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A localization algorithm for D-modules and its application

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First, we review the localization algorithm given in a joint paper with N. Takayama (Kobe) and U. Walther (Minessota/MSRI) [6] with slightly different reasoning of the correctness. The latter part applies this algorithm to the problem of finding the annihilator ideal of some elementary functions.

1 A localization algorithm

We work entirely in the algebraic category. Put $X = \mathbb{C}^n$ and $Y = \{x = (x_1, \ldots, x_n) \in X \mid f(x) = 0\}$ with a nonzero polynomial $f \in \mathbb{C}[x]$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on X. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module on X such that $\mathcal{M}$ is holonomic on $X \setminus Y$. Then Kashiwara ([2]) proved that the localization $\mathcal{M}[1/f] := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[1/f]$ of $\mathcal{M}$ by f is a holonomic $\mathcal{D}_X$-module on X, where $\mathcal{O}_X$ is the sheaf of regular functions on X. (In fact he proved this fact in the analytic category, which is a stronger statement.) Here we remark that starting from an algebraic (i.e., a $\mathcal{D}_X$-) module, the localization is the same both in the algebraic and in the analytic category. More precisely, if we denote by $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{D}_X^{\mathrm{an}}$ the sheaves of holomorphic functions and of holomorphic differential operators on X respectively, then we have an isomorphism

$$\mathcal{O}_X^{\mathrm{an}}[1/f] \otimes_{\mathcal{O}_X^{\mathrm{an}}} (\mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}) \simeq \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}[1/f].$$

Our claim is that $\mathcal{M}[1/f]$ is computable if the input, i.e. both $\mathcal{M}$ and f are defined over a computable subfield of $\mathbb{C}$ (e.g. over $\mathbb{Q}$).

Now let us explain our algorithm. Introcducing an auxiliary variable t, put

$$W := \{(t, x) \in \mathbb{C} \times X \mid tf(x) = 1\}$$

and let $\iota : W \longrightarrow \mathbb{C}^{n+1}$ be the natural embedding. Let $p : W \longrightarrow X$ be the projection $p(t, x) = x$ and $\varphi : X \setminus Y \longrightarrow W$ be the isomorphism defined by $\varphi(x) = (1/f(x), x)$. Let

$j : X \setminus Y \longrightarrow X$ be the natural embedding. Thus we have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\iota} & \mathbb{C} \times X \\ {\scriptstyle\varphi}\uparrow & & \downarrow{\scriptstyle p} \\ X \setminus Y & \xrightarrow{j} & X \end{array}$$

Then by using the integration functor (in the algebraic category) we get

$$\mathcal{M}[1/f] = \int_j j^{-1}\mathcal{M} = \int_p \int_\iota \int_\varphi j^{-1}\mathcal{M}.$$

(See e.g. [7].) Our algorithm simply performs the rightmost successive integration step by step. (For the sake of simplicity we describe the algorithm in the case where $\mathcal{M}$ is generated by one element.) We denote by $D_n = D_n(\mathbb{C})$ the Weyl algebra on n variables x with coefficients in $\mathbb{C}$. Then we can regard D_n as the set of global sections $\Gamma(X, \mathcal{D}_X)$ of $\mathcal{D}_X$. In general, for a left coherent $\mathcal{D}_X$-module $\mathcal{M}$, its global sections $M := \Gamma(X, \mathcal{M})$ is a finitely generated left D_n-module and its correspondence yields an equivalence of the category of left coherent $\mathcal{D}_X$-modules on X and that of finitely generated left D_n-modules since X is affine. The converse correspondence is given by the sheafification (or the 'localization') $\mathcal{M} = \mathcal{D}_X \otimes_{D_n} M$.

Algorithm-Theorem 1
Input: A polynomial $f \in \mathbb{Q}[x]$ and a finite subset $\{P_1, \ldots, P_r\}$ of $D_n(\mathbb{Q})$ which generates a left ideal I of D_n such that the sheafification $\mathcal{M}$ of $M := D_n/I$ is holonomic on $X \setminus Y$.

(1)
  (a) Put $\vartheta_i := \partial_i - t^2 f_i \partial_t$ with $f_i := \partial f / \partial x_i$, $\partial_i := \frac{\partial}{\partial x_i}$, and $\partial_t := \frac{\partial}{\partial t}$.
  (b) Compute $\widetilde{P}_i := P_i(x, \vartheta_1, \ldots, \vartheta_n)$. More precisely, Writing P_i in the normal form (i.e. make the derivations first and then multiply by polynomials) and substitute ϑ_i for ∂_i in P_i. (Note that $\vartheta_1, \ldots, \vartheta_n$ commute with one another.)
  (c) Let J be the left ideal of D_{n+1} generated by $\widetilde{P}_1, \ldots, \widetilde{P}_r$ and $1 - tf(x)$ and put $N := D_{n+1}/J$

(2) Compute $N/\partial_t N$ as left D_n-module as follows:
  (a) Let G be an involutive basis of J with respect to the weight vector

$$w = (1, 0, \ldots, 0; -1, 0, \ldots, 0)$$

  for $(t, x_1, \ldots, x_n; \partial_t, \partial_1, \ldots, \partial_n)$.
  (b) Compute a generator $b(s)$ of the ideal

$$\{b(s) \in \mathbb{C}[s] \mid b(t\partial_t) + Q \in J \text{ with some } Q \in D_{n+1} \text{ such that } \mathrm{ord}_w(Q) \leq -1\}$$

of $\mathbb{C}[s]$, where $\mathrm{ord}_w(Q)$ denotes the maximum weight of the terms of Q with respect to the weight w. $b(s)$ can be computed by eliminating x and $\partial_1, \ldots, \partial_n$ from the highest weight parts (w.r.t. w) of elements of G. Find the largest nonnegative integer root k_1 of $b(s) = 0$. If there is no nonnegative integer root, then we have $M[1/f] = 0$.

(c) In general, for $P \in D_{n+1}$, there exist unique $Q \in D_{n+1}$ and $R \in D_n[t]$ such that

$$P = \partial_t Q + R.$$

Let us denote this R by $R = \rho(P)$. Then R can be regarded as a relation among the residue classes $\overline{1}, \overline{t}, \overline{t^2}, \ldots$ in $N/\partial_t N$. Let L be the left D_n-submodule of $D_n + tD_n + \cdots + t^{k_1} D_n \simeq D_n^{k_1+1}$ generated by

$$\{\rho(t^j P) \mid P \in G,\ \mathrm{ord}_w(P) + j \leq k_1\}.$$

Then L defines a system of linear differentieal equations for $\overline{1}, \overline{t}, \cdots \overline{t^{k_1}}$ in $N/\partial_t N$.

(d) Eliminate $\overline{1}, \overline{t}, \cdots \overline{t^{k_1-1}}$ from L and obtain an ideal L_0 of D_n which annihilates $\overline{t^{k_1}}$.

Output: $\mathcal{M}[1/f]$ is isomorphic to the sheafification of D_n/L_0. More precisely the ideal L_0 is the annihilator ideal of $f^{-k_1-2}u$, which generates $M[1/f]$ (here u is the residue class of 1 in M).

Proof: First we have

$$j^{-1}\mathcal{M} = \mathcal{D}_X[1/f]/\left(\mathcal{D}_X[1/f]P_1 + \cdots + \mathcal{D}_X[1/f]P_r\right),$$

which is a holonomic $\mathcal{D}_X[1/f]$-module. Let A_W be the subring of D_{n+1} generated by $\mathbb{C}[t, x]$ and $\vartheta_1, \ldots, \vartheta_n$. Then $A_W(tf(x) - 1)$ is a two-sided ideal of A_W and $D_W := A_W/A_W(1 - tf(x))$ is the set of global sections of the sheaf $\mathcal{D}_X$ of algebraic differential operators on W (note that W is affine). Then we have an isomorphism (see [4])

$$\int_\varphi j^{-1}\mathcal{M} \simeq \mathcal{D}_W/(\mathcal{D}_W\widetilde{P}_1 + \cdots + \mathcal{D}_W\widetilde{P}_r).$$

Next, the integration along ι is nothing but the so-called Kashiwara equivalence and in view of Proposition A.1 of [4] we have

$$\int_\iota\int_\varphi j^{-1}\mathcal{M} = \mathcal{D}_{\mathbb{C}\times X} \otimes_{D_{n+1}} N,$$

which is a holonomic $\mathcal{D}_{\mathbb{C}\times X}$-module. Next by the definition of the integration we have

$$\int_p N = N/\partial_t N.$$

Let $\mathcal{F}(J)$ be the partial Fourier transform of J with respect to t, which is the ring isomorphism of D_{n+1} that sends t to $-\partial_t$, ∂_t to t, and leaves x_i, ∂_i unchanged. Put $\mathcal{F}(N) := D_{n+1}/\mathcal{F}(J)$. Then we have

$$N/\partial_t N \simeq \mathcal{F}(N)/t\mathcal{F}(N)$$

and the step (2) is nothing but (the Fourier transform of) the restriction algorithm (Theorem 5.7) of [5]. Note that $N/\partial_t N$ is a holonomic D_n-module since N is holonomic on $\mathbb{C} \times X$. Thus we have proved that

$$M[1/f] \simeq N/\partial_t N \simeq (D_n)^{k_1+1}/L. \tag{1}$$

Let us describe the first isomorphism of (1) more explicitly. First note the isomorphism

$$N/\partial_t N \simeq D_{n+1}/(J + \partial_t D_{n+1}).$$

For an arbitrary $P \in D_{n+1}$, there exist unique $R_0, R_1, \ldots \in D_n$ and $S \in D_{n+1}$ such that

$$P = \sum_{j \geq 0} t^j R_j(x, \vartheta_1, \ldots, \vartheta_n) + \partial_t S. \tag{2}$$

Then we define

$$\begin{aligned} \psi(P) &:= \sum_{j\geq 0} f^{-j-2} R_j(x, \partial_1, \ldots, \partial_n) u \\ &= \sum_{j\geq 0} R_j \left(x, \partial_1 + (j+2)\frac{f_1}{f}, \ldots, \partial_n + (j+2)\frac{f_n}{f} \right) f^{-j-2} u \qquad \in M[1/f]. \end{aligned}$$

Note that since the commutation relation of x_i and ϑ_i is the same as that of x_i and ∂_i, the above (non-commutative) substitution makes sense irrespective of the actual expression of R_j. This defines a left D_n-homomorphism $\psi : D_{n+1} \longrightarrow M[1/f]$. In fact, we have $\psi(\partial_i P) = \partial_i \psi(P)$ since

$$\begin{aligned} \psi(\partial_i t^j R_j(x, \vartheta_1, \ldots, \vartheta_n)) &= \psi(t^j(\vartheta_i + t^2 f_i \partial_t) R_j(x, \vartheta_1, \ldots, \vartheta_n)) \\ &= \psi(t^j(\vartheta_i - (j+2)t f_i) + \partial_t t^{j+2} f_i) R_j(x, \vartheta_1, \ldots, \vartheta_n)) \\ &= \psi(t^j(\vartheta_i - (j+2)t f_i) R_j(x, \vartheta_1, \ldots, \vartheta_n)) \\ &= f^{-j-2} \partial_i R_j(x, \partial_1, \ldots, \partial_n) u - (j+2) f^{-j-3} f_i R_j(x, \partial_1, \ldots, \partial_n) \\ &= \partial_i f^{-j-2} R_j(x, \partial_1, \ldots, \partial_n) u. \end{aligned}$$

Since $P_1, \ldots, P_r$ annihilate u, we get

$$\psi(t^j \widetilde{P}_i) = f^{-j-2} P_i(x, \partial_1, \ldots, \partial_n) u = 0.$$

It is easy to see that $\psi(t^j(1-tf(x))=0$. Hence $J+\partial_t D_{n+1}$ is contained in the kernel of ψ.

Conversely, suppose that P of the form (2) is contained in the kernel of ψ. Then there exist $Q_1(t,x,\partial),\dots,Q_r(t,x,\partial)\in D_n[t]$ such that

$$\sum_{j\geq 0} f^{-j}R_j(x,\partial_1,\dots,\partial_n)=\sum_{i=1}^{r} f^2 Q_i(1/f,x,\partial_1,\dots,\partial_n)P_i(x,\partial_1,\dots,\partial_n)$$

holds in $D_n[1/f]$. Then the Hilbert Nullstellensatz assures that

$$\sum_{j\geq 0} t^j R_j(x,\vartheta_1,\dots,\vartheta_n)-\sum_{i=1}^{r} f^2 Q_i(t,x,\vartheta_1,\dots,\vartheta_n)P_i(x,\vartheta_1,\dots,\vartheta_n) \quad \in (1-tf(x))D_{n+1}$$

since $1-tf(x)$ is irreducible. Noting $(1-tf(x))\vartheta_i\in D_{n+1}(1-tf(x))$, we conclude that $P\in J+\partial_t D_{n+1}$. Thus ψ gives the first isomorphism of (1). This implies that $M[1/f]$ is generated by $f^{-2}u,\dots,f^{-k_1-2}u$, and hence only by $f^{-k_1-2}u$. This completes the proof.

2 An application to holonomic functions

Let u be a (possibly multivalued) analytic function defined on $\mathbb{C}^n$ minus an algebraic set. Supppose that u is hyperexponential ([1]); i.e., $g_i:=\partial_i u/u$ is a rational function for any $i=1,\dots,n$. For example, if $f_1,\dots,f_m,g$ are rational functions and $\alpha_1,\dots,\alpha_m$ are complex numbers, then

$$u=f_1^{\alpha_1}\cdots f_m^{\alpha_m}\exp(g(x))$$

is a hyperexponential function. Then we can find the annihilator ideal

$$\mathrm{Ann}(u):=\{P\in D_n \mid Pu=0\}$$

of u exactly by applying the localization algorithm.

Algorithm-Theorem 2
Input: Let u be a (possibly) multi-valued analytic function such that $g_i:=\partial_i u/u\in\mathbb{Q}(x)$ for any $i=1,\dots,n$.

(1) Let $g\in\mathbb{Q}[x]$ be the least common multiple of the denominators of $g_1,\dots,g_n$. Let $f(x)$ be the square-free part of g.
(2) Put $I:=D_n(g\partial_1-gg_1)+\cdots+D_n(g\partial_n-gg_n)$.
(3) Apply Algorithm-Theorem 1 with input D_n/I and f, and let L_0 be the output ideal with the integer k_1.

(4) Compute the ideal quotient

$$L_1 := L_0 : (f^{k_1+2}) = \{P \in D_n \mid Pf^{k_1+2} \in L_0\}$$

by syzygy computation through Gröbner basis.

Output: $L_1 = \mathrm{Ann}(u)$. In particular, u is a holonomic function, i.e., $D_n/\mathrm{Ann}(u)$ is a holonomic system.

Proof: Put $\mathcal{L} := \mathcal{D}_X u$, which is a sheaf of multivalued analytic functions, and define the sheaf

$$\mathcal{A}nn(u) := \{P \in \mathcal{D}_X \mid Pu = 0\},$$

which is the sheafification of $\mathrm{Ann}(u)$. Then we have $\mathcal{L} \simeq \mathcal{D}_X/\mathcal{A}nn(u)$. Let $\mathcal{M}$ be the sheafification of $M = D_n/I$. It is easy to see that $\mathcal{M}$ is a holonomic system of rank one outside of $Y := \{x \in X = \mathbb{C}^n \mid f(x) = 0\}$. This implies that the two sheaves $\mathcal{M}$ and $\mathcal{L}$ coincide on $X \setminus Y$. Hence in view of the Hilbert Nullstellensatz, we have

$$\mathcal{M}[1/f] = \mathcal{L}[1/f]. \tag{3}$$

By Algorithm-Theorem 1, $M[1/f]$ is generated by $f^{-k_1-2}\overline{1}$ whose annihilator ideal is L_0. Hence L_1 is the annihilator ideal of $\overline{1}$ in $M[1/f]$.

On the other hand, since $\mathcal{L}$ is a set of analytic functions, the natural homomorphism

$$\mathcal{L} \longrightarrow \mathcal{L}[1/f] = \mathcal{O}_X[1/f] \otimes_{\mathcal{O}_X} \mathcal{L}$$

induced by the embbeding of $\mathcal{O}_X$ to $\mathcal{O}_X[1/f]$ is injective. In fact, this follows from the fact that $f\cdot : \mathcal{L} \longrightarrow \mathcal{L}$ is injective. By the isomorphism (3), $\overline{1} \in M[1/f]$ corresponds to $u \in \mathcal{L}$, and its annihilator ideal in $\mathcal{L}[1/f]$ is given by L_1. Since $\mathcal{L}$ is a submodule of $\mathcal{L}[1/f]$, the annihilator ideal of u in $\mathcal{L}[1/f]$ coincides with that in $\mathcal{L}$. This implies that $L_1 = \mathrm{Ann}(u)$. This completes the proof.

Example 1 Put $X := \{(x,y,z) \in \mathbb{C}^3\}$ and $f(x,y,z) := x^3 - y^2z^2$. Let us find the annihilator ideal of the function $u := \exp(1/f(x))$. The following computations were performed by computer algebra systems kan/sm1 [9] and Risa/Asir [8] which are connected via open xxx protocol [10]. First let I be the left ideal of D_3 generated by

$$f^2\partial_x - f_x, \quad f^2\partial_y - f_y, \quad f^2\partial_z - f_z$$

with $\partial_x = \partial/\partial x$, $f_x = \partial f/\partial x$, and so on. By computing the characteristic variety $\mathrm{Char}(M)$ of $M := D_3/I$ (see [3] for an algorithm) and by decomposing it to prime (or primary) factors, we know that

$$\mathrm{Char}(M) \supset \{(x,y,z;\xi,\eta,\zeta) \in T^*\mathbb{C}^3 \mid x = y = 0\} \cup \{x = z = 0\}.$$

In particular, M is not holonomic on $\mathbb{C}^3$. Next by using Algorithm-Theorem 1 with I and f as input, we know that $\mathrm{Ann}(u)$ is generated by the following eight operators:

$$
\begin{aligned}
&36y\partial_y - 36z\partial_z,\\
&-24yz^2\partial_x - 36x^2\partial_y,\\
&-24y^2z\partial_x - 36x^2\partial_z,\\
&-24z^3\partial_x\partial_z - 36x^2\partial_y^2 - 24z^2\partial_x,\\
&24x^4\partial_x^2 + 72x^3z\partial_x\partial_z + 54x^2z^2\partial_z^2 + 96x^3\partial_x + 162x^2z\partial_z + 72\partial_x,\\
&36y^2z^3\partial_z - 24x^4\partial_x - 72x^3z\partial_z - 72,\\
&-36yz^4\partial_z^2 + 24x^4\partial_x\partial_y + 72x^3z\partial_y\partial_z - 108yz^3\partial_z + 72\partial_y,\\
&36z^5\partial_z^3 - 24x^4\partial_x\partial_y^2 - 72x^3z\partial_y^2\partial_z + 216z^4\partial_z^2 + 216z^3\partial_z - 72\partial_y^2.
\end{aligned}
$$

We can verify that $D_3/\mathrm{Ann}(u)$ is in fact holonomic and that I is contained in $\mathrm{Ann}(u)$.

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