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Generalized Whittaker functions for cohomological representations of $SU(2,1)$ and $SU(3,1)$ *

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§1 Introduction.

For investigation of automorphic form F, its Fourier expansion is a fundamental and important tool. Let f belong to an automorphic representation $\pi = \pi_\infty \otimes \pi_{\mathrm{fin}} \in \mathcal{A}(\mathbf{G}(\mathbb{A}))$ of a reductive group $\mathbf{G}$. When π_∞ is discrete series representation of $G = SU(2,1)$ or $SU(3,1)$, we investigated Fourier component of f, and reported "what kind of special functions appear as the generalized Whittaker functions for π_∞" in [I2], [I3] respectively. As for ordinary Whittaker functions, see [K-O], [Ta]. In view of application to arithmetic of automorphic forms or of the problem of realization of representations, investigation of generalized Whittaker model for π_∞ which contributes NON-middle degree cohomology is very interesting. This corresponds to a study of Fourier component of f belonging to the so-called "thin" representation $A_{\mathfrak{q}}(\lambda)$. Here we are led to two natural questions:

I) Comparing to the case of discrete series π_∞, how many Fourier components which appear in expansion of f decrease?

II) How do the special functions appearing in expansion degenerate?

In this short note, we report some results for these questions in the case of easy groups in the title. This problem is purely archimedean local. So we omit the subscript $_\infty$. We realize the special unitary group of signature $(n+, 1-)$ as

$$G = SU(n,1) := \{g \in SL(n+1,\mathbb{C}) | {}^t\bar{g} I_{n,1} g = I_{n,1}\}.$$

Here $I_{n,1}$ is $\mathrm{diag}(I_n, -1)$. Let $G = NAK$ be the Iwasawa decomposition. In our realization,

$$K = \{\begin{pmatrix} k & \\ & \det k^{-1} \end{pmatrix} | \ k \in U(n)\} \ \text{: maximal compact subgroup},$$

$$A = \{a_r := \begin{pmatrix} 1_n & \\ & h_r \end{pmatrix} | \ h_r = \begin{pmatrix} \frac{r+r^{-1}}{2} & \frac{r-r^{-1}}{2} \\ \frac{r-r^{-1}}{2} & \frac{r+r^{-1}}{2} \end{pmatrix} r \in \mathbb{R}_{>0}\} \cong \mathbb{R}_>,$$

$$N \cong H(\mathbb{C}^{n+1}) \ \text{: real}(2n+1)\text{dimensional Heisenberg group.}$$

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The unitary dual $\widehat{N}$ of N consists unitary characters ψ and infinite dimensional irreducible unitary representations ρ. Fourier component of f indexed by ψ corresponds to the ordinary Whittaker model $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^{\infty}_{f,K}, C^{\infty}\text{-}\mathrm{Ind}_N^G(\psi)_K)$ of π_f. Here $\pi^{\infty}_{f,K}$ is the underling $(\mathfrak{g}_{\mathbb{C}}, K)$-module of π_f generated by f. This model was investigated by [K-O] and [Ta], when π^{∞} is discrete series representations of $SU(2,1)$ and $SU(3,1)$ respectively.

§2 Generalized Whittaker functions.

Now we recall Kostant's fundamental result:

Proposition 1 ([Ko]) *When G is a connected quasi-split semi-simple Lie group and π^{∞}_K is an irreducible Harish-Chandra module, the followings are equivalent.*
i) *The Whittaker model of π is not vacant*: $\dim_{\mathbb{C}}\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^{\infty}_K, C^{\infty}\text{-}\mathrm{Ind}_N^G(\psi)_K) \neq 0$.
ii) *The Gel'fand-Kirillov dimension of π^{∞}_K is maximal*: $\mathrm{Dim}\pi^{\infty}_K = \dim_{\mathbb{C}}\mathrm{Lie}N$. □

Therefore in order to obtain fully developed Fourier expansion of automorphic forms, investigating only the Whittaker models is not sufficient. In fact, there are many important representations with non maximal Gel'fand-Kirillov dimension. In our situation, we also have to consider Fourier component which is indexed by infinite dimensional representation, that is $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^{\infty}_{f,K}, C^{\infty}\text{-}\mathrm{Ind}_N^G(\rho)_K)$. However this is not appropriate object for investigation. The space is of infinite dimension. So we cut this intertwining space into smaller pieces by introducing a larger group R.

Let $P = L \times N$ be the Levi decomposition of the minimal subgroup P. The Levi part L acts on N by conjugation, hence naturally on $\widehat{N}$ also. We put $S := \mathrm{Stb}_L([\rho])$, which is $\mathrm{Stb}_L(Z(N)) \cong U(n-1)$, since ρ is determined by its central character (:Stone-von Neuman's theorem). Using S we define R by $S \ltimes N$. Next we extend ρ to an irreducible representation η of R by the theory of Weil representation: $\eta := \widetilde{\sigma}_\mu \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$. Here $\widetilde{R}$ is the pullback of R by the metaplectic covering $\widetilde{Sp}_{n-1}(\mathbb{R}) \ltimes H(\mathbb{R}^{2n-2}) \to Sp_{n-1}(\mathbb{R}) \ltimes H(\mathbb{R}^{2n-2})$ and $\widetilde{\sigma}_\mu$ is a genuine representation of $\widetilde{U}(n-1)$: μ belongs to $\mathbb{Z}^{n-1}_{>} + \frac{1}{2}(1,\dots,1)$. By a theorem of Wolf [Wolf], the unitary representations of R with non-trivial central character are exhausted by these η. Our main object of investigation is *the generalized Whittaker model* of π

$$I(\pi|\eta) := \mathrm{Hom}_{(\mathfrak{g},K)}(\pi^{\infty}_{f,K}, C^{\infty}\text{-}\mathrm{Ind}_R^G(\eta)_K)$$

and the image of non-trivial elements of this intertwining space

$$GWh_\eta(\pi) := \mathbb{C}\text{-span}\{\ell(v) | v \in \mathcal{H}^{\infty}_{\pi}, \ell \in I(\pi|\eta)\}.$$

If we fix the K-type of π to the minimal one τ_λ, then generalized Whittaker function $W_\eta \in GWh_\eta(\pi)$ has R- and K- equivariances: $W_\eta(rgk) = \eta(r)\tau_\lambda(k)^{-1}.W_\eta(g)$. By the Iwasawa decomposition $G = RAK$, we only have to determine the A-radial part $W_\eta|_A$. By our fortunate situation that all admissible representations π of $SU(n,1)$ has multiplicity one property with respect to their K-types: $[\pi : \tau] \leq 1$, the system of differential equations which are induced from the so-called "Shcmid operators" and projections to Clebsh-Gordan components characterize the A-radial part $W_\eta|_A$.

Proposition 2 *Let π be an irreducible admissible representation but a principal series of $G = SU(n,1)$. Then the space of generalized Whittaker functions is characterized as follows.*

$$GWh_\eta(\pi) \cong \bigcap_{\beta \in J(\pi)} \mathrm{Ker}\mathcal{D}^{-\beta}.$$

Here $\mathcal{D}^{-\beta}$ is a differential operator which shifts K-types to the direction $-\beta$ and $J(\pi)$ is the set of "negative directions" for π. □

We note here that when π is a discrete series representation satisfying a regularity condition, this is a special case of Yamashita's theorem, which is applicable to very general situation [Ya].

§3 Cohomological representations.

Let $H^i_{dR}(\Gamma, X; E)$ denote the i-th de Rham cohomology for a complex of E-valued differential forms on $X := G/K$ and $H^i(\mathfrak{g}, K; V)$ do the i-th relative Lie algebra cohomology for a $(\mathfrak{g}, K)$-module V. Here E is a finite dimensional complex representation E of G. Then the Matsushima isomorphism tells

$$H^i_{dR}(\Gamma, X; E) \cong H^i(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes_{\mathbb{C}} E).$$

When we decompose $L^2(\Gamma\backslash G)$ into discrete and continuous spectrum parts: $L^2(\Gamma\backslash G) \cong L^2_{\mathrm{disc}}(\Gamma\backslash G) \oplus L^2_{\mathrm{conti}}(\Gamma\backslash G)$, the continuous part $L^2_{\mathrm{conti}}(\Gamma\backslash G)$ does not contribute to cohomologies. For general G satisfying $\mathrm{rk}G = \mathrm{rk}K$, this is shown by Borel, Casselman. Hence we consider the natural mapping

$$\iota^*_{L^2} : H^i_{dR}(\Gamma, X; E) \leftarrow H^i(\mathfrak{g}, K; L^2_{\mathrm{disc}}(\Gamma\backslash G)^\infty \otimes_{\mathbb{C}} E)$$

induced from $(\mathfrak{g}, K)$-stable embedding $\iota_{L^2} : C^\infty(\Gamma\backslash G) \hookleftarrow L^2_{\mathrm{disc}}(\Gamma\backslash G)^\infty$. Here $L^2_{\mathrm{disc}}(\Gamma\backslash G)^\infty$ is the smooth vectors in $L^2_{\mathrm{disc}}(\Gamma\backslash G)$. By the G-irreducible decomposition

$$L^2_{\mathrm{disc}}(\Gamma\backslash G) \cong \tilde{\bigoplus_{\pi \in \widehat{G}}} m(\pi; \Gamma)\mathcal{H}_\pi,$$

we have an isomorphism

$$\mathrm{Img}\ \iota^*_{L^2} \cong \bigoplus_{\pi \in \widehat{G}} m(\pi; \Gamma) H^i(\mathfrak{g}, K; \mathcal{H}^\infty_\pi \otimes_{\mathbb{C}} E).$$

This is a higher dimensional generalization of the Eichler isomorphism of one variable case: $G = SU(1,1) = SL(2;\mathbb{R})$. All cohomological unitary representations π ($i.e. H^i(\mathfrak{g}, K; \mathcal{H}^\infty_\pi \otimes_{\mathbb{C}} E) \neq \{0\}$) are classified [Vo-Zu],[Wal]: $\pi \cong A_{\mathfrak{q}}(\lambda)$. We shortly recall these representations for our groups.

<$G = SU(n,1)$ **case**>

As for $\pi \in \widehat{G}$, take discrete series representation $D^J_\lambda, (J = 1, \ldots, n+1)$. Here λ is the Blattner parameter of D^J_λ: $\lambda \in \mathbb{Z}^n_>$, $\lambda_J > 0 > \lambda_{J+1}$. Then there is an appropriate finite dimensional representation E_λ and the contribution to cohomologies are given as follows.

$$H^{p,q}(\mathfrak{su}(n,1),U(n);D^J_\lambda\otimes E_\lambda)\cong\begin{cases}\mathbb{C} & p=n-J+1,q=J-1,\\ \{0\} & \text{otherwise.}\end{cases}$$

$$H^{p,q}(\mathfrak{su}(n,1),U(n);E_\lambda\otimes E_\lambda)\cong\begin{cases}\{0\} & \text{otherwise,}\\ \mathbb{C} & \text{when } p=q=0,\dots,n\end{cases}$$

For instance $n=2$ case, in the Hodge diagram

$$\begin{array}{ccccc} & & H^{2,2} & & \\ & H^{2,1} & & H^{1,2} & \\ H^{2,0} & & H^{1,1} & & H^{0,2} \\ & H^{1,0} & & H^{0,1} & \\ & & H^{0,0} & & \end{array}\quad (\text{: } \textit{only D.S. reps. appear})$$

the groups in problem are $H^{1,0}$ and $H^{0,1}$ by the Poincaré duality. We denote the representation which contribute to $H^{1,0}$ $J^{1,0}_\lambda$. The composition series of principal series for real rank one group is completely understood. For $G=SU(2,1)$, there is an exact sequence:

$$0\to D^{2,0}_\lambda\oplus D^{1,1}_\lambda\to \mathrm{Ind}^G_P(1_N\otimes e^\nu\otimes\chi_{\lambda_0})\to J^{1,0}_\lambda\to 0.$$

On the other hand, when $J^{p,q}_\lambda$ is unitarizable is also known for our group.

Proposition 3 ([Kra]) *For the group $SU(n,1)$, admissible representation $J^{p,q}_\lambda$ is unitarizable exactly when the two "fundamental corners" of $J^{p,q}_\lambda$ coincide.* □

We can draw a picture of K-type distribution of cohomological representations of $SU(2,1)$.

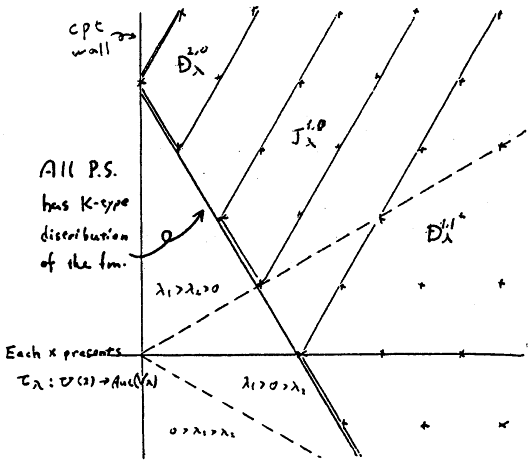

Fig. 1

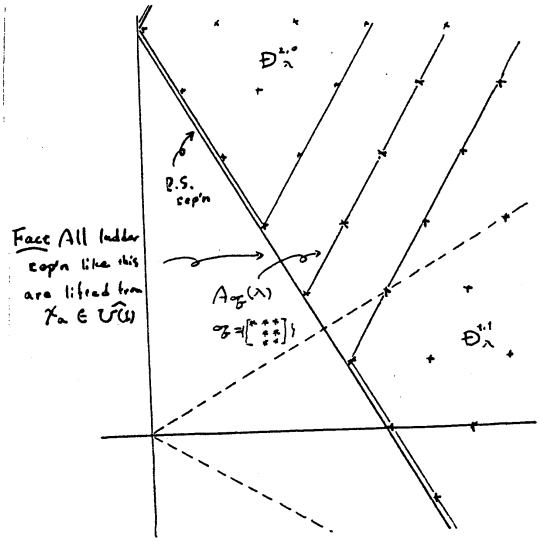

Fig. 2

§4 An explicite form of W_η and Fourier expansion of f.

<$G = SU(2,1)$ case>

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_>$ $(i.e. \lambda_1 > \lambda_2, \lambda_i \in \mathbb{Z})$, we realize $(\tau_\lambda, V_\lambda)$ as in [K-O], [I]. As for ρ, we realize this representation as follows. Let $\psi_s : Z(N) \ni t \mapsto e^{\sqrt{-1}st} \in \mathbb{C}^{(1)}$ be the central character of ρ. Then infinitesimally ρ_{ψ_s} : LieN $\to \mathrm{End}(\mathcal{F}_J)$,

$$\begin{array}{lcl|lcl}
\text{when } s>0 & & & \text{when } s<0 & & \\
\rho_{\psi_s}(E_{2,+}) & := & -s\frac{\partial}{\partial z_i} + z_i, & \rho_{\psi_s}(E_{2,+}) & := & -s\frac{\partial}{\partial z_i} + z_i, \\
\rho_{\psi_s}(E_{2,-}) & := & \sqrt{-1}(s\frac{\partial}{\partial z_i} + z_i), & \rho_{\psi_s}(E_{2,-}) & := & -\sqrt{-1}(s\frac{\partial}{\partial z_i} + z_i),
\end{array}$$

$$\rho_{\psi_s}(E_1) := \sqrt{-1}s.$$

We chose the monomials $f^s_j := z^j,\ j = 0,1,2,\dots$ $(f^s_j := (-1)^j z^j,\ j = 0,-1,-2,\dots)$ as a Hilbert base of $\mathcal{F}_J$ when $s > 0$ (when $s < 0$).

By a compatibility between S-type and K-type: $\eta(m)\tau_\lambda(m)^{-1}W_\eta(g) = W_\eta(mgm)$ $= W_\eta(g)$, we get a linear relation between indices of bases: $j = -k + \frac{2\lambda_1-\lambda_2}{3} - (\frac{1}{2} + \mu) =: j_k$. Moreover the expansion of $W_\eta|_A$ with respect to bases $\{f^s_j\}$ and $\{v^\lambda_k\}$ reduce to the following finite sum.

$$W_\eta|_A(a) = \sum_{k=0}^{\lambda_1-\lambda_2} c_k(a)\left(f^s_j \otimes v^\lambda_k\right).$$

Therefore what we have to do is writing down the differential equations of Proposition 2 in terms of c_k's. Recall the shape of K-type distribution of cohomological representations (figure 2). Then we can read off the "negative directions" $J(\pi)$ for π from the picture:

$$J(\pi^{2,0}) = \{\beta_{32}, \beta_{31}\}, \quad J(\pi^{1,1}) = \{\beta_{31}, \beta_{23}\}, \quad J(\pi^{0,2}) = \{\beta_{23}, \beta_{13}\},$$

$$J(\pi^{1,0}) = \{\beta_{32}, \beta_{31}, \beta_{23}\}, \quad J(\pi^{0,1}) = \{\beta_{31}, \beta_{23}, \beta_{13}\}.$$

In the case of discrete series π, we have already obtained the moderate growth solutions for $\bigcap_{\beta \in J(\pi)} \mathrm{Ker}\mathcal{D}^{-\beta}$ [I]§3.3. Here we records the result for readers' comvenience.

$$c_k(a_r) \sim \begin{cases} r^{\lambda_2+k}e^{sr^2/2} \quad (s<0) & \text{when } \pi = \pi^{2,0}, \\ r^{\lambda_1-\lambda_2+1}W_{\kappa,(k-\lambda_1)/2}(|s|r^2) & \text{when } \pi = \pi^{1,1}, \\ r^{-\lambda_2-k}e^{-sr^2/2} \quad (s>0) & \text{when } \pi = \pi^{0,2}, \end{cases}$$

where $\kappa = \frac{3}{2}k - \frac{\lambda_1}{2} + \frac{2\lambda_1-\lambda_2}{3} - \mu$.

When π is a "thin" representation $\pi^{1,0}$, $\bigcap_{\beta \in J(\pi)} \mathrm{Ker}\mathcal{D}^{-\beta}$ is an over-determining system whose solutions coincide with c_k's which satisfy the third difference-differential equation $\mathcal{D}^{-\beta_{23}}c_k = 0$. By this third equation, we have an extra relation among parameters: $\mu + \frac{1}{2} = \frac{2\lambda_1-\lambda_2}{3} - k$. Therefore $j_k = 0$ is independent upon k. Simultaneously the equation forces λ_2 must equal 1. This agree with the fact $\pi \cong A_{\mathfrak{q}}(\lambda)$, where $\mathfrak{q} = \{X \in M_3(\mathbb{C}) | X_{21} = X_{31} = 0\}$. The case of $\pi^{0,1}$ can be treated by the same way.

Fix normalization of constant multiples as in [I], we obtain an explicit form of the A-radial part of generalized Whittaker functions for cohomological representations.

$$W_\eta|_A(a_r) = \begin{cases} \sum \gamma_k^I r^{\lambda_2+k} e^{sr^2/2}\left(f_j^s \otimes v_k^\lambda\right) \quad (s<0) & \text{when } \pi = \pi^{2,0}, \\ \sum \gamma_k^{II} r^{\lambda_1-\lambda_2+1} W_{\kappa,(k-\lambda_1)/2}(|s|r^2)\left(f_j^s \otimes v_k^\lambda\right) & \text{when } \pi = \pi^{1,1}, \\ \sum \gamma_k^{III} r^{-\lambda_2-k} e^{-sr^2/2}\left(f_j^s \otimes v_k^\lambda\right) \quad (s>0) & \text{when } \pi = \pi^{0,2}, \\ \sum \gamma_k^I r^{1+k} e^{sr^2/2}\left(f_j^s \otimes v_k^\lambda\right) \quad (s<0) & \text{when } \pi = \pi^{1,0}, \\ \sum \gamma_k^{III} r^{\lambda_1-k} e^{sr^2/2}\left(f_j^s \otimes v_k^\lambda\right) \quad (s<0) & \text{when } \pi = \pi^{0,1}. \end{cases}$$

After some discussion on Γ-invariantness (see [I]§5), we obtain an explicit form of Fourier expansion of automorphic form f combining a result of [K-O].

Theorem 4 *Let f be an L^2-automorphic form on $SU(2,1)$ belonging to π with minimal K-type τ_λ. Then the Fourier expansion of f is given as follows.*
i) *When π is a discrete series representation $D_\lambda^{p,q}$ with Blattner parameter $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^{\oplus 2}$, put $j_k = -k + \frac{2\lambda_1-\lambda_2}{3} - (\frac{1}{2}+\mu)$.*
i-1) *The case of large discrete series i.e. contributes to $H^{(1,1)}$*

$$\begin{aligned} f(na_r) \;&=\; \sum_{(\ell,\ell')\in\mathbb{Z}^2\setminus(0,0)} C^f_{\ell,\ell'}\Big(\sum_{k=0}^{\lambda_1-\lambda_2} \gamma_k r^{\lambda_1-\lambda_2+\frac{3}{2}} W_{0,k-\lambda_1}(2\pi\sqrt{\ell^2+\ell'^2}r)\cdot\psi_{2\pi\ell,2\pi\ell'}(n)v_k^\lambda\Big) \\ &+ \sum_{\ell\in\mathbb{Z}\setminus\{0\}}\sum_{i=1}^{2|\ell|}\sum_{\mu\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^f_{\mu,\ell,(i)}\Big(\sum_{k=0}^{\lambda_1-\lambda_2}\gamma_k^{II} r^{\lambda_1-\lambda_2+1} W_{\kappa,\frac{k-\lambda_1}{2}}(2\pi|\ell|r^2)\cdot\theta^{\ell,(i)}_{j_k}(n)v_k^\lambda\Big), \end{aligned}$$

where

$$\kappa = \frac{3}{2}k - \frac{\lambda_1}{2} + \frac{2\lambda_1-\lambda_2}{3} - \mu.$$

i-2) *The case of holomorphic discrete series i.e. contributes to $H^{(2,0)}$*

$$f(na_r) \;=\; \sum_{-\ell=1}^{\infty}\sum_{i=1}^{2|\ell|}\sum_{\mu\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^f_{\mu,\ell,(i)}\Big(\sum_{k=0}^{\lambda_1-\lambda_2}\gamma_k^I r^{\lambda_2+k}e^{\pi\ell r^2}\cdot\theta^{\ell,(i)}_{j_k}(n)v_k^\lambda\Big).$$

i-3) *The case of anti-holomorphic discrete series i.e. contributes to $H^{(0,2)}$*

$$f(na_r) \;=\; \sum_{\ell=1}^{\infty}\sum_{i=1}^{2\ell}\sum_{\mu\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} C^f_{\mu,\ell,(i)}\Big(\sum_{k=0}^{\lambda_1-\lambda_2}\gamma_k^{III} r^{-\lambda_2-k}e^{-\pi\ell r^2}\cdot\theta^{\ell,(i)}_{j_k}(n)v_k^\lambda\Big).$$

ii) *When π is a "thin" cohomological representation, that is $\pi \cong A_{\mathfrak{q}}(\lambda)$.*
ii-1) *The case of lowest weight module, i.e. contributes to $H^{(1,0)}$*

$$f(na_r) \;=\; \sum_{-\ell=0}^{\infty}\sum_{i=1}^{2|\ell|} C^f_{\ell,(i)}\Big(\sum_{k=0}^{\lambda_1-\lambda_2}\gamma_k^I r^{1+k}e^{\pi\ell r^2}\cdot v_k^\lambda\Big)\theta_0^{\ell,(i)}(n).$$

ii-2) *The case of highest weight module, i.e. contributes to* $H^{(0,1)}$

$$f(na_r) \;=\; \sum_{\ell=0}^{\infty}\sum_{i=1}^{2\ell} C^f_{\ell,(i)}\Big(\sum_{k=0}^{\lambda_1-\lambda_2} \gamma_k^{III} r^{\lambda_1-\lambda_2+1-k} e^{-\pi\ell r^2}\cdot v_k^{\lambda}\Big)\theta_0^{\ell,(i)}(n).$$

Here $C^f_{\ell,\ell'}$, $C^f_{\mu,\ell,(i)}$ *and* $C^f_{\ell,(i)}$ *are the Fourier coefficient of* f. □

<$G = SU(3,1)$ **case**>

Former group $SU(2,1)$ has only highest weight modules as "thin" representations. However, in the case of $SU(3,1)$, there are interesting "thin" representations which contribute to $H^{(1,1)}$ and are not highest weight representations. Since our strategy of computation is exactly similar to the case of $SU(2,1)$, we omit the details, which will appear elsewhere. For notation and realization of groups and representations, see [I3].

Theorem 5 *Let* π *be a unitarizable representation of* $SU(3,1)$. *The minimal* K*-type generalized Whittaker function* W_η *indexed by an infinite dimensional representation* η *is given as follows. Let*

$$W_\eta|_A(a_r) \;=\; \sum_{k'=0}\sum_{Q\in GZ(\lambda)}\sum_{j\in SK(\mu',\lambda)} c_{j,Q}(r)\cdot\Big((w_{k'}^{\mu'}\otimes f_j)\otimes v(Q)\Big)$$

be an expansion with respect to bases of R*- and* K*- types.* i) *When* π *is a discrete series representation* $D_\lambda^{p,q}$ *with Blattner parameter* λ.

i-1) *The case of holomorphic discrete series* $D_\lambda^{3,0}$ *i.e. contributes to* $H^{(3,0)}$

$$W_\eta|_A(a_r) \;=\; \sum_{k'=0}\sum_{Q\in GZ(\lambda)}\sum_{j\in SK(\mu',\lambda)} \gamma r^{|\lambda|-|\mu|} e^{sr^2/2}\cdot\Big((w_{k'}^{\mu'}\otimes f_j)\otimes v(Q)\Big).$$

i-2) *The case of large discrete series* $D_\lambda^{2,1}$ *i.e. contributes to* $H^{(2,1)}$
At the K*-finite vector in* π *indexed by extremal Gel'fand-Zetlin schemata of the form*

$$Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_2 & & \mu_2 & \\ & & \lambda_2 & & \end{pmatrix}, \quad c_{j,Q}(r) \sim r^{\lambda_1-\lambda_3+2} W_{\kappa,\frac{\mu_2-1}{2}}(|s|r^2),$$

with $\kappa = -\frac{\mu_2}{2} - \frac{(j_1+\lambda_2-\mu_2)(j_2+1)}{\lambda_2-\mu_2}$.

ii) *When* π *is a cohomological unitarizable representation* $A_{\mathfrak{q}}(\lambda)$ *which contributes to* H^2, *we have also obtained an explicit form of the generalized Whittaker function* W_η *of* π *under some condition.*

ii-1) *The case of lowest weight module, i.e. contributes to* $H^{(2,0)}$
The generalized Whittaker model exists only when $s<0$. λ_3 *must equals to* 2.

$$W_\eta|_A(a_r) \;=\; \sum_{k'=0}\sum_{Q\in GZ(\lambda)}\sum_{j\in SK\cap SK^{\mathrm{extra}}} \gamma r^{\lambda_1+\lambda_2+2-|\mu|} e^{sr^2/2}\cdot\Big((w_{k'}^{\mu'}\otimes f_j)\otimes v(Q)\Big).$$

Moreover an extra relation between Gel'fand-Zetlin parameters and j.

ii-2) *The case of $J_\lambda^{1,1}$ i.e. contributes to $H^{(1,1)}$*

At the K-finite vector in π indexed by extremal Gel'fand-Zetlin schemata of the form

$$Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_2 & & \mu_2 & \\ & & \lambda_2 & & \end{pmatrix}, \quad c_{j,Q}(r) \sim r^{\lambda_1-\lambda_3+2}\sqrt{\frac{s}{\pi}}\, r K_{\frac{\mu_2-1}{2}}\left(\frac{|s|}{2}r^2\right)$$

under assumption that $\kappa = -\frac{\mu_2}{2} - \frac{(j_1+\lambda_2-\mu_2)(j_2+1)}{\lambda_2-\mu_2}$ is an integer.

iii) *When π is a cohomological unitarizable representation $A_{\mathfrak{q}}(\lambda)$ which contributes to H^1* i.e. *the theta lift image from $U(1)^{\sim}$ which is non-tempered ladder representation.*

iii-1) *The case of lowest weight module, i.e. contributes to $H^{(1,0)}$*

Only when the parameter s of central character of η is negative, the generalized Whittaker model exists. (λ_2, λ_3) must equals to $(1,2)$. Moreover two extra relations between Gel'fand-Zetlin parameters and j. □

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