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OPTIMAL STOPPING PROBLEMS RELATED TO THE BALLOT PROBLEM

愛知大学 経営学部　　玉置光司 (Mitsushi Tamaki)
Department of Business Administration, Aichi University

1 Introduction

Suppose that we have an urn containing m minus balls and p plus balls in it, where the value -1 is attached to minus ball and the value $+1$ to plus ball. We draw balls one at a time randomly, without replacement until we wish to stop. We know the values of m and p and are also allowed not to draw at all. Let X_k be the value of the ball chosen at the k-th draw, $1 \leq k \leq m+p$, and define

$$Z_0 = 0, \qquad Z_n = \sum_{k=1}^{n} X_k, \quad 1 \leq n \leq m+p. \tag{1.1}$$

Z_n can be interpreted, for example, as a net gain we can earn if we choose to stop after the n-th draw.

Figure 1 depicts a typical sample path of the process $\{Z_n\}_{n=0}^{m+p}$ by connecting Z_k and Z_{k+1} by a straight line when $m=6$ and $p=5$. Each time a ball is drawn, we observe the

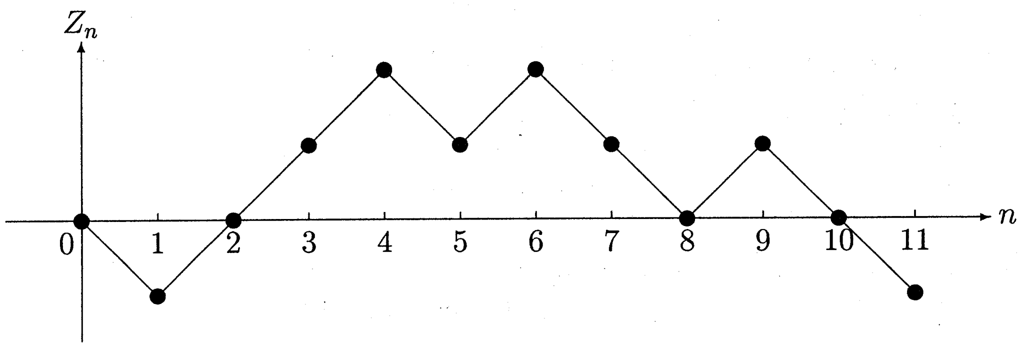

Fig 1
A sample path for $\{Z_n\}_{n=0}^{m+p}$ when $m=6$ and $p=5$.

value of the ball and decide either to stop or continue drawing. The objective of this paper is to find a stopping policy that will maximize the probability of *success*. We consider in Section 3 the problem where the trial is regarded as successful if we could attain the largest value of $\{Z_n\}_{n=0}^{m+p}$ upon stopping. This is a problem posed by Sakaguchi in his paper[7] but left unsolved. In Section 4, we also consider the problem where the trial is regarded as successful if we could attain the largest or the second largest vakue of $\{Z_n\}_{n=0}^{m+p}$. On the sample path given in Figure 1, stopping at time 4 or 6 only leads to success in Section 3, while stopping at time 3,5,7 or 9 also leads to success is the problem treated in Section 4. In Section 2, we present as preliminaries some results on the ballot problem, because they are crucial to our analysis.

Shepp[8] considered the optimal stopping problem related to the urn among other things, but his main concerns were to examine what urns are fovourable. That is, for what m and p is there a stopping policy that will make thte expected net gain positive ? Boyce further studied the Shepp's urn problem in [8] and generalized it to allow a probability distribution on m (though the total number of balls $m+p$ in the urn is assumed to be known) to analyze the bond-selling problem in [2].

2 Preliminaries

Before formulating our problems, we give several preliminary lemmas.

Lemma 2.1 (ballot problem) In an election, candidate A receives a votes and candidate B receives b votes where $a \geq b$. If all orderings for counting the votes are equally likely, then we have the following results.
(i) Let $P_{a,b}$ denote the probability that candidate A will always lead (at least by one vote) throughout the counting process. Then

$$P_{a,b} = \frac{a-b}{a+b}.$$

(ii) Let $\bar{P}_{a,b}$ denote the probability that candidate A never falls behind throughout the counting process. Then

$$\bar{P}_{a,b} = \frac{a+1-b}{a+1}.$$

Proof. These results are known under the name of the *ballot problem.* For a proof, see, e.g., Karlin and Taylor[5](Sec.13.2 and problem 45, p.134).

Let $\{Z_n\}_{n=0}^{m+p}$ be the process as defined in (1.1), and assume $m \geq p$ through this section unless otherwise specified. Let $Q_k(m,p)$ denote the probability that Z_n ever exceeds k $(1 \leq k \leq p)$, that is,

$$Q_k(m,p) = P_r\left\{\max_{1\leq n\leq m+p} Z_n \geq k\right\}.$$

Obviously $Q_1(m,p) = 1 - \bar{P}_{m,p}$ from the ballot problem. For general k, we have the following lemma.

Lemma 2.2 For $1 \leq k \leq p$,

$$Q_k(m,p) = \frac{p(p-1)\cdots(p-k+1)}{(m+1)(m+2)\cdots(m+k)}.$$

Proof. See Karlin and Taylor[5] (problems 46 and 48, p.135).

Define $T_k(m,p), 1 \leq k \leq p$, as the first time Z_n attains k, if any, that is,

$$T_k(m,p) = \min\{n : Z_n = k, \quad 1 \leq n \leq m+p\}.$$

Obviously the possible values $T_k(m,p)$ can take are $k+2i$ for $i = 0, 1, \cdots, p-k$. If we denote by $p_j^{(k)}(m,p)$ the probability mass function of $T_k(m,p)$,i.e., $p_j^{(k)}(m,p) = P_r\{T_k(m,p) = j\}$, then this is given in the following lemma.

Lemma 2.3 For $1 \leq k \leq p$, the probability mass function of $T_k(m,p)$ is given by

$$p_{2i+k}^{(k)}(m,p) = \frac{k}{2i+k}\frac{\binom{m}{i}\binom{p}{i+k}}{\binom{m+p}{2i+k}}, \qquad i = 0, 1, \cdots, p-k.$$

Proof. $p_{2i+k}^{(k)}(m,p)$ can be obtained by first conditioning on the total number of minus balls during the first $2i+k$ drawings. This yields

$$p_{2i+k}^{(k)}(m,p) = P_r\{T_k(m,p) = 2i+k \mid i \text{ minus balls and } i+k \text{ plus balls during the first } 2i+k \text{ drawings}\} \times \frac{\binom{m}{i}\binom{p}{i+k}}{\binom{m+p}{2i+k}}.$$

Given that a total of i minus balls during the first $2i+k$ drawings, it is easy to see that all possible orderings of the i minus balls and $i+k$ plus balls are equally likely and thus the above conditional probability turns out to be $P_{i+k,i}$ from the ballot problem if we trace the process going backwords in time. Thus the proof is completed.

Remark : Lemma 2.3 is also valid for $m < p$.

The following lemma gives two identities related to the probability mass function $p_j^{(k)}(m,p)$.

Lemma 2.4 For $1 \leq k \leq p$,

(i) $$\sum_{i=0}^{p-k} p_{2i+k}^{(k)}(m,p) = \frac{p(p-1)\cdots(p-k+1)}{(m+1)(m+2)\cdots(m+k)}$$

(ii) $$\sum_{i=0}^{p-k} \frac{m+1+k-p}{m+1-i}\, p_{2i+k}^{(k)}(m,p) = \frac{p(p-1)\cdots(p-k+1)(m+1+2k-p)}{(m+1)(m+2)\cdots(m+k)(m+k+1)}.$$

Proof. (i) The event that Z_n ever exceeds the value k can be broken up in terms of the first time Z_n attains the value k, that is, the event $\{\max_{1\le n\le m+p} Z_n \ge k\}$ is equivalent to the event $\cup_{i=0}^{p-k}\{T_k(m,p)=2i+k\}$. We have thus

$$\sum_{i=0}^{p-k} p_{2i+k}^{(k)}(m,p) = Q_k(m,p),$$

which, combined with Lemma 2.2, yields (i).
(ii) $P_r\{\max_{1\le n\le m+p} Z_n = k\}$ can be expressed in two ways. One way is to express it in terms of $Q_k(m,p)$;

$$\begin{aligned} P_r\left\{\max_{1\le n\le m+p} Z_n = k\right\} &= P_r\left\{\max_{1\le n\le m+p} Z_n \ge k\right\} - P_r\left\{\max_{1\le n\le m+p} Z_n \ge k+1\right\} \\ &= Q_k(m,p) - Q_{k+1}(m,p). \end{aligned} \tag{2.1}$$

Another expression can be obtained by conditioning on $T_k(m,p)$ as follows.

$$P_r\left\{\max_{1\le n\le m+p} Z_n = k\right\} = \sum_{i=0}^{p-k} P_r\left\{\max_{1\le n\le m+p} Z_n = k \mid T_k(m,p) = 2i+k\right\} p_{2i+k}^{(k)}(m,p). \tag{2.2}$$

The above conditional probability is equivalent to the probability that, in an election where candidate A has $m-i$ votes (which correspond to minus balls) and candidate B has $p-(i+k)$ votes (which correspond to plus balls), candidate A never falls behind in the counting until the last vote. Thus, from the ballot problem

$$P_r\left\{\max_{1\le n\le m+p} Z_n = k \mid T_k(m,p) = 2i+k\right\} = \bar{P}_{m-i,p-i-k}. \tag{2.3}$$

From (2.1)-(2.3), we now have the following identity;

$$\sum_{i=0}^{p-k} \bar{P}_{m-i,p-i-k}\; p_{2i+k}^{(k)}(m,p) = Q_k(m,p) - Q_{k+1}(m,p),$$

which, combined with Lemmas 2.1 and 2.2, yields (ii).

We further introduce a random variable $T(m,p)$ that represents the first time Z_n becomes non-negative, if any, namely

$$T(m,p) = \min\{n : Z_n \ge 0, \quad 1 \le n \le m+p\}.$$

$T(m,p)$ takes the values of $1, 2, 4, \cdots, 2p$, and if we let $p_j(m,p) = P_r\{T(m,p) = j\}$, this is given in the following lemma.

Lemma 2.5 The probability mass function of $T(m,p)$ is given by

$$\begin{aligned} p_1(m,p) &= \frac{p}{m+p} \\ p_{2i}(m,p) &= \frac{1}{2(2i-1)} \frac{\binom{m}{i}\binom{p}{i}}{\binom{m+p}{2i}}, \quad i = 1, 2, \cdots, p. \end{aligned}$$

Proof. Omitted because the proof is similar to that given in Lemma 2.3.

The following lemma gives the identities related to $p_j(m,p)$.

Lemma 2.6

(i) $$\sum_{i=1}^{p} \frac{m+1-p}{m+1-i}\, p_{2i}(m,p) = \frac{p(m+2-p)}{(m+1)(m+p)}$$

(ii) $$\sum_{i=1}^{p} \frac{(m+2-p)(m+1+p-2i)}{(m+1-i)(m+2-i)}\, p_{2i}(m,p) = \frac{p(m+3-p)}{(m+1)(m+2)}.$$

Proof. Observe that, from Lemmas 2.3 and 2.5, $p_{2i+1}^{(1)}(m,p)$ can be expressed in terms of $p_{2(i+1)}(m,p)$ as follows.

$$p_{2i+1}^{(1)}(m,p) = \frac{m+p-(2i+1)}{m-i} p_{2(i+1)}(m,p), \qquad i = 0, 1, \cdots, p-1.$$

Thus substituting these into (i) and (ii) of Lemma 2.4 (for $k = 1$) yields the desired results.

Remark : identity (i) can be found in Karlin and Taylor(see problem 47(i), p.135).

3 Stopping at the largest

Suppose that we have drawn k balls and recognized $Z_1, \cdots, Z_k$ through the observed values of $X_1, \cdots, X_k$. Also suppose that we know there still remain m minus balls and p plus balls in the urn. If $Z_k < \max\{Z_0, Z_1, \cdots, Z_k\}$, we do not stop drawing because Z_k cannot be the largest among all. If $m < p$, we do not stop drawing because drawing all the remaining balls is better than stopping immediately. Thus the serious decision of either stop or continue takes place only when $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\}$ and $m \geq p$. Let this state be described as (m,p) regardless of k because, as a bit of consideration shows, the decision depends only on the remaining numbers of minus balls and plus balls in the urn but not on the number of balls already drawn.

Let $v(m,p)$ be the probability of success starting from state (m,p), and let $s(m,p)$ and $c(m,p)$ be respectively the probability of success when we stop drawing and continue drawing in an optimal manner in state (m,p); then, from the principle of optimality,

$$v(m,p) = \max\{s(m,p), c(m,p)\}, \qquad m \geq p \geq 0, \tag{3.1}$$

where

$$s(m,p) = \bar{P}_{m,p}, \qquad m \geq p \geq 0, \tag{3.2}$$

and

$$c(m,p) = p_1(m,p)v(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)v(m-i,p-i), \tag{3.3}$$

for $m \geq p \geq 1$ with the boundary condition $c(m,0) \equiv 0$ for $m \geq 0$. Eq.(3.2) is immediate from the ballot problem because we succeed only when stopping at the largest value of z. Eq.(3.3) follows because the time duration until the next decision epoch is distributed as $T(m,p)$ and the state makes a transition into $(m,p-1)$ or $(m-i,p-i)$ with probability $p_1(m,p)$ or $p_{2i}(m,p), 1 \leq i \leq p$.

Let $\tilde{c}(m,p)$ be the probability of success attainable by continuing drawing (starting from state (m,p)) until the next decision epoch is reached and then stopping. Then, from (3.3) and Lemma 2.6(i)

$$\begin{aligned}
\tilde{c}(m,p) &= p_1(m,p)s(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)s(m-i,p-i) \\
&= \frac{p(m+2-p)}{(m+1)(m+p)} + \sum_{i=1}^{p} \frac{m+1-p}{m+1-i} p_{2i}(m,p) \\
&= \frac{2p(m+2-p)}{(m+1)(m+p)}.
\end{aligned} \tag{3.4}$$

Let B be the stopping region derived by the one-stage look-ahead (OLA) stopping policy, that is, the set of states for which stopping immediately is at least as good as continuing one more transition and then stopping ; then from (3.2) and (3.4),

$$\begin{aligned}
B &= \{(m,p) : s(m,p) \geq \tilde{c}(m,p)\} \\
&= \{(m,p) : m^2 - (2p-1)m + p(p-3) \geq 0\} \\
&= \{(m,p) : m \geq m^*(p)\},
\end{aligned}$$

where

$$m^*(p) = p + \frac{\sqrt{8p+1}-1}{2}. \tag{3.5}$$

The following theorem states that the OLA stopping region B in fact gives the optimal stopping region.

Theorem 3.1 The optimal policy stops drawing as soon as the state enters the set B.

Proof. It is well known (see Ross[7] or Chow,Robbins and Ziegmund[4]) that the region B becomes the optimal stopping region if B is *closed* in a sense that once the state enters B, then the process never leaves B. To show that B is closed, it suffices to show that if $(m,p) \in B$ then $(m,p-1) \in B$ and $(m-i,p-i) \in B$ for $1 \leq i \leq p$. This property is immediate from the easily verifiable fact that $m^*(p+1) - m^*(p) \geq 1$.

Let $PS(m,p)$ denote the probability of success when we start with an urn having m minus balls and p plus balls. Then we have

Corollary 3.2

$$PS(m,p) = \begin{cases} v(m,p), & \text{if } m \geq p \\ \sum_{i=0}^{m} p_{2i+p-m}^{(p-m)}(m,p)v(m-i,m-i), & \text{if } m < p. \end{cases}$$

Proof. Obviously if $m \geq p$ $PS(m,p) = v(m,p)$ from the definition. If $m < p$, we must at any rate continue drawing until the remaining number of minus balls equals that of plus balls and so the result follows from the remark of Lemma 2.3. Before concluding this section, we give some comments on the asymptotics of the problem. We easily see the following result from (3.5).

The following asymptotic result is immediate from (3.5).

Corollary 3.3

$$\lim_{p\to\infty} \frac{m^*(p) - p}{\sqrt{2p}} = 1. \tag{3.6}$$

As mentioned before, Shepp[8] considered the similar urn problem with the objective of maximizing the expected net gain. Shepp shows that there exists $m^{**}(p)$, a non-decreasing function of p, such that when the remaining numbers of minus balls and plus balls in the urn are m and p respectively, then the optimal policy stops drawing if and only if $m \geq m^{**}(p)$. In addition Shepp derived

$$\lim_{p\to\infty} \frac{m^{**}(p) - p}{\sqrt{2p}} = \alpha, \tag{3.7}$$

where $\alpha \approx 0.83992\cdots$ is the unique root of the integral equation

$$(1-\alpha^2)\int_0^\infty \exp\{\alpha x - \frac{x^2}{2}\}\, dx = \alpha.$$

Comparison between (3.6) and (3.7) shows that the optimal stopping region of our problem becomes narrower than that of Shepp's problem asymptotically.

The process $\{Z_n\}_{n=0}^{m+p}$ defined in (1.1) can be approximated as a Brownian bridge process if we let m and p tend to infinity in an appropriate way.. Fix m and p, and define, for $0 \leq n \leq m+p$ and $n < (m+p)t \leq n+1$

$$W_{m,p}(t) = \frac{Z_n}{\sqrt{m+p}}, \quad 0 \leq t \leq 1.$$

Let u be defined by

$$u = \frac{m-p}{\sqrt{m+p}}.$$

Then $W_{m,p}(1) = -u$ and if u is fixed, $W_{m,p}(t)$ converges in distribution as $p \to \infty$ to $W(t)$ the Brownian bridge process pinned to $-u$ at $t = 1$(see Shepp, p.1001).

Suppose that we have just observed $Z_n = \max_{0 \leq j \leq n} Z_j$ at time n. Then from Theorem 3.1 we stop drawing if

$$m - \frac{n - Z_n}{2} \geq m^* \left(p - \frac{n + Z_n}{2}\right)$$

or equivalently

$$\frac{Z_n}{\sqrt{m+p}} \geq \sqrt{1 - \frac{n-1}{m+p}} - \frac{m-p+1}{\sqrt{m+p}}.$$

Thus asymptotically the optimal policy τ^* can be described as

$$\tau^* = \min\left\{t : \max_{0 \leq s \leq t} W(s) \geq \sqrt{1-t} - u\right\},$$

and the asymptotic success probability $V(u)$ is expressed as

$$V(u) = P_r\left\{W(\tau^*) = \max_{0 \leq t \leq 1} W(t)\right\}.$$

Solving this asymptotic problem is left for a future study.

4 Stopping at the largest or the second largest

Suppose that we have just observed $Z_1, \cdots, Z_k$ and that we know that the urn contains m minus balls and p plus balls. Since the objective is now to maximize the probability of attaining either the largest or the second largest value of $\{Z_n\}_{n=0}^{m+p}$, serious decision of stop or continue takes place only when $Z_k \geq \max\{Z_0, Z_1, \cdots, Z_k\} - 1$ and $m + 1 \geq p$. We denote this state by $(m, p\ ; 1)$ or $(m, p\ ; 2)$ regardless of k, distinguishing between $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\}$ and $Z_k = \max\{Z_0, Z_1, \cdots, Z_k\} - 1$.

Let $v_i(m,p), i = 1,2$, be the probability of success starting from state $(m,p\ ;i)$, and let $s_i(m,p)$ and $c_i(m,p)$ be respectively the probability of success when we stop drawing and continue drawing in an optimal manner in state $(m,p\ ;i)$; then, from the principle of optimality,

$$v_i(m,p) = \max\{s_i(m,p), c_i(m,p)\}, \qquad m+1 \geq p, \quad i = 1,2, \tag{4.1}$$

where

$$s_1(m,p) = s_2(m,p) = \frac{(m+1+p)(m+2-p)}{(m+1)(m+2)}, \quad m+1 \geq p, \tag{4.2}$$

$$c_1(m,p) = \frac{p}{m+p} v_1(m,p-1) + \frac{m}{m+p} v_2(m-1,p), \tag{4.3}$$

and

$$c_2(m,p) = p_1(m,p)v_1(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)v_2(m-i,p-i), \tag{4.4}$$

for $m+1 \geq p \geq 1, m \geq 1$ with the boundary conditions $c_1(0,0) = 0, c_1(0,1) = c_1(m,0) = 1$ for $m \geq 1$ and $c_2(0,1) = 1, c_2(m,0) = 0$ for $m \geq 0$. Eq.(4.2) follows from Lemma 2.2 since $s_1(m,p) = s_2(m,p) = 1 - Q_2(m,p)$. Eq.(4.3) is immediate and Eq.(4.4) is obtained in a similar way as Eq.(3.3) was obtained.

Assume that we choose to continue drawing in state $(m,p\ ;1)$ when $s_1(m,p) = c_1(m,p)$. Then, as the next lemma shows, we never stop in state $(m,p\ ;1)$ except for the last stage.

Lemma 4.1 For $m+1 \geq p$ (except for $m = p = 0$),

$$c_1(m,p) \geq s_1(m,p).$$

Proof. This lemma holds for $p = 0$ because $c_1(m,0) \equiv 1$ for $m \geq 1$. For $p \geq 1$, we have from (4.3) and (4.2)

$$c_1(m,p) \geq \frac{p}{m+p} s_1(m,p-1) + \frac{m}{m+p} s_2(m-1,p) = s_1(m,p),$$

which completes the proof.

From Lemma 4.1, we are only concerned with the optimal decision in state $(m,p\ ;2)$. The following lemma attempts to express $c_2(m,p)$ in terms of v_2.

Lemma 4.2

$$c_2(m,p) = \sum_{i=0}^{p-1} \left\{ \left(\frac{m}{m+i}\right) \prod_{j=i+1}^{p} \left(\frac{j}{m+j}\right) \right\} v_2(m-1,i) + \sum_{i=1}^{p} p_{2i}(m,p)v_2(m-i,p-i).$$

Proof. From Lemma 4.1, Eqs.(4.3) and (4.4) are reduced to

$$c_1(m,p) = \frac{p}{m+p} c_1(m,p-1) + \frac{m}{m+p} v_2(m-1,p), \tag{4.5}$$

and

$$c_2(m,p) = p_1(m,p)c_1(m,p-1) + \sum_{i=1}^{p} p_{2i}(m,p)v_2(m-i,p-i) \tag{4.6}$$

respectively. Then, since the repeated use of (4.5) yields

$$c_1(m,p) = \sum_{i=0}^{p} \left\{ \left(\frac{m}{m+i}\right) \prod_{j=i+1}^{p} \left(\frac{j}{m+j}\right) \right\} v_2(m-1,i),$$

substituting this form into (4.6) gives the desired result.

Let $\tilde{c}_2(m,p)$ be the probability of success attainable by continuing drawing (starting from state $(m,p\,;2)$ until the next decision epoch is reached and then stopping. Then, from Lemma 4.2

$$\tilde{c}_2(m,p) = \sum_{i=0}^{p-1} \left\{ \left(\frac{m}{m+i}\right) \prod_{j=i+1}^{p} \left(\frac{j}{m+j}\right) \right\} s_2(m-1,i) + \sum_{i=1}^{p} p_{2i}(m,p)s_2(m-i,p-i), \tag{4.7}$$

which can be further simplified as follows.

Lemma 4.3

$$\tilde{c}_2(m,p) = \frac{2p(m+3-p)}{(m+1)(m+2)}.$$

Proof. We have from (4.2) and Lemma 2.6 (ii)

$$\sum_{i=1}^{p} p_{2i}(m,p)s_2(m-i,p-i) = \frac{p(m+3-p)}{(m+1)(m+2)}. \tag{4.8}$$

Thus, from (4.7) and (4.8), to prove the lemma it suffices to show

$$\sum_{i=0}^{p-1} \left\{ \left(\frac{m}{m+i}\right) \prod_{j=i+1}^{p} \left(\frac{j}{m+j}\right) \right\} s_2(m-1,i) = \frac{p(m+3-p)}{(m+1)(m+2)}. \tag{4.9}$$

We show the validity of (4.9) by induction on p. Let the left side of (4.9) be denoted by $f(p)$, that is,

$$f(p) = \sum_{i=0}^{p-1} B_i(p),$$

where

$$\begin{aligned} B_i(p) &= \left\{ \left(\frac{m}{m+i}\right) \prod_{j=i+1}^{p} \left(\frac{j}{m+j}\right) \right\} s_2(m-1,i) \\ &= \left(\frac{m+1-i}{m+1}\right) \frac{p!(m+i)!}{i!(m+p)!}. \end{aligned}$$

For $p=1$, (4.9) holds because $f(1)=B_0(1)=\frac{1}{m+1}$.
Assume

$$f(p-1)=\frac{(p-1)(m+4-p)}{(m+1)(m+2)}$$

from the induction hypothesis. Then considering $B_i(p)=(\frac{p}{m+p})B_i(p-1)$, we have

$$\begin{aligned} f(p) &= \sum_{i=0}^{p-1}\left(\frac{p}{m+p}\right)B_i(p-1) \\ &= \left(\frac{p}{m+p}\right)[f(p-1)+B_{p-1}(p-1)] \\ &= \left(\frac{p}{m+p}\right)\left[\frac{(p-1)(m+4-p)}{(m+1)(m+2)}+\frac{m+2-p}{m+1}\right] \\ &= \frac{p(m+3-p)}{(m+1)(m+2)}, \end{aligned}$$

which completes the induction.

Let B be the stopping region derived by the OLA stopping region. Then, from (4.2) and Lemma 4.3,

$$\begin{aligned} B &= \{(m,p\,;2) : s_2(m,p)\geq \tilde{c}_2(m,p)\} \\ &= \{(m,p\,;2) : (m+1)^2-(2p-1)(m+1)+p(p-3)\geq 0\} \\ &= \{(m,p\,;2) : m\geq m^*(p)-1\}, \end{aligned}$$

where $m^*(p)$ is given in (3.5).

Theorem 4.4 The optimal policy stops drawing as soon as the state $(m,p\,;2)$ enters the set B.

Proof. From the construction of B, it is easy to see that, if $(m,p\,;2)\in B$, then $(m-1,i-1\,;2)\in B$ and $(m-i,p-i\,;2)\in B$ for $1\leq i\leq p$. Thus B proves to be closed and hence becomes the optimal stopping region since, from Lemma 4.2, the next possible state that the process can visit after leaving state $(m,p\,;2)$ is either $(m-1,i-1\,;2)$ or $(m-i,p-i\,;2)$ for $1\leq i\leq p$.

Let $PS(m,n)$ denote the probability of success when we start with the urn having m minus balls and p plus balls. Then we have

$$PS(m,n)=\begin{cases} v_1(m,p), & \text{if} \quad m+1\geq p \\ \displaystyle\sum_{i=0}^{m} p_{2i+p-m-1}^{(p-m-1)}(m,p)v_1(m-i,m+1-i), & \text{if} \quad m+1<p. \end{cases}$$

The problem considered in this section can be further generalized to stop at one of r largest. That is, the trial is regarded as successful if we could attain either largest, 2nd largest,$\cdots$, or rth largest of Z_n upon stopping. Though we omit the detail, we can show (in a similar manner as in Lemma 4.1) by using an easily verifiable identity

$$Q_k(m,p) = \frac{p}{m+p}Q_k(m,p-1) + \frac{m}{m+p}Q_k(m-1,p)$$

that the optimal policy never stops drawing except for the last stage if the observed value is ith largest($1 \leq i < r$). Consequently the serious decision of either stop or continue takes place only when the observed value is relatively rth largest.

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