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ON THE CONVERGENCE OF FEYNMAN PATH INTEGRALS THROUGH BROKEN LINES

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0. Introduction

Let $K(T,x,y)$ be the integral kernel of the fundamental solution for Schödinger equation such that

$$\Big[i\hbar\partial_T+\frac{1}{2}\hbar^2\triangle-V(T,x)\Big]K(T,x,y)=0\,,$$
$$\lim_{T\downarrow 0}K(T,x,y)=\delta(x-y)\,.$$

In 1948, Feynman expressed the integral kernel $K(T,x,y)$ using path integral as follows:

$$K(T,x,y)=\int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]\,.$$

Here $\gamma:[0,T]\to \boldsymbol{R}^d$ is a path which starts from y at time 0 and arrives at x at time T, i.e., $\gamma(0)=y$, $\gamma(T)=x$. $S[\gamma]$ is the action of path γ defined by

$$S[\gamma]=\int_0^T\frac{|\dot\gamma|^2}{2}-V(\tau,\gamma)d\tau\,.$$

Path integral is the sum of the exponential of $\frac{i}{\hbar}S[\gamma]$ over all path satisfying these conditions (R. P. Feynman [1]).

Path integral was a new idea. Feynman explained path integral as the limit of finite dimensional integral which is called time slicing approximation.

Let $\Delta:0=T_0<T_1<\cdots<T_L<T_{L+1}=T$ be a division of the interval $[0,T]$ into subintervals. γ_Δ be a broken line path (or a piecewise classical path) satisfying $\gamma_\Delta(T_j)=x_j$ for any $j=0,1,2,\ldots,L+1$. Let $t_j=T_j-T_{j-1}$, $|\Delta|$ be the size of a division $|\Delta|=\max t_j$. Then $S[\gamma_\Delta]$ is a finite variable function of x_{L+1}, x_L, …, x_1, x_0, i.e.,

$$S[\gamma_\Delta]=S(x_{L+1},x_L,\ldots,x_1,x_0)\,,$$

in the case of broken line paths,

$$=\sum_{j=1}^{L+1}\frac{(x_j-x_{j-1})}{2t_j}-t_j\int_0^1 V(\theta t_j+T_{j-1},\theta(x_j-x_{j-1})+x_{j-1})d\theta\,.$$

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Time slicing approximation is an expression as follows:

$$\int e^{i\nu S[\gamma]}\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0}\prod_{j=1}^{L+1}\left(\frac{\nu}{2\pi i t_j}\right)^{d/2}\int_{\boldsymbol{R}^{dL}} e^{i\nu S(x_{L+1},x_L,\dots,x_1,x_0)}\prod_{j=1}^{L}dx_j\,, \tag{0}$$

with $\nu = \hbar^{-1}$.

Now our problem is whether time slicing approximation really converge or not. Since $|e^{i\nu S}| = 1$, this integral does not converge absolutely. L^2-convergence by E. Nelson is most famous (E. Nelson[14]). But under smooth condition, time slicing approximation really convege on $\boldsymbol{R}^d\times\boldsymbol{R}^d$.

Under the conditon $|\partial_x^\alpha V(t,x)| \le C_\alpha$ for any $|\alpha|\ge 2$, the time slicing approximation throgh piecewise classical paths converges uniformly on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ (D. Fujiwara[2],[4]), and the time slicing approximation throgh broken line paths converges uniformly on any compact sets on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ (D. Fujiwara[3]).

Under the more general condition with electromagnetic fields, the time slicing approximation throgh piecewise classical paths converges uniformly on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ (K. Yajima[16], T. Tsuchida[15]), but it is open whether the time slicing approximation throgh broken line paths converges any compact set or not, though it converges in L^2-sense (W. Ichinose[8]).

In discussing the convergence of Feynman path integrals, we use Fujiwara's skip method (D. Fujiwara[5], T. Tsuchida[15]) and the estimate of H. Kumano-go-Taniguchi type that we can control the multi integral of the right hand side of (0) by dimensional power of a constant (H. Kumano-go and K. Taniguchi [11], [10]). However, the original estimate was made for hyperbolic equations of Hamiltonian type, not for Feynman path integrals of Lagrangian type. Therefore, the size of time to satisfy the estimate must be very short and phase function must be very smooth. Recently D. Fujiwara, N. Kumano-go and K. Taniguchi gave another proof for the estimate of Hamiltonian type by integration by parts and by change of variables (D. Fujiwara, N. Kumano-go and K. Taniguchi [6], N. Kumano-go[13]). In this paper, using this idea, we give another proof for the estimate for Feynman path integrals of Lagrangian type and show that the size of time is determined by the derivatives of phase function of 2-nd order.

Remark. 1° In the strict sense of the word, Nelson's line path is not broken line paths (W. Ichinose[8]). 2° In the Hamiltonian path integral case, the time slicing approximation through piecewise classical paths conveges uniformly on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ (H. Kitada and H. Kumano-go[9]), the time slicing approximation through broken line paths (of Nelson's type) converges uniformly on any compact set on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ (N. Kumano-go[12]).

1. Statement of results

We consider the following oscillatory integrals:

$$I(S,p,\nu)(x_{L+1},x_0) = \mathcal{N}\int_{\boldsymbol{R}^{dL}} e^{i\nu S(x_{L+1},x_L,\dots,x_1,x_0)}p(x_{L+1},x_L,\dots,x_1,x_0)\prod_{j=1}^{L}dx_j\,. \tag{1.1}$$

Here each x_j, $j=0,1,\dots,L+1$, runs in $\boldsymbol{R}^d$, $\nu>1$ is a parameter, t_j, $j=1,2,\dots,L+1$, are positive constants, $T_j=t_1+t_2+\cdots+t_j$, $j=1,2,\dots,L+1$, and

$$\mathcal{N}=\Big(\frac{2\pi i T_{L+1}}{\nu}\Big)^{d/2}\prod_{j=1}^{L+1}\Big(\frac{\nu}{2\pi i t_j}\Big)^{d/2}. \tag{1.2}$$

Our assumption for the phase function $S(x_{L+1},x_L,\dots,x_1,x_0)$ is the following:

Assumption 1. *$S(x_{L+1},x_L,\dots,x_1,x_0)$ is a real valued function of the form*

$$S(x_{L+1},x_L,\dots,x_1,x_0)=\sum_{j=1}^{L+1}\Big(\frac{(x_j-x_{j-1})^2}{2t_j}+\omega_j(x_j,x_{j-1})\Big), \tag{1.3}$$

and $\omega_j(x_j,x_{j-1})$, $j=1,2,\dots,L+1$ satisfy the following condtions:

(1) *For any integer $m\ge 2$ there exists a constant κ_m such that*

$$|\partial_x^\alpha\partial_y^\beta\omega_j(x,y)|\le\kappa_m, \tag{1.4}$$

for any $2\le|\alpha+\beta|\le m$, x, $y\in\boldsymbol{R}^d$ and $j=1,2,\dots,L+1$.

(2) *For any integer $m\ge 2$ there exists a constant $\mathcal{K}_m$ such that*

$$\sum_{j=1}^{L+1}|(\partial_{x_{j-1}}+\partial_{x_j}+\partial_{x_{j+1}})^\alpha\{\partial_{x_j}^\beta\omega_j(x_j,x_{j-1})+\partial_{x_j}^\beta\omega_{j+1}(x_{j+1},x_j)\}|\le\mathcal{K}_m, \tag{1.5}$$

for any $2\le|\alpha+\beta|\le m$, $|\beta|=1$, and $x_j\in\boldsymbol{R}^d$, $j=0,1,2,\dots,L+1$.

Our assumption for the amplitude function $p(x_{L+1},x_L,\dots,x_1,x_0)$ is the following:

Assumption 2. *For any integer $m\ge 0$,*

$$\max_{|\beta_0|,|\beta_1|,\dots,|\beta_{L+1}|\le m}\ \sup_{x_0,x_1,\dots,x_{L+1}\in\boldsymbol{R}^d}|(\prod_{j=0}^{L+1}\partial_{x_j}^{\beta_j})p(x_{L+1},x_L,\dots,x_1,x_0)|<\infty. \tag{1.6}$$

We denote the left-hand side of this by $|p|_m$.

Proposition. *Assume that $(4\kappa_2+\mathcal{K}_2)d^2T_{L+1}<1/2$. Then there exists a unique solution $(x_1^*,x_2^*,\dots,x_L^*)$ such that*

$$\frac{x_{j+1}^*-x_j^*}{t_{j+1}}-\frac{x_j^*-x_{j-1}^*}{t_j}=\partial_{x_j}\omega_j(x_j^*,x_{j-1}^*)+\partial_{x_j}\omega_{j+1}(x_{j+1}^*,x_j^*). \tag{1.7}$$

where $x_{L+1}^*=x_{L+1}$ and $x_0^*=x_0$.

Furtheremore, for any β_{L+1}, β_0 with $|\beta_{L+1}+\beta_0|\ge 1$, there exists a constant C_{β_{L+1},β_0} such that

$$|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{x_0}^{\beta_0}(x_j^*-x_j^\circ)|\le C_{\beta_{L+1},\beta_0}, \tag{1.8}$$

for $j=1,2,\dots,L$, where

$$x_j^\circ=\frac{(T_{L+1}-T_j)x_0+T_jx_{L+1}}{T_{L+1}}. \tag{1.9}$$

Set

$$S^*(x_{L+1},x_0)=S(x_{L+1},x_L^*,\dots,x_1^*,x_0), \tag{1.10}$$

Then our main theorem is the following :

Theorem. *Assume that* $(4\kappa_2 + \mathcal{K}_2)d^2T_{L+1} < 1/2$ *and define* $q(x_{L+1}, x_0)$ *by*

$$I(S,p,\nu)(x_{L+1},x_0) = e^{i\nu S^*(x_{L+1},x_0)}q(x_{L+1},x_0), \tag{1.11}$$

Then for any integer $m \geq 0$, *there exists a positive constant* C_m *independent of* L *such that*

$$|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{x_0}^{\beta_0} q(x_{L+1},x_0)| \leq (C_m)^L |p|_{2m+d+1}, \tag{1.12}$$

for $|\beta_{L+1}+\beta_0| \leq m$.

2. Some Lemmas

In this section, we state two important lemmas needed later. First lemma is found in H. Kumano-go and K. Taniguchi [10], [11].

Lemma 2.1. *Let* $A = (a_{jk})$ *be an* $L \times L$ *real matrix. If there exists a positive constant* $0 \leq c < 1$ *such that*

$$\sum_{k=1}^{L} |a_{jk}| \leq c, \tag{2.1}$$

for any $j = 1, 2, \ldots, L$, *then we have*

$$(1-c)^L \leq \det(I_L - A) \leq (1+c)^L, \tag{2.2}$$

where I_L *denotes the* $L \times L$ *unit matrix.*

Second lemma is found in D. Fujiwara, N. Kumano-go and K. Taniguchi [6].

Let N and L be positive integers and $x \in \boldsymbol{R}^N$. For $j = 1, 2, \ldots, L+1$, let P_j be the first-order partial differential operator with smooth coefficients given by

$$P_j = \sum_{\beta_j \leq \gamma_j,\, |\beta_j| \leq 1} a_{j,\beta_j}(x)\partial_x^{\beta_j}, \tag{2.3}$$

where $\gamma_j \in \{0,1\}^N \subset \boldsymbol{N}_0^N$ and $a_{j,\beta_j}(x) \in C^\infty(\boldsymbol{R}^N)$. Furthermore, we assume the following properties:

1°. There exists a positive integer Γ independent of N and of L such that

$$|\gamma_j| \leq \Gamma, \tag{2.4}$$

for $j = 1, 2, \ldots, L+1$.

2°. There exists a positive integer K independent of N and of L such that

$$\sharp\Big\{j = 1, 2, \ldots, k;\ \partial_x^{\beta_{k+1}} a_{j,\beta_j}(x) \not\equiv 0\Big\} \leq K, \tag{2.5}$$

for $k = 1, 2, \cdots, L$, $\beta_j \leq \gamma_j$, $|\beta_j| \leq 1$, $j = 1, 2, \ldots, k$ and $0 \neq \beta_{k+1} \leq \gamma_{k+1}$.

Then we get the following lemma:

Lemma 2.2.

(1) *The product of operators $P_{L+1}P_L\cdots P_1$ is of the form*

$$
\begin{aligned}
&P_{L+1}P_L\cdots P_1\\
&=\sum_{\{\beta_j\}_{j=1}^{L+1}}{}' \sum_{\{\alpha_j\}_{j=0}^{L+1}}{}'' C(\{\beta_j\}_{j=1}^{L+1},\{\alpha_j\}_{j=0}^{L+1})\Big(\prod_{j=1}^{L+1}\partial_x^{\alpha_j}a_{j,\beta_j}(x)\Big)\partial_x^{\alpha_0}\,, \qquad (2.6)
\end{aligned}
$$

where $\sum'_{\{\beta_j\}_{j=1}^{L+1}}$ *is the summation with respect to* $\{\beta_j\}_{j=1}^{L+1}$ *such that* $\beta_j\le\gamma_j$ *and* $|\beta_j|\le 1$ *for* $j=1,2,\ldots,L+1$, $\sum''_{\{\alpha_j\}_{j=0}^{L+1}}$ *is the summation with respect to* $\{\alpha_j\}_{j=0}^{L+1}$ *such that* $\sum_{j=0}^{L+1}\alpha_j=\sum_{j=1}^{L+1}\beta_j$ *and* $\alpha_{L+1}=0$, *and* $C(\{\beta_j\}_{j=1}^{L+1},\{\alpha_j\}_{j=0}^{L+1})$ *is a non-negative integer.*

(2) *Furthermore, there exists a positive integer C independent of N and of L such that*

$$
\sum_{\{\beta_j\}_{j=1}^{L+1}}{}' \sum_{\{\alpha_j\}_{j=0}^{L+1}}{}'' C(\{\beta_j\}_{j=1}^{L+1},\{\alpha_j\}_{j=0}^{L+1})\le C^{L+1}\,. \qquad (2.7)
$$

We can choose $C\le(1+\Gamma(K+1))$.

3. Proof of Theorem in the case $|\beta_{L+1}+\beta_0|=0$

1°. Note that

$$
\begin{aligned}
&\Big(\frac{\nu}{2\pi i t_{j+1}}\Big)^{d/2}e^{i\nu\frac{(x_{j+1}-x_j)^2}{2t_{j+1}}}\Big(\frac{\nu}{2\pi i T_j}\Big)^{d/2}e^{i\nu\frac{(x_j-x_0)^2}{2T_j}}\\
&=\Big(\frac{\nu}{2\pi i T_{j+1}}\Big)^{d/2}e^{i\nu\frac{(x_{j+1}-x_0)^2}{2T_{j+1}}}\\
&\times\Big(\frac{\nu}{2\pi}\Big)^{d}\int_{\boldsymbol{R}^d}\exp i\nu\Big(-\frac{t_{j+1}T_j}{2T_{j+1}}\xi_j^2+(x_j-\frac{T_j}{T_{j+1}}x_{j+1}-\frac{t_{j+1}}{T_{j+1}}x_0)\xi_j\Big)d\xi_j\,. \qquad (3.1)
\end{aligned}
$$

Using this inductively, we get

$$
I(S,p,\nu)(x_{L+1},x_0)=e^{i\nu\frac{(x_{L+1}-x_0)^2}{2T_{L+1}}}\,\mathbb{I}(\Phi,p,\nu)(x_{L+1},x_0)\,, \qquad (3.2)
$$

where

$$
\mathbb{I}(\Phi,p,\nu)(x_{L+1},x_0)=\Big(\frac{\nu}{2\pi}\Big)^{dL}\int_{\boldsymbol{R}^{2dL}}e^{i\nu\Phi}p(x_{L+1},x_L,\ldots,x_1,x_0)\prod_{j=1}^{L}dx_jd\xi_j\,, \qquad (3.3)
$$

and

$$\Phi = \sum_{j=1}^{L} \Big(x_j - \frac{T_j}{T_{j+1}} x_{j+1} - \frac{t_{j+1}}{T_{j+1}} x_0 \Big) \xi_j - \sum_{j=1}^{L} \frac{t_{j+1} T_j}{2T_{j+1}} \xi_j^2 + \sum_{j=1}^{L+1} \omega_j (x_j, x_{j-1}) . \tag{3.4}$$

2°. For $j = 1, 2, \dots, L$, we have

$$\begin{aligned} \partial_{\xi_j} \Phi &= x_j - \frac{T_j}{T_{j+1}} x_{j+1} - \frac{t_{j+1}}{T_{j+1}} x_0 - \frac{t_{j+1} T_j}{T_{j+1}} \xi_j , \\ \partial_{x_j} \Phi &= \xi_j - \frac{T_{j-1}}{T_j} \xi_{j-1} + \partial_{x_j} \omega_j (x_j, x_{j-1}) + \partial_{x_j} \omega_{j+1} (x_{j+1}, x_j) . \end{aligned} \tag{3.5}$$

with $\xi_0 = 0$. Set

$$\begin{aligned} M_j &= \frac{1 - i(\partial_{\xi_j} \Phi) \partial_{\xi_j}}{1 + \nu |\partial_{\xi_j} \Phi|^2} , \\ N_j &= \frac{1 - i(\partial_{x_j} \Phi) \partial_{x_j}}{1 + \nu |\partial_{x_j} \Phi|^2} . \end{aligned} \tag{3.6}$$

We denote the adjoint operators of M_j and of N_j respectively by M_j^* and by N_j^*. Then we can write

$$\begin{aligned} M_j^* &= a_j^1 (x_{j+1}, \xi_j, x_j, x_0) \partial_{\xi_j} + a_j^0 (x_{j+1}, \xi_j, x_j, x_0) , \\ N_j^* &= b_j^1 (x_{j+1}, \xi_j, x_j, \xi_{j-1}, x_{j-1}) \partial_{x_j} + b_j^0 (x_{j+1}, \xi_j, x_j, \xi_{j-1}, x_{j-1}) , \end{aligned} \tag{3.7}$$

For any β_{j+1}, α_j, β_j, there exists a constant $C_{\beta_{j+1}, \alpha_j, \beta_j}$ independent of j such that

$$\begin{aligned} &|\partial_{x_{j+1}}^{\beta_{j+1}} \partial_{\xi_j}^{\alpha_j} \partial_{x_j}^{\beta_j} a_j^1 (x_{j+1}, \xi_j, x_j, x_0)| \\ &\le C_{\beta_{j+1}, \alpha_j, \beta_j} \frac{1}{(1 + \nu |\partial_{\xi_j} \Phi|^2)^{1/2}} \nu^{-1/2 + |\beta_{j+1} + \alpha_j + \beta_j|/2} , \\ &|\partial_{x_{j+1}}^{\beta_{j+1}} \partial_{\xi_j}^{\alpha_j} \partial_{x_j}^{\beta_j} a_j^0 (x_{j+1}, \xi_j, x_j, x_0)| \\ &\le C_{\beta_{j+1}, \alpha_j, \beta_j} \frac{1}{(1 + \nu |\partial_{\xi_j} \Phi|^2)^{1/2}} \nu^{|\beta_{j+1} + \alpha_j + \beta_j|/2} . \end{aligned} \tag{3.8}$$

For any β_{j+1}, β_j, there exists a constant C_{β_{j+1}, β_j} independent of j such that

$$\begin{aligned} &|\partial_{x_{j+1}}^{\beta_{j+1}} \partial_{x_j}^{\beta_j} b_j^1 (x_{j+1}, \xi_j, x_j, \xi_{j-1}, x_{j-1})| \\ &\le C_{\beta_{j+1}, \beta_j} \frac{1}{(1 + \nu |\partial_{x_j} \Phi|^2)^{1/2}} \nu^{-1/2 + |\beta_{j+1} + \beta_j|/2} , \\ &|\partial_{x_{j+1}}^{\beta_{j+1}} \partial_{x_j}^{\beta_j} b_j^0 (x_{j+1}, \xi_j, x_j, \xi_{j-1}, x_{j-1})| \\ &\le C_{\beta_{j+1}, \beta_j} \frac{1}{(1 + \nu |\partial_{x_j} \Phi|^2)^{1/2}} \nu^{|\beta_{j+1} + \beta_j|/2} . \end{aligned} \tag{3.9}$$

Integrating by parts, we have

$$\mathbb{I}(\Phi, p, \nu)(x_{L+1}, x_0) = \mathbb{I}(\Phi, p^\circ, \nu)(x_{L+1}, x_0)\,, \tag{3.10}$$

where

$$\begin{aligned} p^\circ = {} & (N_L^*)^{d+1}(N_{L-1}^*)^{d+1}\cdots(N_1^*)^{d+1} \\ & \circ (M_L^*)^{d+1}(M_{L-1}^*)^{d+1}\cdots(M_1^*)^{d+1}p\,. \end{aligned} \tag{3.11}$$

By Lemma 2.2, there exists a positive constant C_1 such that

$$|p^\circ| \leq (C_1)^L |p|_{d+1} \prod_{j=1}^{L} \frac{1}{(1+\nu|\partial_{\xi_j}\Phi|^2)^{(d+1)/2}} \prod_{j=1}^{L} \frac{1}{(1+\nu|\partial_{x_j}\Phi|^2)^{(d+1)/2}}\,. \tag{3.12}$$

3°. For $j = 1, 2, \ldots, L$, let

$$z_j = \partial_{\xi_j}\Phi\,, \qquad \zeta_j = \partial_{x_j}\Phi\,. \tag{3.13}$$

For simplicity, we set

$$\begin{aligned} \widetilde{x}_{1,L} &= (x_1, x_2, \ldots, x_L)\,, \quad \widetilde{\xi}_{1,L} = (\xi_1, \xi_2, \ldots, \xi_L)\,, \\ \widetilde{z}_{1,L} &= (z_1, z_2, \ldots, z_L)\,, \quad \widetilde{\zeta}_{1,L} = (\zeta_1, \zeta_2, \ldots, \zeta_L)\,. \end{aligned} \tag{3.15}$$

Then we have

$$\frac{\partial(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L})}{\partial(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})} = \begin{pmatrix} \Delta_1 & \Lambda_1 \\ \Lambda_2 & \Delta_2 \end{pmatrix}, \tag{3.15}$$

where Δ_1, Δ_2, Λ_1 and Λ_2 are $dL \times dL$ matrices defined by

$$\Delta_1 = \begin{pmatrix} I_d & -\frac{T_1}{T_2}I_d & 0 & \cdots \\ 0 & I_d & -\frac{T_2}{T_3}I_d & \ddots \\ 0 & 0 & I_d & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.16}$$

$$\Delta_2 = \begin{pmatrix} I_d & 0 & 0 & \cdots \\ -\frac{T_1}{T_2}I_d & I_d & 0 & \ddots \\ 0 & -\frac{T_2}{T_3}I_d & I_d & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.17}$$

$$\Lambda_1 = \begin{pmatrix} -\frac{t_2T_1}{T_2}I_d & 0 & 0 & \cdots \\ 0 & -\frac{t_3T_2}{T_3}I_d & 0 & \ddots \\ 0 & 0 & -\frac{t_4T_3}{T_4}I_d & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.18}$$

$$\Lambda_2 = \begin{pmatrix} \partial^2_{x_1}\omega_1 + \partial^2_{x_1}\omega_2 & \partial_{x_2}\partial_{x_1}\omega_2 & 0 & \cdots \\ \partial_{x_1}\partial_{x_2}\omega_2 & \partial^2_{x_2}\omega_2 + \partial^2_{x_2}\omega_3 & \partial_{x_3}\partial_{x_2}\omega_3 & \ddots \\ 0 & \partial_{x_2}\partial_{x_3}\omega_3 & \partial^2_{x_3}\omega_3 + \partial^2_{x_3}\omega_4 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.19}$$

Furthermore, we can write

$$\det \frac{\partial(\widetilde{z}_{1.L}, \widetilde{\zeta}_{1,L})}{\partial(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})} = \det \begin{pmatrix} I_{dL} & \Delta_1^{-1}\Lambda_1 \\ \Delta_2^{-1}\Lambda_2 & I_{dL} \end{pmatrix} = \det(I_{dL} - \Delta_1^{-1}\Lambda_1\Delta_2^{-1}\Lambda_2). \tag{3.20}$$

We denote the (j,k)-component of a matrix A by $[A]_{j,k}$. Note that

$$\begin{aligned} &\frac{T_{k-1}}{T_j}\partial_{x_k}\partial_{x_{k-1}}\omega_k + \frac{T_k}{T_j}(\partial^2_{x_k}\omega_k + \partial^2_{x_k}\omega_{k+1}) + \frac{T_{k+1}}{T_j}\partial_{x_k}\partial_{x_{k+1}}\omega_{k+1} \\ &= -\frac{t_k}{T_j}\partial_{x_k}\partial_{x_{k-1}}\omega_k + \frac{t_{k+1}}{T_j}\partial_{x_k}\partial_{x_{k+1}}\omega_{k+1} \\ &+ \frac{T_k}{T_j}\,{}^t(\partial_{x_{k-1}} + \partial_{x_k} + \partial_{x_{k+1}})(\partial_{x_k}\omega_k + \partial_{x_k}\omega_{k+1}), \end{aligned} \tag{3.21}$$

for any $k \le j-1$. Then we get the following estimates for any j :

$$\begin{aligned} &\sum_{k=1}^{dL} \left| [\Delta_2^{-1}\Lambda_2]_{j,k} \right| \le (4\kappa_2 + \mathcal{K}_2)d, \\ &\sum_{k=1}^{dL} \left| [\Delta_1^{-1}\Lambda_1\Delta_2^{-1}\Lambda_2]_{j,k} \right| \le T_{L+1}(4\kappa_2 + \mathcal{K}_2)d^2. \end{aligned} \tag{3.22}$$

By Lemma 2.1, we have

$$(1 - T_{L+1}(4\kappa_2 + \mathcal{K}_2)d^2)^{dL} \le \left| \det \frac{\partial(\widetilde{z}_{1.L}, \widetilde{\zeta}_{1,L})}{\partial(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})} \right|. \tag{3.23}$$

Therefore, there exists a constant C_2 such that

$$\begin{aligned} &\left| p^\circ \det \frac{\partial(\widetilde{x}_{1.L}, \widetilde{\xi}_{1,L})}{\partial(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L})} \right| \\ &\le (C_2)^L |p|_{d+1} \prod_{j=1}^{L} \frac{1}{(1+\nu|z_j|^2)^{(d+1)/2}} \prod_{j=1}^{L} \frac{1}{(1+\nu|\zeta_j|^2)^{(d+1)/2}}. \end{aligned} \tag{3.24}$$

Changing the variables: $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) \longrightarrow (\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L})$, we integrate by $\widetilde{z}_{1,L}$ and $\widetilde{\zeta}_{1,L}$. Then there exists a constant C_3 such that

$$|q(x_{L+1}, x_0)| = |\mathbb{I}(\Phi, p^\circ, \nu)(x_{L+1}, x_0)| \le (C_3)^L |p|_{d+1}. \tag{3.25}$$

4. Proof of Proposition

1°. For $\widetilde{x}_{1,L} \in \boldsymbol{R}^{dL}$, we introduce the norm $||\widetilde{x}_{1,L}||_\infty$ given by

$$||\widetilde{x}_{1,L}||_\infty = \max_{j=1,2,\dots,L} |x_j| \,. \tag{4.1}$$

We consider the mapping $\mathcal{F} : \widetilde{x}_{1,L} \to \widetilde{y}_{1,L}$ given by

$${}^t\widetilde{y}_{1,L} = \Delta_1^{-1}\Lambda_1\Delta_2^{-1}\lambda(x_{L+1}, \widetilde{x}_{1,L}, x_0) + {}^t\widetilde{x}^\circ_{1,L} \,, \tag{4.2}$$

where

$$\lambda(x_{L+1}, \widetilde{x}_{1,L}, x_0) = \begin{pmatrix} \partial_{x_1}\omega_1(x_1, x_0) + \partial_{x_1}\omega_2(x_2, x_1) \\ \partial_{x_2}\omega_2(x_2, x_1) + \partial_{x_2}\omega_3(x_3, x_2) \\ \vdots \\ \partial_{x_L}\omega_L(x_L, x_{L-1}) + \partial_{x_L}\omega_{L+1}(x_{L+1}, x_L) \end{pmatrix} . \tag{4.3}$$

For $\widetilde{x}_{1,L}$, $\widetilde{x}'_{1,L} \in \boldsymbol{R}^{dL}$, let

$$\begin{aligned} {}^t\widetilde{y}_{1,L} &= \Delta_1^{-1}\Lambda_1\Delta_2^{-1}\lambda(x_{L+1}, \widetilde{x}_{1,L}, x_0) + {}^t\widetilde{x}^\circ_{1,L} \,, \\ {}^t\widetilde{y}'_{1,L} &= \Delta_1^{-1}\Lambda_1\Delta_2^{-1}\lambda(x_{L+1}, \widetilde{x}'_{1,L}, x_0) + {}^t\widetilde{x}^\circ_{1,L} \,. \end{aligned} \tag{4.4}$$

Then we can write

$$\begin{aligned} &{}^t(\widetilde{y}'_{1,L} - \widetilde{y}_{1,L}) \\ &= \Delta_1^{-1}\Lambda_1\Delta_2^{-1} \int_0^1 \Lambda_2(x_{L+1}, \widetilde{x}_{1,L} + \theta(\widetilde{x}'_{1,L} - \widetilde{x}_{1,L}), x_0) d\theta \, {}^t(\widetilde{x}'_{1,L} - \widetilde{x}_{1,L}) \,. \end{aligned} \tag{4.5}$$

Hence we have

$$||\widetilde{y}'_{1,L} - \widetilde{y}_{1,L}||_\infty \le (4\kappa_2 + \mathcal{K}_2) d^2 T_{L+1} ||\widetilde{x}'_{1,L} - \widetilde{x}_{1,L}||_\infty \,. \tag{4.6}$$

Hence $\mathcal{F}$ is a contraction and has a unique fixed point $\widetilde{x}^*_{1,L}$ such that

$${}^t(\widetilde{x}^*_{1,L} - \widetilde{x}^\circ_{1,L}) = \Delta_1^{-1}\Lambda_1\Delta_2^{-1}\lambda(x_{L+1}, \widetilde{x}^*_{1,L}, x_0) \,, \tag{4.7}$$

Therefore, there exists a unique solution $(x_1^*, x_2^*, \dots, x_L^*)$ such that

$$\frac{x^*_{j+1} - x^*_j}{t_{j+1}} - \frac{x^*_j - x^*_{j-1}}{t_j} = \partial_{x_j}\omega_j(x^*_j, x^*_{j-1}) + \partial_{x_j}\omega_{j+1}(x^*_{j+1}, x^*_j) \,. \tag{4.8}$$

where $x^*_{L+1} = x_{L+1}$ and $x^*_0 = x_0$.

Furtheremore, for any β_{L+1}, β_0 with $|\beta_{L+1} + \beta_0| \ge 1$, there exists a constant C_{β_{L+1},β_0} such that

$$|\partial^{\beta_{L+1}}_{x_{L+1}} \partial^{\beta_0}_{x_0} (x^*_j - x^\circ_j)| \le C_{\beta_{L+1},\beta_0} T_{L+1} \,, \tag{4.9}$$

for $j = 1, 2, \dots, L$.

5. Proof of Theorem in the case $|\beta_{L+1}+\beta_0| \neq 0$

1°. For $j = 1, 2, \dots, L$, let

$$y_j = x_j - x_j^* \,. \tag{5.1}$$

Then we can rewrite

$$S - S^* = \sum_{j=1}^{L} y_j \Big(\psi_j^1 y_{j+1} + \psi_j^2 y_j + \psi_j^3 y_{j-1} \Big) \,, \tag{5.2}$$

with

$$\begin{aligned} \psi_j^1 &= -\frac{1}{2t_{j+1}} + \int_0^1 (1-\theta)(\partial_{x_{j+1}} \partial_{x_j} \omega_{j+1})(x_{j+1}^* + \theta y_{j+1}, x_j^* + \theta y_j) d\theta \,, \\ \psi_j^2 &= \frac{1}{2t_{j+1}} + \int_0^1 (1-\theta)(\partial_{x_j}^2 \omega_{j+1})(x_{j+1}^* + \theta y_{j+1}, x_j^* + \theta y_j) d\theta \\ &\quad + \frac{1}{2t_j} + \int_0^1 (1-\theta)(\partial_{x_j}^2 \omega_j)(x_j^* + \theta y_j, x_{j-1}^* + \theta y_{j-1}) d\theta \,, \\ \psi_j^3 &= -\frac{1}{2t_j} + \int_0^1 (1-\theta)(\partial_{x_{j-1}} \partial_{x_j} \omega_j)(x_j^* + \theta y_j, x_{j-1}^* + \theta y_{j-1}) d\theta \,, \end{aligned} \tag{5.3}$$

where $y_{L+1} = y_0 = 0$.

Changing the variables: $(x_1, x_2, \dots, x_L) \to (y_1, y_2, \dots, y_L)$, we have

$$q(x_{L+1}, x_0) = \mathcal{N} \int_{\boldsymbol{R}^{dL}} e^{i\nu\Psi} r(x_{L+1}, y_L, \dots, y_1, x_0) \prod_{j=1}^{L} dy_j \,, \tag{5.4}$$

where

$$\Psi = \sum_{j=1}^{L} y_j \Big(\psi_j^1 y_{j+1} + \psi_j^2 y_j + \psi_j^3 y_{j-1} \Big) \,, \tag{5.5}$$

and

$$r(x_{L+1}, y_L, \dots, y_1, x_0) = p(x_{L+1}, x_L^* + y_L, \dots, x_1^* + y_1, x_0) \,. \tag{5.6}$$

For β_{L+1}, β_0, we define $r_{\beta_{L+1},\beta_0}(x_{L+1}, y_L, \dots, y_1, x_0)$ such that

$$\partial_{x_{L+1}}^{\beta_{L+1}} \partial_{x_0}^{\beta_0} q(x_{L+1}, x_0) = \mathcal{N} \int_{\boldsymbol{R}^{dL}} e^{i\nu\Psi} r_{\beta_{L+1},\beta_0}(x_{L+1}, y_L, \dots, y_1, x_0) \prod_{j=1}^{L} dy_j \,, \tag{5.7}$$

For any β'_{L+1}, β'_0 and non-negative integer K, there exists a positive constant $C_{\beta_{L+1},\beta_0,\beta'_{L+1},\beta'_0,K}$ such that

$$\begin{aligned} &\Big| \partial_{x_{L+1}}^{\beta'_{L+1}} \partial_{x_0}^{\beta'_0} \Big(\prod_{j=1}^{L} \partial_{y_j}^{\beta_j} \Big) r_{\beta_{L+1},\beta_0}(x_{L+1}, y_L, \dots, y_1, x_0) \Big| \\ &\le (C_{\beta_{L+1},\beta_0,\beta'_{L+1},\beta'_0,K})^L |p|_{|\beta_{L+1}+\beta_0+\beta'_{L+1}+\beta'_0|+K} \\ &\quad \times \nu^{\sum_{j=1}^{L} |\beta_j|/2} (1 + \nu^{1/2} ||\widetilde{y}_{1,L}||_\infty)^{2|\beta_{L+1}+\beta_0|} \,, \end{aligned} \tag{5.8}$$

for any $|\beta_j| \le K$, $j = 1, 2, \dots, L$.
2°. We restore the variables:

$$x_j = y_j + x_j^*, \tag{5.9}$$

for $j = 1, 2, \dots, L$. Then we have

$$\begin{aligned} &\partial_{x_{L+1}}^{\beta_{L+1}} \partial_{x_0}^{\beta_0} q(x_{L+1}, x_0) \\ &= \mathcal{N} \int_{\mathbf{R}^{dL}} e^{i\nu(S - S^*)} p_{\beta_{L+1},\beta_0}(x_{L+1}, x_L, \dots, x_1, x_0) \prod_{j=1}^{L} dx_j \\ &= e^{-i\nu S^*} e^{i\nu \frac{(x_{L+1} - x_0)^2}{2T_{L+1}}} \mathbb{I}(\Phi, p_{\beta_{L+1},\beta_0}, \nu), \end{aligned} \tag{5.10}$$

where

$$p_{\beta_{L+1},\beta_0}(x_{L+1}, x_L, \dots, x_1, x_0) = r_{\beta_{L+1},\beta_0}(x_{L+1}, x_L - x_L^*, \dots, x_1 - x_1^*, x_0). \tag{5.11}$$

For any non-negative integer K, there exists a positive constant $C_{\beta_{L+1},\beta_0,K}$ such that

$$\begin{aligned} &\Big|\big(\prod_{j=1}^{L} \partial_{x_j}^{\beta_j}\big) p_{\beta_{L+1},\beta_0}(x_{L+1}, x_L, \dots, x_1, x_0)\Big| \\ &\le (C_{\beta_{L+1},\beta_0,K})^L |p|_{|\beta_{L+1}+\beta_0|+K} \nu^{\sum_{j=1}^{L} |\beta_j|/2} (1 + \nu^{1/2} \|\widetilde{x}_{1,L} - \widetilde{x}_{1,L}^*\|_\infty)^{2|\beta_{L+1}+\beta_0|}, \end{aligned} \tag{5.12}$$

for any $|\beta_j| \le K$, $j = 1, 2, \dots, L$.
3°. For $j = 1, 2, \dots, L$, let

$$z_j = \partial_{\xi_j} \Phi, \qquad \zeta_j = \partial_{x_j} \Phi. \tag{5.13}$$

Since

$$\begin{aligned} {}^t\widetilde{x}_{1,L} &= \Delta_1^{-1} \Lambda_1 \Delta_2^{-1} \lambda(x_{L+1}, \widetilde{x}_{1,L}, x_0) + {}^t\widetilde{x}_{1,L}^{\circ} \\ &\quad + \Delta_1^{-1}\, {}^t\widetilde{z}_{1,L} + \Delta_1^{-1} \Lambda_1 \Delta_2^{-1}\, {}^t\widetilde{\zeta}_{1,L}, \end{aligned} \tag{5.14}$$

and

$$ {}^t\widetilde{x}_{1,L}^* = \Delta_1^{-1} \Lambda_1 \Delta_2^{-1} \lambda(x_{L+1}, \widetilde{x}_{1,L}^*, x_0) + {}^t\widetilde{x}_{1,L}^{\circ}, \tag{5.15}$$

we can write

$$\begin{aligned} &- \Delta_1^{-1} \Lambda_1 \Delta_2^{-1}\, {}^t\widetilde{\zeta}_{1,L} + \Delta_1^{-1}\, {}^t\widetilde{z}_{1,L} \\ &= \Big(I_{dL} + \Delta_1^{-1} \Lambda_1 \Delta_2^{-1} \int_0^1 \Lambda_2(x_{L+1}, \widetilde{x}_{1,L} + \theta(\widetilde{x}_{1,L} - \widetilde{x}_{1,L}^*), x_0) d\theta \Big) \\ &\quad \times {}^t(\widetilde{x}_{1,L} - \widetilde{x}_{1,L}^*). \end{aligned} \tag{5.16}$$

Hence we have

$$||\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}||_\infty \leq 2(\sum_{j=1}^{L} |z_j| + \sum_{j=1}^{L} |\zeta_j|). \tag{5.17}$$

Therefore,

$$\begin{aligned}(1 + \nu^{1/2}&||\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}||_\infty) \\ &\leq 2 \cdot 2^L \prod_{j=1}^{L} (1 + \nu|z_j|^2)^{1/2} \prod_{j=1}^{L} (1 + \nu|\zeta_j|^2)^{1/2}. \end{aligned} \tag{5.18}$$

4°. Integrating by parts, we have

$$\mathbb{I}(\Phi, p_{\beta_{L+1},\beta_0}, \nu) = \mathbb{I}(\Phi, p^\circ_{\beta_{L+1},\beta_0}, \nu), \tag{5.19}$$

where

$$\begin{aligned} p^\circ_{\beta_{L+1},\beta_0} = &(N^*_L)^{2|\beta_{L+1}+\beta_0|+d+1} \dots (N^*_1)^{2|\beta_{L+1}+\beta_0|+d+1} \\ &\circ (M^*_L)^{2|\beta_{L+1}+\beta_0|+d+1} \dots (M^*_1)^{2|\beta_{L+1}+\beta_0|+d+1} p_{\beta_{L+1},\beta_0}. \end{aligned} \tag{5.20}$$

Hence, there exists a positive constant C_{β_{L+1},β_0} such that

$$\begin{aligned} |\partial^{\beta_{L+1}}_{x_{L+1}} \partial^{\beta_0}_{x_0} q(x_{L+1}, x_0)| &= |\mathbb{I}(\Phi, p^\circ_{\beta_{L+1},\beta_0}, \nu)(x_{L+1}, x_0)| \\ &\leq (C_{\beta_{L+1},\beta_0})^L |p|_{3|\beta_{L+1}+\beta_0|+d+1}, \end{aligned} \tag{5.21}$$

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