{-1} = \frac{1}{p(p + a_1(0))} \begin{pmatrix} p + a_1(0) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p + a_1(0) & 0 \\ -a_m(0) & \cdots & \cdots & -a_2(0) & p \end{pmatrix}$$

for $p \geq 1$, and so

$$\begin{aligned} |(p - A_0)^{-1}| &\leq \max\left\{\frac{1}{p}, \frac{p + |a_2(0)| + \cdots + |a_m(0)|}{p|p + a_1(0)|}\right\} \\ &\leq \max\left\{1, \frac{p + M}{|p + a_1(0)|}\right\} \\ &\leq \max\left\{1, \frac{3M}{\delta}, \sup_{p \geq 2M} \frac{p + M}{p - M}\right\} \\ &\leq \max\left\{\frac{3M}{\delta}, 3\right\} \leq 3\big(1 + \frac{M}{\delta}\big) =: K. \end{aligned}$$

Hence,

$$|\overrightarrow{x_p}| \leq K\left(M \sum_{q=0}^{p-1} |\overrightarrow{x_q}| + \|f\|\right) \quad (\forall p \geq 1). \tag{2.14}$$

Therefore, putting

$$y_p = \sum_{q=0}^{p} |\overrightarrow{x_q}|,$$

we have an estimation

$$y_p \leq (KM + 1)y_{p-1} + K\|f\| \leq \frac{(KM + 1)^p - 1}{M}\|f\| + (KM + 1)^p|\overrightarrow{x_0}|$$

for $\forall p \geq 1$, and so

$$|\overrightarrow{x_p}| \leq y_p \leq (KM + 1)^p\left(\frac{\|f\|}{M} + |\overrightarrow{x_0}|\right) \quad \text{for } \forall p \geq 1. \tag{2.15}$$

Further from

$$-A_0\overrightarrow{x_0} = \overrightarrow{b_0}$$

we obtain that

$$|u^{(m-1)}(0)| \leq \frac{1}{\delta}\Big\{|f(0)| + M\big(|u(0)| + \cdots + |u^{(m-2)}(0)|\big)\Big\}.$$

Hence

$$|\overrightarrow{x_0}| \leq \frac{1}{\delta}\|f\| + K\big(|u(0)| + \cdots + |u^{(m-2)}(0)|\big).$$

Consequently

$$|\overrightarrow{x_p}| \le (KM+1)^p \Big\{ \big(\frac{1}{M}+\frac{1}{\delta}\big)\|f\| + K\big(|u(0)|+\cdots+|u^{(m-2)}(0)|\big) \Big\}$$

for $\forall p \ge 0$, and so we have

$$\sup\left\{ |\overrightarrow{x(z)}|; |z| \le \frac{1}{2(KM+1)} \right\} \le 2\Big(K+\frac{1}{M}+\frac{1}{\delta}\Big)\Big(\|f\|+|u(0)|+\cdots+|u^{(m-2)}(0)|\Big). \tag{2.16}$$

Putting $\sigma = 1/\{2(MK+1)\} < 1$, we get an integral inequality similar to (2.9):

$$|\overrightarrow{x(re^{i\theta})}| \le |\overrightarrow{x(\sigma e^{i\theta})}| + \int_\sigma^r \frac{M}{s}|\overrightarrow{x(se^{i\theta})}|ds + \int_\sigma^r \frac{\|f\|}{s}ds$$

for any $r \in [\sigma, 1]$. By Gronwall's inequality and (2.16) we get

$$|\overrightarrow{x(re^{i\theta})}| \le \Big(\frac{r}{\sigma}\Big)^M \Big\{2\big(K+\frac{1}{M}+\frac{1}{\delta}\big)+\log\frac{1}{\sigma}\Big\} \times \Big(\|f\|+|u(0)|+\cdots+|u^{(m-2)}(0)|\Big)$$

for any $r \in [\sigma, 1]$. Therefore

$$\sup_{z\in D}|\overrightarrow{x(z)}| \le \Big(\frac{1}{\sigma}\Big)^M \Big\{2\big(K+\frac{1}{M}+\frac{1}{\delta}\big)+\log\frac{1}{\sigma}\Big\}\Big(\|f\|+|u(0)|+\cdots+|u^{(m-2)}(0)|\Big).$$

Note that

$$\sup_{z\in D}|u^{(m)}(z)| = \sup_{|z|=1}\left|\frac{-a_1(z)u^{(m-1)}(z)-\cdots-a_m(z)u(z)}{z}\right| \le M\sup_{z\in D}|\overrightarrow{x(z)}|.$$

Therefore since $M \ge 1$,

$$\|u\|' \le M\sup_{z\in D}|\overrightarrow{x(z)}| \le C\Big(\|f\|+|u(0)|+\cdots+|u^{(m-2)}(0)|\Big)$$

with

$$C = M\Big\{2(KM+1)\Big\}^M\Big[2\big(K+\frac{1}{M}+\frac{1}{\delta}\big)+\log\{2(KM+1)\}\Big] \tag{2.17}$$

and

$$K = 3\Big(1+\frac{M}{\delta}\Big).$$

This completes the proof of **Theorem 2.1** .

§3. Estimations of Formal Symbols

We take $U, L, \mathcal{L}, R$ defined in §1 . Hereafter considering a suitable scale transformation in z, we may assume that each $A_k(z,x,D_z,D_x)$ is defined in a conic neighbourhood of

$$\{z \in \mathbb{C}; |z| \leq 1\} \times (x^0, i\eta^0).$$

To show the convergence of series of formal symbols $\sum_{k=0}^{\infty} U_k(z,x,\xi)$, we introduce 2 types of formal norms, which are similar to Boutet-de-Monvel and Kree's one.

(1) Regular type: When each component $u_j(z,x,\xi)$ of U is holomorphic in a neighbourhood of $\{|z| \leq 1\}$, we define a formal power series $N_m(U;X)$ in X with parameters x, ξ by

$$N_m(U;X) \equiv \sum_{p,\alpha,\beta,l} \frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \max_{0\leq j\leq m} \|\partial_z^{j+l}\partial_x^{\alpha}\partial_\xi^{\beta}u_{-p}\|. \tag{3.1}$$

(2) Non-regular type: When each component $u_j(z,x,\xi)$ of U is holomorphic in a neighbourhood of

$$\Omega = \{z \in \mathbb{C}; 0 < |z| \leq 1, |\arg z| \leq \pi - \varepsilon\},$$

we define a formal power series $N_m^{\mu}(U;X)$ in X with parameters x, ξ by

$$N_m^{\mu}(U;X) \equiv$$

$$\begin{cases} \displaystyle\sum_{p,\alpha,\beta,l} \frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \max_{0\leq j\leq m} \|\partial_z^{j+l}\partial_x^{\alpha}\partial_\xi^{\beta}u_{-p}\|_{\mu+j+l+|\alpha+\beta|+p-m+1} & (m \geq 1) \\ \displaystyle\sum_{p,\alpha,\beta,l} \frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \|\partial_z^{l}\partial_x^{\alpha}\partial_\xi^{\beta}u_{-p}\|_{\mu+l+|\alpha+\beta|+p} & (m=0). \end{cases} \tag{3.2}$$

Further, when each component $u_j(x,\xi)$ is not depending on z, we define

$$K(U;X) \equiv \sum_{p,\alpha,\beta} \frac{p!C^{p+|\alpha+\beta|}X^{2p+|\alpha+\beta|}}{(p+|\alpha|)!(p+|\beta|)!} |\partial_x^{\alpha}\partial_\xi^{\beta}u_{-p}|. \tag{3.3}$$

In the approximation process (1.15), we need an a'pri · o'ri estimation for $N_m(U_k;X)$ or $N_m^{\mu}(U_k;X)$. For this purpose, in the symbol equation

$$LU = F \equiv \sum_{p=0}^{\infty} f_{-p} \tag{3.4}$$

we estimate $N_m(U;X)$ by $N_0(F;X)$ and $\sum_{j=0}^{m-2} K(\partial_z^j U(0,x,\xi);X)$; or we estimate $N_m^{\mu}(U;X)$ by $N_0^{\mu}(F;X)$ and $\sum_{j=0}^{m-1} K(\partial_z^j U(1,x,\xi);X)$.

To derive such estimations we apply $\partial_z^l \partial_x^\alpha \partial_\xi^\beta$ to both sides of $Lu_{-p} = f_{-p}$. Then we obtain

$$L(\partial_z^l \partial_x^\alpha \partial_\xi^\beta u_{-p}) = \partial_z^l \partial_x^\alpha \partial_\xi^\beta f_{-p}$$
$$- \sum_{l',l'',\alpha',\alpha'',\beta',\beta''} \sum_{k=0}^{m} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \partial_z^{l'} \partial_x^{\alpha'} \partial_\xi^{\beta'} a_k \cdot \partial_z^{l''+k} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}$$
$$(l = l' + l'', \alpha = \alpha' + \alpha'', \beta = \beta' + \beta'', (l', \alpha', \beta') \neq 0) .$$

Here we employ **Theorem 2.1**. For a sufficiently small $\varepsilon > 0$ we set

$$M_\varepsilon = \max \left\{ 1, \sup_{|z| \leq 1+\varepsilon, (x,\xi) \in V_\varepsilon} \sum_{k=1}^{m} |a_k(z, x, \xi)| \right\} < +\infty$$

and

$$\delta_\varepsilon = \inf\{|p + a_1(0, x, \xi)|; p = 0, 1, 2, \dots, (x, \xi) \in V_\varepsilon\} > 0$$

with

$$V_\varepsilon = \{(x, \xi) \in \mathbb{C}^n \times \mathbb{C}^n; |x - x^0| \leq \varepsilon, |\xi/|\xi| - i\eta^0/|\eta^0|| \leq \varepsilon\}.$$

Then there exists a positive constant C_0 depending only on M_ε and δ_ε, which satisfies some estimations (2.4), (2.5) for

$$L = \sum_{k=0}^{m} a_k(z, x, \xi) \partial_z^{m-k}.$$

In particular we have the following estimation on $|\xi| = 1$:

$$|\partial_z^l \partial_x^\alpha \partial_\xi^\beta a_k(z, x, \xi)| \leq l! \alpha! \beta! \left(\frac{2}{\varepsilon}\right)^{l+|\alpha|+|\beta|} M_\varepsilon \quad (|z| \leq 1, (x, \xi) \in V_{\varepsilon/2}).$$

Hereafter we fix a $(x, \xi) \in V_{\varepsilon/2}$ and set

$$C_1 = \max\{M_\epsilon, \frac{2}{\varepsilon}\}.$$

(1) Regular type case:

$$\max_{0 \leq j \leq m} \|\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}\| \leq C_0 \Bigg(\|\partial_z^l \partial_x^\alpha \partial_\xi^\beta f_{-p}\| + \sum_{j=0}^{m-2} |\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}(0, x, \xi)|$$
$$+ (m+1) \sum_{(l',\alpha',\beta') \neq 0} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} l'! \alpha'! \beta'! C_1^{l'+|\alpha'|+|\beta'|+1} \max_{0 \leq j \leq m} \|\partial_z^{j+l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}\| \Bigg).$$

Then, we obtain

$$N_m(U; X) \ll C_0 \Bigg\{ N_0(F; X) + \sum_{j=0}^{m-2} K(\partial_z^j U(0, x, \xi); X) + C(m-1)X \cdot N_m(U; X)$$
$$+ (m+1) C_1 N_m(U; X) \sum_{(l',\alpha',\beta') \neq 0} (C_1 C X)^{l'+|\alpha'+\beta'|} \Bigg\}.$$

That is, letting

$$\psi(X) \equiv \sum_{(l',\alpha',\beta')\neq 0} (C_1 C X)^{l'+|\alpha'+\beta'|} \tag{3.5}$$

and

$$\Phi(X) \equiv \frac{C_0}{1-(m+1)C_0C_1\psi(X)-(m-1)C_0CX}, \tag{3.6}$$

we get the following proposition:

Proposition 3.1. *If each component of F and U is holomorphic on a neighbourhood of $\{|z| \leq 1\}$, we have on $|\xi| = 1$*

$$N_m(U;X) \ll \Phi(X)\Big\{N_0(F;X) + \sum_{j=0}^{m-2} K(\partial_z^j U(0,x,\xi);X)\Big\}. \tag{3.7}$$

(2) Non-regular type case:

For $m \geq 1$, we obtain

$$\begin{aligned}
&\max_{0\leq j\leq m} \|\partial_z^{j+l}\partial_x^\alpha\partial_\xi^\beta u_{-p}\|_{\mu+j+l+p+|\alpha+\beta|-m+1} \\
&\leq C_0\Big\{\|\partial_z^l\partial_x^\alpha\partial_\xi^\beta f_{-p}\|_{\mu+l+|\alpha+\beta|+p} + \sum_{j=0}^{m-1} |\partial_z^{j+l}\partial_x^\alpha\partial_\xi^\beta u_{-p}(1,x,\xi)| \\
&+ (m+1)\sum_{(l',\alpha',\beta')\neq 0} \binom{l}{l'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'} l'!\alpha'!\beta'! C_1^{l'+|\alpha'+\beta'|+1} \\
&\times \max_{0\leq j\leq m} \|\partial_z^{j+l''}\partial_x^{\alpha''}\partial_\xi^{\beta''} u_{-p}\|_{\mu+l+|\alpha+\beta|+p}\Big\}.
\end{aligned}$$

Since $\mu + l + |\alpha+\beta| + p \geq \mu + j + l'' + |\alpha''+\beta''| + p - m + 1$, we obtain

$$\max_{j=0,\dots,m} \|\partial_z^{j+l''}\partial_x^{\alpha''}\partial_\xi^{\beta''} u_{-p}\|_{\mu+l+|\alpha+\beta|+p} \leq \max_{j=0,\dots,m} \|\partial_z^{j+l''}\partial_x^{\alpha''}\partial_\xi^{\beta''} u_{-p}\|_{\mu+j+l''+|\alpha''+\beta''|+p-m+1}.$$

In the same way as the regular type case, we obtain the following proposition:

Proposition 3.2. *If each component of F and U is holomorphic on a neighbourhood of Ω, we have on $|\xi| = 1$*

$$N_m^\mu(U;X) \ll \Phi(X)\Big\{N_0^\mu(F;X) + \sum_{j=0}^{m-1} K(\partial_z^j U(1,x,\xi);X)\Big\} \tag{3.8}$$

with $\forall\mu \geq M_\varepsilon + m + 1$.

In the last part of this section we estimate the formal norms of the remaining terms

$$(\mathcal{L}-L)U \quad \text{and} \quad R\circ U$$

by those of U.

Proposition 3.3.

Set

$$\psi_1(X) \equiv \sum_{l'=0}^{\infty}(CC_1X)^{l'}\sum_{\alpha'}(CC_1X)^{|\alpha'|}\sum_{\beta'}(2CC_1X)^{|\beta'|}\sum_{|r|\geq 1}C_1(2C_1X)^{|r|}. \quad (3.9)$$

(1) Regular type case: If each component of U is holomorphic on a neighbourhood of $\{|z| \leq 1\}$, we have on $|\xi| = 1$

$$N_0((\mathcal{L} - L)U; X) \ll \psi_1(X)N_m(U; X). \quad (3.10)$$

(2) Non-regular type case: If each component of U is holomorphic on a neighbourhood of Ω, we have on $|\xi| = 1$

$$N_0^{\mu}((\mathcal{L} - L)U; X) \ll \psi_1(X)N_m^{\mu}(U; X) \quad (3.11)$$

with $\forall \mu \geq M_\varepsilon + m + 1$.

[Proof]
(1) Regular type case:

$$\begin{aligned}
N_0((\mathcal{L} - L)U; X) &= \sum_{p,\alpha,\beta,l} \frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \\
&\times \|\partial_z^l\partial_x^\alpha\partial_\xi^\beta\Big(\sum_{p=|r|+q,|r|>0,k=0,\dots,m}\frac{1}{r!}\partial_\xi^r a_k\cdot\partial_z^{m-k}\partial_x^r u_{-q}\Big)\| \\
&\ll \sum_{l',l'',\alpha',\alpha'',\beta',\beta'',q,|r|>0}\frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!}\frac{1}{r!}\binom{l}{l'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}l'!\alpha'!(\beta'+r)! \\
&\times C_1^{|\alpha'+\beta'+r|+l'+1}\max_{j=0,\dots,m}\|\partial_z^{j+l''}\partial_x^{\alpha''+r}\partial_\xi^{\beta''}u_{-q}\| \\
&\ll \cdots \ll N_m(U;X)\Big\{\sum_{l'=0}^{\infty}(CC_1X)^{l'}\sum_{\alpha'}(CC_1x)^{|\alpha'|}\sum_{\beta'}(2CC_1X)^{|\beta'|}\sum_{|r|\geq 1}C_1(2C_1X)^{|r|}\Big\} \\
&= N_m(U;X)\psi_1(X).
\end{aligned}$$

(2) Non-regular type case:

$$\begin{aligned}
N_0^{\mu}((\mathcal{L} - L)U; X) &\ll \sum_{l',l'',\alpha',\alpha'',\beta',\beta'',q,|r|>0}\frac{p!C^{p+l+|\alpha+\beta|}X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!}\frac{1}{r!}\binom{l}{l'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \\
&\times l'!\alpha'!(\beta'+r)!C_1^{1+l'+|\alpha'+\beta'+r|}\max_{j=0,\dots,m-1}\|\partial_z^{j+l''}\partial_x^{\alpha''+r}\partial_\xi^{\beta''}u_{-q}\|_{\mu+p+l+|\alpha+\beta|},
\end{aligned}$$

where we use the fact that $\partial_\xi^r a_0 \equiv 0$ for any $|r| > 0$.

Since $\mu+p+l+|\alpha+\beta| \geq \mu+j+l''+|\alpha''+r|+|\beta''|+q-m+1$, we have

$$\max_{j=0,\dots,m-1} \|\cdot\|_{\mu+p+l+|\alpha+\beta|} \leq \max_{j=0,\dots,m} \|\cdot\|_{\mu+j+l''+|\alpha''+r|+|\beta''|+q-(m-1)}.$$

Therefore the same argument as in the regular type case leads to the conclusion

$$N_0^\mu((\mathcal{L}-L)U;X) \ll \psi_1(X)N_m^\mu(U;X). \quad \square$$

Note that

$$R\circ U = \sum_{k=1}^{m} A'_k \circ (\partial_z^{m-k}U), \tag{3.12}$$

where

$$A'_k = \sum_{p=0}^{\infty} a'_{k,-p}(z,x,\zeta,\xi) \tag{3.13}$$

are microdifferential operators of $ord(A'_k) \leq 0$ defined in a neighbourhood of $\{|z| \leq 1\} \times (x^0; i\eta^0)$ satisfying

$$a'_{k,0}(z,x,0,\xi) = 0 \quad \forall k.$$

Moreover, there exists a constant $C_2 > 0$ such that on

$$\left\{|z| \leq 1, |x-x^0| \leq \varepsilon, |\zeta| \leq \varepsilon|\xi|, |\xi/|\xi| - i\eta^0/|\eta^0|| \leq \varepsilon\right\} \cap \{|\xi| = 1\}$$

we have

$$|\partial_z^l \partial_x^\alpha \partial_\zeta^s \partial_\xi^\beta a'_{k,-t}(z,x,\zeta,\xi)| \leq t!l!\alpha!\beta!s!C_2^{l+s+|\alpha+\beta|+t}. \tag{3.14}$$

By the similar argument due to Boutet-de-Monvel and Kree we get the following estimation

Lemma 3.4.

There exists a convergent majorant series $\psi_2(X)$ with $\psi_2(0) = 0$ depending only on C,C_2 and n such that on $|\xi| = 1$
(1) Regular type case:

$$N_0(A'_k \circ U;X) \ll \psi_2(X)N_0(U;X), \tag{3.15}$$

(2) Non-regular type case:

$$N_0^\mu(A'_k \circ U;X) \ll \psi_2(X)N_0^\mu(U;X) \tag{3.16}$$

with $\forall\mu \geq M_\varepsilon + m + 1$.

Proposition 3.5.

We have the following estimation on $|\xi| = 1$:
(1) Regular type case:

$$N_0(R\circ U;X) \ll m\psi_2(X)N_m(U;X), \tag{3.17}$$

(2) Non-regular type case:

$$N_0^\mu(R\circ U;X) \ll m\psi_2(X)N_m^\mu(U;X) \tag{3.18}$$

with $\forall\mu \geq M_\varepsilon + m + 1$.

§4. Construction of Solutions

We consider the following relation:

$$\begin{cases} LU_0 = 0 \\ LU_{k+1} = (L-\mathcal{L})U_k - R\circ U_k \quad (k=0,1,2,\dots). \end{cases}$$

(1) Regular solution

We obtain

$$\begin{aligned} N_m(U_{k+1};X) &\ll \Phi(X)\Big\{N_0((L-\mathcal{L})U_k - R\circ U_k;X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0,x,\xi);X)\Big\} \\ &\ll \Phi(X)\Big\{N_0(R\circ U_k;X) + N_0((\mathcal{L}-L)U_k;X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0,x,\xi);X)\Big\} \\ &\ll \Phi(X)\Big\{m\psi_2(X)N_m(U_k;X) + \psi_1(X)N_m(U_k;X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0,x,\xi);X)\Big\} \\ &= \Phi(X)\Big\{(\psi_1(X)+m\psi_2(X))N_m(U_k;X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0,x,\xi);X)\Big\}. \end{aligned}$$

With the condition $U_{k+1}(0) = \cdots = U_{k+1}^{(m-2)}(0) = 0$, we obtain

$$\begin{aligned} N_m(U_{k+1};X) &\ll \{\Phi(X)(\psi_1(X)+m\psi_2(X))\}N_m(U_k;X) \\ &\ll \cdots \ll \{\Phi(X)(\psi_1(X)+m\psi_2(X))\}^{k+1}N_m(U_0;X). \end{aligned}$$

That is,

$$N_m(\sum_{k=0}^{\infty} U_k;X) \ll \sum_{k=0}^{\infty} N_m(U_k;X) \ll \sum_{k=0}^{\infty}\{\Phi(X)(\psi_1(X)+m\psi_2(X))\}^{k+1}N_m(U_0;X).$$

Since $LU_0 = 0$, we obtain

$$\begin{aligned} N_m(U_0;X) &\ll \Phi(X)\Big\{N_0(0;X) + \sum_{j=0}^{m-2} K(\partial_z^j U_0(0,x,\xi);X)\Big\} \\ &= \Phi(X)\Big\{0 + \sum_{j=0}^{m-2} K(\partial_z^j U_0(0,x,\xi);X)\Big\} < +\infty. \end{aligned}$$

Therefore, we get

$$N_m(\sum_{k=0}^{\infty} U_k;X) \ll \sum_{k=0}^{\infty}\big\{\Phi(X)(\psi_1(X)+m\psi_2(X))\big\}^k N_m(U_0;X) < +\infty,$$

that is, $\sum_{k=0}^{\infty} U_k$ is convergent in N_m norm.

(2) Non-regular solution

As the same as regular's case, we obtain

$$N_m^\mu(U_{k+1};X) \ll \Phi(X)\{(\psi_1(X)+m\psi_2(X))N_m^\mu(U_k;X)+\sum_{j=0}^{m-1} K(\partial_z^j U_{k+1}(1,x,\xi);X)\}.$$

Let U_0 be non-regular function and homogeous of degree 0 with respect to ξ. From now, we solve a Cauchy problem;

$$\begin{cases} LU_{k+1} = (L-\mathcal{L})U_k - R\circ U_k \\ \partial_z^j U_{k+1}(1,x,\xi) = 0 \quad (j=0,1,\dots,m-1). \end{cases}$$

Then, we obtain

$$\begin{aligned} N_m^\mu(U_{k+1};X) &\ll \{\Phi(X)(\psi_1(X)+m\psi_2(X))\}N_m^\mu(U_k;X) \\ &\ll \cdots \ll \{\Phi(X)(\psi_1(X)+m\psi_2(X))\}^{k+1}N_m^\mu(U_0;X). \end{aligned}$$

As same as the regular case, we obtain

$$N_m^\mu(\sum_{k=0}^{\infty} U_k;X) \ll \sum_{k=0}^{\infty} \{\Phi(X)(\psi_1(X)+m\psi_2(X))\}^k N_m^\mu(U_0;X) < +\infty,$$

that is, $\sum_{k=0}^{\infty} U_k$ is convergent in N_m^μ norm.

Thus we obtained $m-1$ regular solutions and a non-regular solution, which span the full solutions of $(\mathcal{L}+R\circ)U=0$.

References

[1]. K.Kataoka, *Microlocal Analysis of Boundary Value Problems with Regular or Fractional Power Singularities,in Structure of Solutions of Differetial Equations,edited by M.Morimoto and T.kawai,World Scientific Publ Co.,Singapore-New Jersey-London-Hong Kong* (1996), 215-225..

[2]. K.Kataoka, *Micro-local theory of boundary value problems I*, J. Fac. Sci. Univ. Tokyo Sect. IA **27** (1980), 355-399.

[3]. R.Ishimura,Y.Okada,and Y.Hino, *Differetial Equation*, A series of Mathematical and Information Science, vol. 11, Makino.

[4]. T.Kawai, *Introduction of Linear Partial Differential Equation*, Seminary Note 30 in Mathematical Class of Univ. Tokyo (1973).