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Quantile Hedging for Defaultable Securities in an Incomplete Market

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Abstract: In this paper, we aim at

1. giving formulas of prices and replicating-strategies of defaultable securities(e.g., bonds, swaps, derivatives) in incomplete market, and
2. giving "solvable" examples of quantile hedging strategies in incomplete market.

Considering an incomplete market that consists of tradable assets and an unhedgeable defaultable security, whose non-predictable default time has stochastic intensity correlated with the tradable assets-price-processes, we treat the problem of pricing and hedging of the defaultable security on it. We employ the quantile hedging strategy (cf., [F-L]) to replicate "the cumulative dividend process" of the defaultable security by an admissible strategy among the tradable assets. The strategy that maximize the success probability of hedge under the given initial capital and the strategy that minimize the initial capital under the given success probability of hedge are calculated explicitly.

Keywords: quantile-hedging, defaultable security, incomplete market, Neyman-Pearson's lemma

1 Introduction

One of the major approach to pricing defaultable securities, the so-called "reduced-form approach" (or "intensity-model approach") regards the default time τ as "unpredictable" (i.e., totally inaccessible)

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stopping time. (cf., [D-S,D-S-S,J-T,L1,L2].) Therefore, for example, if we start with a filtration generated by continuous assets price processes, a defaultable security expressed as a functional of a discontinuous submartingale $\left(1_{\{\tau<t\}}\right)_{t\geq 0}$ of "default indicator", is unhedgeable by its definition. So, in the referred papers above, and the all existing studies about reduced-form approach as we know, the standpoint that

- some defaultable securities(e.g., bonds) are already marketed and liquid on a market and the completeness of the market is established, or
- an equivalent martingale measure is given a priori and fixed,

is employed and arbitrage-free pricing and hedging formulas of defaultable securities(e.g., bonds, derivatives) are derived under the measure.

In this paper, we will start with an incomplete market setting (not fixing equivalent martingale measure) and price or replicate ("new-introduced") defaultable securities. Typical examples of our setting are perhaps the problems of pricing and hedging of *untraded* or *unliquid* defaultable securities (e.g., loans). Especially, we will employ the quantile hedging strategy for the replication, which has recently introduced by Föllmer and Leukert in [F-L] in place of perfect or super replication. We will seek the strategy that

1. maximize the probability of success of hedge under a given initial capital, or
2. minimize the initial capital under a given lower bound of success probability of hedge.

They can be regarded as dynamic versions of the VaR (i.e., Value at Risk), a globally standard method for the measurement of marketed risks, and still look more realistic than the perfect or the super replication, although some drawbacks have been pointed out. (cf., [A-D-E-H],[F-L].) In Corollary 2, as a simplest example, we give a Jarrow-Turnbull-type defaultable-bond model with deterministic hazard-rate process (cf., [J-T]) and a constant risk-premium parameter; in this case the only random variable Z_T is lognormal distributed, and very explicit expressions of the solutions are obtained. To obtain the explicit optimal solutions, the Neyman-Pearson's fundamental lemma in hypothesis testing has been effectively utilized (at least in complete market cases) in [F-L], while it might not be so effective in general incomplete market cases. Fortunately, in our defaultable security models, since the equivalent martingale measure that realizes "the worst scenario for hedging" can be characterized explicitly (cf., Lemma 4, and the proof of Lemma 5 in Section 3), we can also obtain the explicit solutions via the Neyman-Pearson's lemma (by solving a statistical-test-type problem against *a simple* alternative iteratedly). Financial theoretically, our defaultable security model is an unsatisfactory deformed one as stated in Assumption 2 in the next section.

We will restrict the behavior of the security-holder *after the default*, which enables us to concentrate to hedge the "payoff" at the *terminal-date* T:

$$H_T := \begin{cases} d_T, & \text{if default occurs before } T, \\ D, & \text{if default does not occur before } T \end{cases}$$

of the security, and as a result, the problems are simplified and the explicit solutions for this "European-type" defaultable security can be obtained. More proper model(or problem) is may be the one stated in the remark after Problem 1-2, for example, though it remains unsolved.

In the next section, we will state our setup and our main results, and in Section 3, we prove them.

2 Setup and Results

For a fixed constant $T(>0)$, let us prepare a complete probability space, $(\Omega, \mathcal{F}, P)$, a d-dimensional Brownian motion on it, $w := (w_t)_{t\in[0,T]}$, the augmented Brownian filtration, $(\mathcal{G}_t)_{t\in[0,T]}$, (i.e., $\mathcal{G}_t := \sigma\text{-}\{w_s; s \in [0,t]\} \vee \{A \subset \Omega; \exists B \in \mathcal{F} \text{ with } A \subset B, P(B) = 0\}$), and a random variable e that is independent of $\mathcal{G}_T$ and exponentially distributed (with intensity 1).

Now, consider a financial market on a time interval $[0,T]$ consists of the following elements:

1. the $(d+1)$-assets-price-processes:

$$p := (p_t)_{t\in[0,T]}, q^1 := (q^1_t)_{t\in[0,T]}, \dots, q^d := (q^d_t)_{t\in[0,T]}$$

   that are $\mathcal{G}_t$-adapted processes, in particular, p is the price process of a default-free bond maturing at T, i.e., it holds that $p_t > 0$ for all $t \in [0,T]$ and $p_T = 1$ P-a.e.,

2. a defaultable security, expressed as the triplet: (τ, d, D) (cf., [D-S-S]), i.e.,

   (a) the default time, τ, defined by the formula:

$$\tau := \inf\left\{t > 0\,;\, e \le \int_0^t \lambda_u du\right\}$$

   with a nonnegative $\mathcal{G}_t$-adapted process, $\lambda := (\lambda_t)_{t\in[0,T]}$, which satisfies

$$\Pr\left(\int_0^t \lambda_u du < \infty\right) = 1,$$

   P-a.e. for all $t \in [0,T]$,

   (b) the payoff upon default, d_τ, which is determined by the default time τ above and a nonnegative $\mathcal{G}_t$-predictable process $d := (d_t)_{t\in[0,T]}$,

(c) the payoff at the terminal date T, say D, which is nonnegative, $\mathcal{G}_T$-measurable and provided if there has been no default.

Let us denote the default indicator function by

$$N_t := 1_{\{\tau \le t\}} \quad (t \in [0,T]),$$

set the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ by

$$\mathcal{F}_t := \mathcal{G}_t \vee \sigma\text{-}\{N_s\,;\; s \in [0,t]\},$$

and interpret it as the whole information on the market (along the time-evolution). This is a way of introducing *reduced-form* defaultable security model, which follows [L1-2], especially. More generally, [D-S-S] and [K] are referred for example. For simplicity, we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$. By definition above, it is easy to see that the relation

$$E\left[1 - N_t \mid \mathcal{G}_t\right] = \Lambda_t := \exp\left\{-\int_0^t \lambda_u du\right\}$$

holds for $t \in [0,T]$ and that the process: $(M_t)_{t\in[0,T]}$, where

$$M_t := N_t - \int_0^t (1 - N_u)\lambda_u du$$

is an $\mathcal{F}_t$-martingale obtained from the Doob-Meyer decomposition of the submartingale $(N_t)_{t\in[0,T]}$. Moreover, let us recall the following, which shall be used in the proof of our results:

Lemma 1 *(Corollary 3.8 in [K], or Proposition 3.1 in [L2]) For any $\mathcal{G}_T$-measurable and $L^1(P)$-random variable F, we have*

$$E\left[F(1 - N_T) \mid \mathcal{F}_t\right] = (1 - N_t)E\left[\Lambda_T \Lambda_t^{-1} F \mid \mathcal{G}_t\right] \quad \textit{for any } t \in [0,T].$$

Throughout this paper, we assume the following:

Assumption 1 *The normalized assets-prices-process:*

$$X = \left(X^1, \ldots, X^d\right)' := \left(q^1/p, \ldots, q^d/p\right)',$$

($(\cdot)'$ denotes the transposition of a vector) with a numéraire p satisfies the following stochastic differential equation:

$$dX_t^i \;=\; X_t^i \left[\sum_{j=1}^d \sigma_t^{ij}\left(dw_t^j + \gamma_t^j dt\right)\right] \quad (i = 1, \ldots, d,\, t \in [0,T]),$$

with a $d \times d$-matrix-valued $\sigma = \left(\sigma^{ij}\right)_{1\le i,j\le d}$ and an $\mathbb{R}^d$-valued $\gamma = \left(\gamma^1, \ldots, \gamma^d\right)'$ are $\mathcal{G}_t$-adapted satisfying

1. $c|x|^2 \le (\sigma_t(\omega)x, x) \le C|x|^2$, $P \times dt$-a.e., *for all* $x \in \mathbb{R}^d$ *and for some* $0 < c \le C$,

2. *the space of the probability measures on* $(\Omega, \mathcal{F})$:

$$\mathcal{P} := \{Q \ : \ \textit{equivalent to } P, \textit{ and } X \textit{ is a martingale under } Q\}$$

contains $\hat{P}$, *given by the formula:*

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}_t\left(-\int \gamma dw\right) =: Z_t \quad \textit{for all } t \in [0,T].$$

The cumulative dividend process $H := (H_t)_{t\in[0,\tau\wedge T]}$ of the defaultable security (τ, d, D) is defined by

$$\begin{aligned} H_t &:= d_\tau N_t + D(1 - N_T)1_{\{t\ge T\}} \\ &= \int_0^{\tau\wedge t} d_u dN_u + D(1 - N_T)1_{\{t \ge T\}}, \end{aligned}$$

as in [D-S-S]; we will deform the definition:

Assumption 2 *The process* d/p *is a* $\hat{P}$*-martingale, which means that a holder of the security receives some tradable (and priced arbitrage-freely) asset in the case of default. After the default* $t > \tau$, *we will assume that the holder keeps the tradable* d, *so, we will interpret the value of the cumulative dividend* $H_t := d_t$; *we will extend the cumulative divided process* H *on* $[0, \tau \wedge T]$ *to* $[0, T]$ *by redefining*

$$\begin{aligned} H_t &:= d_t N_t + D(1 - N_T)1_{\{t\ge T\}} \\ &= \begin{cases} 0 & \textit{on } \{0 \le t < (\tau \wedge T)\}, \\ d_t & \textit{on } \{\tau \le t \le T\}, \\ D & \textit{on } \{t = T < \tau\}. \end{cases} \end{aligned}$$

Assumption 3 *One of the following is satisfied:*

$$\textit{(A)} \quad D > d_T \ge 0 \quad \hat{P}\textit{-a.e.},$$

$$\textit{(B)} \quad 0 \le D \le d_T \quad \hat{P}\textit{-a.e.}.$$

Remark: Assumption 3 can be removed. It is just for the simplicity of the presentation of our results, and it is satisfied in typical examples: e.g.,

- a defaultable (zero-coupon) bond model: $D = 1 > d_T \ge 0$, P-a.e.. This can be interpreted a generalization of defaultable bond model by Jarrow and Turnbull in [J-T]. Upon default, the bond-holder receives $\delta_\tau p_\tau$, where $\delta_\tau := \hat{E}[d_T|\mathcal{G}_\tau]$ is called the recovery-rate upon default.

- a default-swap model: e.g., an insurance on the defaultable-bond above, i.e., the holder is insured the default-loss: $d_t := (1-\delta_\tau)p_\tau$, and D is set to 0.

Now, consider the situation that a hedger seeks to recover the default-loss of the defaultable security by a self-financing strategy between the assets $p, q^1, \ldots, q^d$, or that a writer of the security who wants to decide the price of this defaultable security. By a standard argument, the value process $(V_t)_{t\in[0,T]}$ of the self-financing hedging portfolio is written as

$$V_t = p_t\left(\frac{V_0}{p_0} + \int_0^t \xi_u dX_u\right),$$

where $V_0 \in \mathbb{R}_+$ is the initial cost and the $\mathcal{G}_t$-predictable (and X-integrable) process $\xi := (\xi_t)_{t\in[0,T]}$ represents the trading-process of the assets. If $V_t \geq 0$, P-a.e. for all $t \in [0,T]$, then, the strategy is called *admissible* in this paper. Obviously, the hedger cannot replicate perfectly the cumulative dividend process $H := (H_t)_{t\in[0,T]}$ of the defaultable security by the admissible strategy between $p, q^1, \ldots, q^d$, i.e., our market is incomplete. We can observe

$$\bar{H}_t := \operatorname*{esssup}_{P^*\in\mathcal{P}} E^*\left[H_T|\mathcal{F}_t\right] = \frac{d_t}{p_t} + (1-N_t)\hat{E}\left[(D-d_T)^+ \mid \mathcal{G}_t\right] \tag{1}$$

(cf., Lemma 6), it provides us the trivial super hedging strategy of H such that:

- starting with the initial cost $\bar{H}_0 = d_0/p_0 + \hat{E}\left[(D-d_T)^+\right] = \hat{E}\left[\max\left(d_T, D\right)\right]$ and choose the trading process $(\xi_t)_{t\in[0,T]}$ of X, such that

$$\frac{V_t}{p_t} = \bar{H}_0 + \int_0^t \xi_u dX_u := \hat{E}\left[\max\left(d_T, D\right) \mid \mathcal{G}_t\right]$$

then, the hedger shall be in the safe-side:

$$V_T \geq H_T,$$

at the terminal-date T with probability 1.

Instead of the trivial strategy above, we will employ the quantile hedging strategy that has been proposed by Föllmer and Leukert in [F-L] as more "*suitable*" strategy and price for the defaultable security; we will seek the following:

Problem 1 *(maximizing the probability of success) Fix $\tilde{V}_0 \leq \bar{H}_0$. Among admissible strategies, solve the following optimization-problem:*

$$\max \Pr\left(\{V_T \geq H_T\}\right) \quad \textit{subject to} \quad V_0 \leq \tilde{V}_0, \tag{2}$$

Problem 2 *(minimizing the cost for a given probability of success) Fix $0 < \alpha < 1$. Among admissible strategies, solve the following optimization-problem:*

$$\min V_0 \quad \textit{subject to} \quad \Pr(\{V_T \geq H_T\}) \geq 1 - \alpha, \tag{3}$$

Remark: It might be more natural to consider the probability *at the default time*:

$$\Pr(\{V_{\tau\wedge T} \geq H_{\tau\wedge T}\})$$

in place of the probability *at the terminal*:

$$\Pr(\{V_T \geq H_T\})$$

in the expression (2) and (3) since the defaultable securities are only defined on the time interval $[0, \tau\wedge T]$; for example, in Problem 1, the inequality:

$$\max \Pr(\{V_{\tau\wedge T} \geq H_{\tau\wedge T}\}) \leq \max \Pr(\{V_T \geq H_T\}),$$

is always satisfied, where the maximization is considered over all admissible strategies with the initial cost $V_0 \leq \tilde{V}_0 (\leq \bar{H}_0)$. Our deformation simplifies our quantile hedging problems, we only have to see the "two states": N_T and $1 - N_T$, i.e., at the terminal T, if the default occurs or not.

Our results are stated as follows:

Theorem 1 (A) *Let (A) in Assumption 3 hold. For a nonnegative constant k, denote*

$$\begin{aligned} A_1(k) &:= \{1 - \Lambda_T > k d_T Z_T, \Lambda_T \leq k(D - d_T) Z_T\}, \\ A_2(k) &:= \{1 > kDZ_T, \Lambda_T > k(D - d_T) Z_T\}, \end{aligned}$$

and assume that there exists $k^* = k^*(\tilde{V}_0)$ satisfying

$$\hat{E}\left[1_{A_1^*} d_T + 1_{A_2^*} D\right] = \tilde{V}_0 / p_0, \tag{4}$$

where we denote by $A_1^* := A_1(k^*), A_2^* := A_2(k^*)$. The super replicating strategy of "the modified claim":

$$\tilde{H}_T := 1_{A_1^* \cup A_2^*} d_T N_T + 1_{A_2^*} D (1 - N_T) \tag{5}$$

is a solution of Problem 1. We have

$$\operatorname*{esssup}_{P^* \in \mathcal{P}} E^*\left[\tilde{H}_T \mid \mathcal{F}_t\right] = \hat{E}\left[1_{A_1^* \cup A_2^*} d_T \mid \mathcal{G}_t\right] + (1 - N_t) \hat{E}\left[\left(1_{A_2^*} D - 1_{A_1^* \cup A_2^*} d_T\right)^+ \mid \mathcal{G}_t\right]$$

$$
\begin{aligned}
&\leq \hat{E}\left[1_{A_1^*\cup A_2^*} d_T \mid \mathcal{G}_t\right] + \hat{E}\left[\left(1_{A_2^*} D - 1_{A_1^*\cup A_2^*} d_T\right)^+ \mid \mathcal{G}_t\right] \\
&= \hat{E}\left[\max\left(1_{A_1^*\cup A_2^*} d_T, 1_{A_2^*} D\right) \mid \mathcal{G}_t\right] \\
&= \hat{E}\left[1_{A_1^*} d_T + 1_{A_2^*} D \mid \mathcal{G}_t\right],
\end{aligned}
$$

and we can construct an optimal strategy $(\tilde{V}_0, \xi^*)$ *by defining*

$$
\hat{E}\left[1_{A_1^*} d_T + 1_{A_2^*} D \mid \mathcal{G}_t\right] = \frac{\tilde{V}_0}{p_0} + \int_0^t \xi_u^* dX_u \quad \textit{for } t \in [0, T].
$$

(B) *Let (B) in Assumption 3 hold. For a nonnegative constant* k*, denote*

$$
\begin{aligned}
B_1(k) &:= \{1 > k d_T Z_T, 1 - \Lambda_T > k(d_T - D) Z_T\}, \\
B_2(k) &:= \{\Lambda_T > k D Z_T, 1 - \Lambda_T \leq k(d_T - D) Z_T\},
\end{aligned}
$$

and assume that there exists $k^* = k^*(\tilde{V}_0)$ *satisfying*

$$
\hat{E}\left[1_{B_1^*} d_T + 1_{B_2^*} D\right] = \tilde{V}_0 / p_0, \tag{6}
$$

where we denote by $B_1^* := B_1(k^*), B_2^* := B_2(k^*)$*. The super replicating strategy of "the modified claim":*

$$
\tilde{H}_T := 1_{B_1^*} d_T N_T + 1_{B_1^* \cup B_2^*} D (1 - N_T) \tag{7}
$$

is a solution of Problem 1. We have

$$
\begin{aligned}
\operatorname*{esssup}_{P^* \in \mathcal{P}} E^*\left[\tilde{H}_T \mid \mathcal{F}_t\right] &= \hat{E}\left[1_{B_1^*} d_T \mid \mathcal{G}_t\right] + (1 - N_t)\hat{E}\left[\left(1_{B_1^*\cup B_2^*} D - 1_{B_1^*} d_T\right)^+ \mid \mathcal{G}_t\right] \\
&\leq \hat{E}\left[1_{B_1^*} d_T \mid \mathcal{G}_t\right] + \hat{E}\left[\left(1_{B_1^*\cup B_2^*} D - 1_{B_1^*} d_T\right)^+ \mid \mathcal{G}_t\right] \\
&= \hat{E}\left[\max\left(1_{B_1^*\cup B_2^*} D, 1_{B_1^*} d_T\right) \mid \mathcal{G}_t\right] \\
&= \hat{E}\left[1_{B_1^*} d_T + 1_{B_2^*} D \mid \mathcal{G}_t\right],
\end{aligned}
$$

and we can construct an optimal strategy $(\tilde{V}_0, \xi^*)$ *by defining*

$$
\hat{E}\left[1_{B_1^*} d_T + 1_{B_2^*} D \mid \mathcal{G}_t\right] = \frac{\tilde{V}_0}{p_0} + \int_0^t \xi_u^* dX_u \quad \textit{for } t \in [0, T].
$$

Theorem 2 **(A)** *Let (A) in Assumption 3 hold and assume that the equation:*

$$
E\left[1_{A_1(k)}(1 - \Lambda_T) + 1_{A_2(k)}\right] = 1 - \alpha \tag{8}
$$

with respect to k *is solved for some* $k^* = k^*(\alpha)$*. Then, the super replicating strategy of "the modified claim" defined by (5) is a solution of Problem 2.*

(B) *Let (B) in Assumption 3 hold and assume that the equation:*

$$E\left[1_{B_1(k)} + 1_{B_2(k)}\Lambda_T\right] = 1-\alpha \tag{9}$$

with respect to k is solved for some $k^* = k^*(\alpha)$. Then, the super replicating strategy of "the modified claim" defined by (7) is a solution of Problem 2.

Remark: 1. The existence of the sets $A_1^*, A_2^*, B_1^*, B_2^* \in \mathcal{G}_T$ satisfying (4),(6),(8), or (9) is assured if, for example,

$$\begin{aligned} & E\left[1_{\partial A_1(k)}\left(1-\Lambda_T\right) + 1_{\partial A_2(k)}\right] = 0 \\ \text{and} \quad & E\left[1_{\partial B_1(k)} + 1_{\partial B_2(k)}\Lambda_T\right] = 0, \\ \text{where} \quad & \partial A_1(k) := \{1-\Lambda_T = kd_T Z_T, \Lambda_T = k(D-d_T)Z_T\}, \\ & \partial A_2(k) := \{1 = kDZ_T, \Lambda_T = k(D-d_T)Z_T\}, \\ & \partial B_1(k) := \{1 = kDZ_T, 1-\Lambda_T = k(d_T-D)Z_T\}, \\ \text{and} \quad & \partial B_2(k) := \{\Lambda_T = kDZ_T, 1-\Lambda_T = k(d_T-D)Z_T\} \end{aligned}$$

are satisfied for arbitrary $k \geq 0$ (cf., e.g., [Sc] Chapter III,3). If the sets do not exist, we can reformulate our quantile-hedging procedure as stated in [F-L]: for instance, in Problem 1, we will modify "the success-set-maximization" to "the success-ratio-maximization".

2. In Theorem 2, the minimal cost of quantile hedging strategy:

$$\tilde{V}_0 := \begin{cases} p_0\hat{E}\left[1_{A_1^*}d_T + 1_{A_2^*}D\right] & \text{in the Case (A),} \\ p_0\hat{E}\left[1_{B_1^*}d_T + 1_{B_2^*}D\right] & \text{in the Case (B)} \end{cases}$$

is reexpressed as

$$\tilde{V}_0 = \begin{cases} \lim_{\epsilon\to 0} p_0 E_\epsilon^{(d_T 1_{A_1^* \cup A_2^*} + D1_{A_2^*})}[H_T] & \text{in the Case (A),} \\ \lim_{\epsilon\to 0} p_0 E_\epsilon^{(d_T 1_{B_1^*} + D1_{B_1^* \cup B_2^*})}[H_T] & \text{in the Case (B),} \end{cases}$$

by using the sequences of equivalent martingale measures (abbrev. EMM, hereafter):

$$\left(Q_\epsilon^{(d_T 1_{A_1^* \cup A_2^*} + D1_{A_2^*})}\right)_{\epsilon>0}, \quad \text{and} \quad \left(Q_\epsilon^{(d_T 1_{B_1^*} + D1_{B_1^* \cup B_2^*})}\right)_{\epsilon>0},$$

and the expectations with respect to them, which shall be defined in Lemma 4 in the next section. An interpretation of the expression above is that the optimal cost $\tilde{V}_0$ is the expectation of the payoff H_T with respect to the EMM that realizes "the worst scenario for hedging".

Further, we add two corollaries of the theorems above without proofs. First, we observe the following "trivialized" situations:

Corollary 1 **(A)** *1. if* $0 < (1-\Lambda_T)D \leq d_T < D$ *holds* $\hat{P}$*-a.e., we have*

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^* D} \right\} \quad \textit{satisfying} \quad \hat{E}\left[1_{A_2^*} D\right] = \tilde{V}_0/p_0,$$

2. if $0 < (1-\Lambda_T)D \leq d_T < D$ *holds* P*-a.e., we have*

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^* D} \right\} \quad \textit{satisfying} \quad \mathrm{P}\left(A_2^*\right) = 1-\alpha,$$

(B) *1. if* $0 < \Lambda_T d_T \leq D \leq d_T$ *holds* $\hat{P}$*-a.e., we have*

$$B_1^* = \left\{ Z_T < \frac{1}{k^* d_T} \right\} \quad \textit{satisfying} \quad \hat{E}\left[1_{B_1^*} d_T\right] = \tilde{V}_0/p_0, \quad B_2^* = \emptyset,$$

2. if $0 < \Lambda_T d_T \leq D \leq d_T$ *holds* P*-a.e., we have*

$$B_1^* = \left\{ Z_T < \frac{1}{k^* d_T} \right\} \quad \textit{satisfying} \quad \mathrm{P}\left(B_1^*\right) = 1-\alpha, \quad B_2^* = \emptyset,$$

In each cases, (conditional) default probability Λ_T *has no effect on the optimal solutions of quantile hedging.*

Secondly, we give an explicit calculation in the case of Jarrow-Turnbull-type defaultable bond model.

Corollary 2 *Let* $0 < d_T = \delta < D = 1$ *and* Λ *(or* λ*) be deterministic. We have*

$$A_1^* = \left\{ \frac{\Lambda_T}{k^*(1-\delta)} \leq Z_T \leq \frac{1-\Lambda_T}{k^*\delta} \right\}, \quad A_2^* = \left\{ Z_T < \frac{\Lambda_T}{k^*(1-\delta)} \right\}$$

in the case of $\Lambda_T + \delta < 1$*, and*

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^*} \right\}$$

in the case of $\Lambda_T + \delta \geq 1$*, as given in Corollary 1 (A). Setting* $d = 1$ *and the risk-premium process* γ *constant, we observe*

1. in Problem 1, the equations (4) is reexpressed as

$$\begin{aligned} (1-\delta)\hat{F}_T^{\gamma}\left(\frac{\Lambda_T}{k^*(1-\delta)}\right) + \delta\hat{F}_T^{\gamma}\left(\frac{1-\Lambda_T}{k^*\delta}\right) &= \tilde{V}_0/p_0 \quad \textit{if } \Lambda_T + \delta < 1, \\ \hat{F}_T^{\gamma}(1/k^*) &= \tilde{V}_0/p_0 \quad \textit{if } \Lambda_T + \delta \geq 1, \end{aligned}$$

where $\hat{F}_T^\gamma$ denote the distribution functions of Z_T under $\hat{P}$, i.e.,

$$\begin{aligned}\hat{F}_T^\gamma(z) &:= \hat{\mathrm{P}}\left(Z_T < z\right) := \int_0^z g_T\left(h_\gamma^-(x)\right)|h_\gamma^-(x)'|dx,\\ g_T(x) &:= \frac{1}{\sqrt{2\pi T}}e^{-x^2/2T}, \quad h_\gamma^-(x) := \frac{1}{\gamma}\left(\log x - \frac{\gamma^2 T}{2}\right),\end{aligned}$$

2. *in Problem 2, the equations (6) is reexpressed as*

$$\begin{aligned}\Lambda_T F_T^\gamma\left(\frac{\Lambda_T}{k^*(1-\delta)}\right) + (1-\Lambda_T)F_T^\gamma\left(\frac{1-\Lambda_T}{k^*\delta}\right) &= 1-\alpha \quad \text{if } \Lambda_T+\delta<1,\\ F_T^\gamma(1/k^*) &= 1-\alpha \quad \text{if } \Lambda_T+\delta\geq 1,\end{aligned}$$

where F_T^γ denote the distribution functions of Z_T under P, i.e.,

$$\begin{aligned}F_T^\gamma(z) &:= \mathrm{P}\left(Z_T < z\right) := \int_0^z g_T\left(h_\gamma^+(x)\right)|h_\gamma^+(x)'|dx,\\ h_\gamma^+(x) &:= \frac{1}{\gamma}\left(\log x + \frac{\gamma^2 T}{2}\right).\end{aligned}$$

The initial cost:

$$\tilde{V}_0(\alpha) := p_0\left[\delta\hat{\mathrm{P}}\left(A_1(k^*(\alpha))\right) + \hat{\mathrm{P}}\left(A_2(k^*(\alpha))\right)\right]$$

of the quantile hedging strategy under the success probability constraint, $\geq 1-\alpha$, is equal to

$$\tilde{V}_0(\alpha) = p_0\left[\delta\hat{F}_T^\gamma\left(\frac{1-\Lambda_T}{k^*(\alpha)\delta}\right) + (1-\delta)\hat{F}_T^\gamma\left(\frac{\Lambda_T}{k^*(\alpha)(1-\delta)}\right)\right]$$

in the case of $\Lambda_T+\delta<1$, or equal to

$$\tilde{V}_0(\alpha) = p_0\hat{F}_T^\gamma\left(\frac{1}{k^*(\alpha)}\right)$$

in the case of $\Lambda_T+\delta\geq 1$, respectively.

Remark: The corollaries above treat only trivialized situations, and computable examples with stochastic Λ_T, (or λ) seem to be necessary to lead more financial implementations. But they may have to be computed through numerical computations or simulations, since we may not obtain the explicit expression of the joint distribution of Z_T and Λ_T generally, (nor even in simplest examples).

3 Proofs

First, we will prove Theorem 1. Let us consider

Problem 1'

$$\max_{A,B\in\mathcal{G}_T} E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right],$$
$$\textit{subject to}\quad E^*\left[1_A d_T N_T + 1_B D(1-N_T)\right] \le \tilde{V}_0/p_0, \quad \textit{for all } P^*\in\mathcal{P},$$

and, via similar discussions to Proposition 2.8 in [F-L], see the following:

Lemma 2 *Let us denote a solution of Problem 1' by A^* and $B^* \in \mathcal{G}_T$. The super replicating strategy of "the modified claim" $H_T^* := 1_{A^*}d_T N_T + 1_{B^*}D(1-N_T)$ is a solution of Problem 1.*

Proof: For any admissible strategy (V_0,ξ) with $V_0 \le \tilde{V}_0$, the associated "success set":

$$1_{\{V_T\ge H_T\}} = 1_{\{V_T\ge d_T\}}N_T + 1_{\{V_T\ge D\}}(1-N_T)$$

satisfies

$$\begin{aligned} V_T &\ge H_T 1_{\{V_T\ge H_T\}} \\ &= 1_{\{V_T\ge d_T\}}d_T N_T + 1_{\{V_T\ge D\}}D(1-N_T) \end{aligned}$$

P-a.s., so,

$$\tilde{V}_0/p_0 \ge V_0/p_0 \ge E^*\left[V_T\right] \ge E^*\left[1_{\{V_T\ge d_T\}}d_T N_T + 1_{\{V_T\ge D\}}D(1-N_T)\right]$$

for any $P^*\in\mathcal{P}$ since nonnegative local P^*-local martingale V/p is a super martingale, this implies

$$E\left[1_{\{V_T\ge H_T\}}\right] \le E\left[1_{A^*}N_T + 1_{B^*}(1-N_T)\right] = E\left[1_{A^*}(1-\Lambda_T)+1_{B^*}\Lambda_T\right]$$

On the other hand, the super replicating strategy of "the modified claim" H_T^* is obviously admissible,

$$\operatorname*{esssup}_{P^*\in\mathcal{P}} E^*\left[H_T^* \mid \mathcal{F}_t\right] = \operatorname*{esssup}_{P^*\in\mathcal{P}} E^*\left[1_{A^*}d_T N_T + 1_{B^*}D(1-N_T) \mid \mathcal{F}_t\right] \ge 0,$$

and has a maximal "success set". i.e., in the expression

$$1_{\{H_T^*\ge H_T\}} = 1_{\{1_{A^*}d_T\ge d_T\}}N_T + 1_{\{1_{B^*}D\ge D\}}(1-N_T),$$

we observe

$$\tilde{A} := \{1_{A^*}d_T \ge d_T\} \supset A^*, \quad \text{and} \quad \tilde{B} := \{1_{B^*}D \ge D\} \supset B^*,$$

hence, $\tilde{A}$ and $\tilde{B}$ are solutions of Problem 1', and the optimality in Problem 1 follows.

□

Moreover, we set

Problem 1"

(A) *In the case of* $D > d_T \geq 0$, $\hat{P}$*-a.e.,*

$$\max_{A,B\in\mathcal{G}_T, A\supset B} E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right], \quad \text{subject to} \quad \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \leq \tilde{V}_0/p_0,$$

(B) *in the case of* $0 \leq d_T \leq D$, $\hat{P}$*-a.e.,*

$$\max_{A,B\in\mathcal{G}_T, A\subset B} E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right], \quad \text{subject to} \quad \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \leq \tilde{V}_0/p_0,$$

and observe

Lemma 3 *Problem 1' is equivalent to Problem 1".*

Proof: We have

$$\begin{aligned} E^*\left[1_A d_T N_T + 1_B D(1-N_T)\right] &= E^*\left[1_A d_T\right] + E^*\left[\left(1_B D - 1_A d_T\right)(1-N_T)\right] \\ &\leq E^*\left[1_A d_T\right] + E^*\left[\left(1_B D - 1_A d_T\right)^+\right] \\ &= \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \end{aligned}$$

for any $P^* \in \mathcal{P}$, since $P^* = \hat{P}$ on $\mathcal{G}_T$. If we use $\left(Q_\epsilon^{(1_B D - 1_A d_T)}\right)_{\epsilon>0}$ $(\subset \mathcal{P})$, which shall be defined in Lemma 4 below, we can approximate the trivial upper bound as

$$\lim_{\epsilon\to 0} E_\epsilon^{(1_B D - 1_A d_T)}\left[\left(1_B D - 1_A d_T\right)(1-N_T)\right] = \hat{E}\left[\left(1_B D - 1_A d_T\right)^+\right],$$

so

$$\sup_{P^*\in\mathcal{P}} E^*\left[1_A d_T N_T + 1_B D(1-N_T)\right] = \hat{E}\left[\left(1_B D - 1_A d_T\right)^+\right].$$

Further, if $D > d_T \geq 0$ $\hat{P}$-a.e., for example, for any $A, B \in \mathcal{G}_T$ satisfying the condition $\hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \leq \tilde{V}_0/p_0$, set $\tilde{A} := A \cup B$, and recall that the relation

$$\begin{aligned} E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right] &\leq E\left[1_{\tilde{A}}(1-\Lambda_T)+1_B\Lambda_T\right], \\ \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] &= \hat{E}\left[\max\left(1_{\tilde{A}} d_T, 1_B D\right)\right] \end{aligned}$$

hold, hence follows the lemma.

□

Lemma 4 *For arbitrary $F \in L^1(\hat{P})$, $\alpha, \beta > 0$ and $\epsilon \in (0,T)$, let us define an equivalent martingale measure Q^F_ϵ by the formula*

$$\begin{aligned}
\left.\frac{dQ^F_\epsilon}{dP}\right|_{\mathcal{F}_t} &= \rho_t, \\
\textit{where} \quad \rho_t &= 1 + \int_0^t \rho_{u-}\left(-\gamma_u dw_u + \kappa_u dM_u\right), \\
\kappa_t &:= \begin{cases} -1 + \epsilon/\lambda_t & \textit{if } t < T - \epsilon \\ -1 + \left(\epsilon^{-\alpha-1} - \left(\epsilon^{-\alpha-1} + \epsilon^{\beta-1}\right)\hat{E}\left[1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}\right]\right)/\lambda_t & \textit{if } T - \epsilon \leq t \leq T, \end{cases}
\end{aligned}$$

Then,

$$\lim_{\epsilon \to 0} E^F_\epsilon \left[F(1 - N_T)\right] = \hat{E}\left[F^+\right] \tag{10}$$

holds, where we have denoted the expectation with respect to Q^F_ϵ by $E^F_\epsilon[\cdot]$.

Proof: We will only show "$\geq$-side" inequality in (10), since the relation

$$\lim_{\epsilon \to 0} E^F_\epsilon \left[F(1 - N_T)\right] \leq E^F_\epsilon\left[F^+\right] = \hat{E}\left[F^+\right]$$

is obvious. Under Q^F_ϵ,

$$\left(N_t - \int_0^t (1 - N_u)(1 + \kappa_u)\lambda_u du\right)_{t \in [0,T]}$$

is a martingale, and the relation

$$E^F_\epsilon \left[F(1 - N_T)\right] = \hat{E}\left[F \tilde{\Lambda}^F_\epsilon\right]$$

holds, where we have denoted

$$\begin{aligned}
\log \tilde{\Lambda}^F_\epsilon &:= -\int_0^T (1 + \kappa_t)\lambda_t dt \\
&= -\epsilon(T - \epsilon) - \left(\epsilon^{-\alpha} - \left(\epsilon^{-\alpha} + \epsilon^{\beta}\right)\hat{E}\left[1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}\right]\right) \\
&= -\epsilon(T - \epsilon) - \epsilon^{-\alpha} 1_{G^c_\epsilon} + \epsilon^{\beta} 1_{G_\epsilon} - \left(\epsilon^{-\alpha} + \epsilon^{\beta}\right) L^F_\epsilon \\
\text{with} \quad G_\epsilon = G(F, \alpha; \epsilon) &:= \left\{\hat{E}\left[1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}\right] \geq 1 - \epsilon^{\alpha+1}\right\} \\
\text{and} \quad L^F_\epsilon &:= 1_{G_\epsilon} - \hat{E}\left[1_{\{F \geq 0\}} \mid \mathcal{G}_{T-\epsilon}\right].
\end{aligned}$$

(cf.,[K], for example.) So, by using the inequality $e^{-x} \geq 1 - x$ ($x \in \mathbb{R}$), we have

$$\begin{aligned}
\tilde{\Lambda}^F_\epsilon &\geq \exp\left\{-\epsilon(T - \epsilon) - \epsilon^{-\alpha} 1_{G^c_\epsilon} + \epsilon^{\beta} 1_{G_\epsilon}\right\}\left(1 - \left(\epsilon^{-\alpha} + \epsilon^{\beta}\right) L^F_\epsilon\right) \\
&= \left[\exp\left(-\epsilon^{-\alpha}\right) 1_{G^c_\epsilon} + \exp\left(\epsilon^{\beta}\right) 1_{G_\epsilon}\right] e^{-\epsilon(T-\epsilon)}\left(1 - \left(\epsilon^{-\alpha} + \epsilon^{\beta}\right) L^F_\epsilon\right)
\end{aligned}$$

therefore,

$$\begin{aligned} E_\epsilon^F\left[F(1-N_T)\right] \;\geq\; & \exp\left\{-\epsilon(T-\epsilon)-\epsilon^{-\alpha}\right\}\hat{E}\left[F1_{G_\epsilon^c}\right] \\ & +\exp\left\{-\epsilon(T-\epsilon)+\epsilon^{\beta}\right\}\hat{E}\left[F1_{G_\epsilon}\right] \\ & -\left(\epsilon^{-\alpha}+\epsilon^{\beta}\right)\exp\left\{-\epsilon(T-\epsilon)-\epsilon^{-\alpha}\right\}\hat{E}\left[FL_\epsilon^F1_{G_\epsilon^c}\right] \\ & -\left(\epsilon^{-\alpha}+\epsilon^{\beta}\right)\exp\left\{-\epsilon(T-\epsilon)+\epsilon^{\beta}\right\}\hat{E}\left[FL_\epsilon^F1_{G_\epsilon}\right] \end{aligned}$$

follows. The second term of the right-hand-side above converges to $\hat{E}[F^+]$ as $\epsilon \to 0$, and the rest of all terms go to 0 as $\epsilon \to 0$, since $xe^{-x} \to 0$ as $x \to \infty$ and since the relation:

$$\hat{E}\left[FL_\epsilon^F1_{G_\epsilon}\right]=\hat{E}\left[F\left(1-\hat{E}\left[1_{\{F\geq 0\}}\mid \mathcal{G}_{T-\epsilon}\right]\right)1_{G_\epsilon}\right]\leq \hat{E}\left[F1_{G_\epsilon}\right]\epsilon^{\alpha+1}$$

is observed. Hence follows the lemma.

□

To obtain Theorem 1, we show the following

Lemma 5 (A) *The sets $A_1^*\sqcup A_2^*$ and A_2^* defined in Theorem 1 (A) is a solution of Problem 1" (A).*

(B) *The sets B_1^* and $B_1^*\sqcup B_2^*$ defined in Theorem 1 (B) is a solution of Problem 1" (B).*

Proof: We only show (A), since (B) can be seen similarly. For the constant $k^*=k^*(\tilde{V}_0)$ given in Theorem 1 (A), define

$$\begin{aligned} B^* &:= \left\{\Lambda_T > k^*(D-d_T)Z_T\right\}, \\ A^* &:= \left\{(1-\Lambda_T+1_{B^*}\Lambda_T) > k^*\left(d_T+1_{B^*}(D-d_T)\right)Z_T\right\}, \end{aligned}$$

and note that the relations

$$A^*\setminus B^* = A_1^* \quad \text{and} \quad A^*\cap B^* = A_2^*$$

hold. For any $\mathcal{G}_T$-measurable $A\supset B$ satisfying $\hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right]\leq \tilde{V}_0/p_0$, we have

$$\begin{aligned} & E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right]-k^*\tilde{V}_0/p_0 \\ \leq\; & E\left[1_A(1-\Lambda_T)+1_B\Lambda_T\right]-k^*\hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \\ =\; & E\left[1_A\left((1-\Lambda_T)+1_B\Lambda_T\right)\right]-k^*\hat{E}\left[1_A\left(d_T+1_B(D-d_T)\right)\right] \\ =\; & E\left[1_A\left\{(1-\Lambda_T)-k^*d_TZ_T+1_B\left(\Lambda_T-k^*(D-d_T)Z_T\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
&\leq E\left[1_A\left\{(1-\Lambda_T) - k^* d_T Z_T + 1_{B^*}\left(\Lambda_T - k^*(D-d_T)Z_T\right)\right\}\right] \\
&= E\left[1_A\left\{(1-\Lambda_T + 1_{B^*}\Lambda_T) - k^*\left(d_T + 1_{B^*}(D-d_T)\right)Z_T\right\}\right] \\
&\leq E\left[1_{A^*}\left\{(1-\Lambda_T + 1_{B^*}\Lambda_T) - k^*\left(d_T + 1_{B^*}(D-d_T)\right)Z_T\right\}\right] \\
&= E\left[1_{A^*}(1-\Lambda_T + 1_{B^*}\Lambda_T)\right] - k^*\hat{E}\left[1_{A^*}\left(d_T + 1_{B^*}(D-d_T)\right)\right] \\
&= E\left[1_{A_1^* \cup A_2^*}(1-\Lambda_T) + 1_{A_2^*}\Lambda_T\right] - k^*\hat{E}\left[1_{A_1^* \cup A_2^*} d_T + 1_{A_2^*}(D-d_T)\right] \\
&= E\left[1_{A_1^* \cup A_2^*}(1-\Lambda_T) + 1_{A_2^*}\Lambda_T\right] - k^*\tilde{V}_0/p_0,
\end{aligned}
$$

therefore the optimality is derived, hence follows the lemma.

$\square$

Now, Theorem 2 can be obtained straightforwardly. Following to the discussion in [F-L], we can reduce solving Problem 2 to solving

Problem 2'

(A) *If $D > d_T \geq 0$, $\hat{P}$-a.e.,*

$$
\min_{A \supset B \in \mathcal{G}_T} \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \quad \text{subject to} \quad E\left[1_A(1-\Lambda_T) + 1_B\Lambda_T\right] \geq 1-\alpha,
$$

(B) *If $0 \leq d_T \leq D$, $\hat{P}$-a.e.,*

$$
\min_{A \subset B \in \mathcal{G}_T} \hat{E}\left[\max\left(1_A d_T, 1_B D\right)\right] \quad \text{subject to} \quad E\left[1_A(1-\Lambda_T) + 1_B\Lambda_T\right] \geq 1-\alpha.
$$

and we can give solutions of these problems via similar "Neyman-Pearson-like" discussion as Lemma 5, although we omit the detail. At the last of this section, we give the relation, which is used to describe the super hedging strategy of H_T.

Lemma 6 *The relation (1) holds.*

Proof: For any $P^* \in \mathcal{P}$, we have

$$
E^*\left[d_T N_T + D(1-N_T) \mid \mathcal{F}_t\right] = \hat{E}\left[d_T \mid \mathcal{G}_t\right] + E^*\left[(D-d_T)(1-N_T) \,\Big|\, \mathcal{G}_t\right],
$$

so obviously,

$$
\bar{H}_t \leq \frac{d_t}{p_t} + (1-N_t)\hat{E}\left[(D-d_T)^+ \mid \mathcal{G}_t\right].
$$

If we prepare $\left(Q_\epsilon^{D-d_T}\right)_{\epsilon>0}$ defined in Lemma 4 above, we observe

$$\operatorname*{esssup}_{\epsilon>0} E_\epsilon^{D-d_T}\left[H_T|\mathcal{G}_t\right] = \frac{d_t}{p_t} + (1-N_t)\hat{E}\left[(D-d_T)^+ \mid \mathcal{G}_t\right],$$

hence actually, the trivial upper bound above is estimated arbitrary.

□

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