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Supercyclic Translation Semigroups of Linear Operators

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1 Introduction

The translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ is characterized to be hypercyclic, chaotic, supercyclic, and so on, according to the property of the admissible weight function. In 1997, W. Desch, W. Schappacher and G. F. Webb gave a necessary and sufficient condition to be hypercyclic for the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ by using the property of an admissible weight function. In 1999, M. Yamada and F. Takeo gave a necessary and sufficient condition to be chaotic for the translation semigroup on the same function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$. D. A. Herrero et al. investigated the spectral properties of hypercyclic and supercyclic operators on a complex, separable infinite dimensional Hilbert space [3, 4]. The definition of a hypercyclic or chaotic operator is consistent with that of topologically transitive or chaotic, respectively in a topological space defined by Devaney [2]. In [5], chaotic semigroups are associated with the idea of exactness and are applied to partial differential equations. C. Read has developed the theory of hypercyclic and chaotic bounded linear operators in connection with the invariant subspace problem of Hilbert spaces [6].

We investigate how the property of an admissible weight function changes according to supercyclic, hypercyclic and chaotic translation semigroups on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$. As for supercyclicity, the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ is always supercyclic if I is an interval $[0, \infty)$(Theorem 1(1)). For $I = (-\infty, \infty)$, the semigroup is not always supercyclic and we give a necessary and sufficient condition to be supercyclic for the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ (Theorem 1(2)). We also construct the special function x such that $\{cT(t)x \mid t \geq 0, c \in \mathbb{R}\}$ is dense in X (Remark).

2 Preliminaries

Let X be a Banach space. A strongly continuous semigroup $\{T(t)\}$ of linear operators on X is called *supercyclic* (resp. *hypercyclic*) if there exists $x \in X$ such that $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ (resp. $\{T(t)x \mid t \geq 0\}$) is dense in X [4]. A strongly continuous semigroup $\{T(t)\}$ is called *chaotic* if $\{T(t)\}$ is hypercyclic and the set $X_{per} = \{x \in X \mid \exists t > 0\ s.t.\ T(t)x = x\}$ of periodic points is dense in X[1]. Let I be the interval $[0, \infty)$ or $(-\infty, \infty)$. By an *admissible weight function* on I we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the conditions:
(i)　$\rho(x) > 0$ for all $x \in I$;

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{\omega t}\rho(t+x)$ for all $x \in I$ and $t > 0$. With an admissible weight function, we construct the following function spaces:

$$L^p_\rho(I,\mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \text{ measurable}, \int_I |u(\tau)|^p \rho(\tau)\, d\tau < \infty \right\}$$

$$\text{with } \|u\| = \left(\int_I |u(\tau)|^p \rho(\tau)\, d\tau \right)^{\frac{1}{p}}, \qquad (p \geq 1)$$

$$C_{0,\rho}(I,\mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \text{ continuous}, \lim_{\tau \to \pm\infty} \rho(\tau)u(\tau) = 0 \right\}$$

$$\text{with } \|u\| = \sup_{\tau \in I} |u(\tau)|\rho(\tau).$$

We consider a (*forward*) *translation semigroup* $\{T(t)\}$ with parameter $t \geq 0$ such as

$$[T(t)u](\tau) = u(\tau + t) \quad \text{for } u \in C_{0,\rho}(I) \text{ or } L^p_\rho(I).$$

When $\rho(\tau) = 1$, weighted function spaces are equal to L^p or C_0 and the translation semigroup is never hypercyclic, since the norm of $T(t)$ is equal to 1 for all $t \geq 0$ in L^p or C_0. Necessary and sufficient conditions for the translation semigroup in L^p_ρ or $C_{0,\rho}$ to be hypercyclic or to be chaotic are known as follows.

Theorem A [1]. *Let X be $L^p_\rho(I)$ or $C_{0,\rho}(I)$ with an admissible weight function ρ. Then the following (1) and (2) are equivalent:*

(1) *the translation semigroup $\{T(t)\}$ on X is hypercyclic;*

(2) (i) *if $I = [0,\infty)$, then $\liminf_{t\to\infty} \rho(t) = 0$ holds.*

(ii) *if $I = (-\infty,\infty)$, then for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j=1}^\infty$ ($t_j \to \infty$ as $j \to \infty$) of positive real numbers such that*

$$\lim_{j\to\infty} \rho(t_j + \theta) = \lim_{j\to\infty} \rho(-t_j + \theta) = 0.$$

Theorem B [7]. *Let $I = (-\infty,\infty)$ (resp. $I = [0,\infty)$) and let X be $L^p_\rho(I)$. Then the translation semigroup $\{T(t)\}$ on X is chaotic if and only if for all $\varepsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that*

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \rho(l + nP) < \varepsilon \quad \left(\text{resp. } \sum_{n=1}^{\infty} \rho(l + nP) < \varepsilon \right).$$

Theorem C [7]. *Let $I = (-\infty,\infty)$ (resp. $I = [0,\infty)$) and let X be $C_{0,\rho}(I)$. Then the following assertions are equivalent:*

(1) *the translation semigroup $\{T(t)\}$ on X is chaotic;*

(2) *for all $\varepsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that*
$\rho(l + nP) < \varepsilon$ *for all $n \in \mathbb{Z} \setminus \{0\}$ (resp. $n \in \mathbb{N}$);*

(3) *there exists $\{l_i\}_{i=1}^\infty \subset \mathbb{R}^+$ ($l_i \to \infty$ as $i \to \infty$) such that for all $\varepsilon > 0$ and for all $i \in \mathbb{N}$ there exists $P > 0$ such that $\rho(l_i + nP) < \varepsilon$ for all $n \in \mathbb{Z} \setminus \{0\}$ (resp. $n \in \mathbb{N}$).*

3 Supercyclic semigroups

As shown in the previous section, necessary and sufficient conditions for the translation semigroup to be hypercyclic or to be chaotic are known. In this section, we shall give a necessary and sufficient condition for the translation semigroup to be supercyclic. In the first subsection we consider a semigroup on a Banach space, and in the next subsection we treat a translation semigroup on weighted function spaces.

3.1 Supercyclic semigroup on a Banach space

Lemma 1. *Let X be a separable infinite dimensional Banach space. Suppose that $\{T(t)\}$ is supercyclic, i.e. there exists $x \in X$ such that the set $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ is dense in X. Then the set $\{cT(t)x \mid t \geq s,\ c \in \mathbb{R}\}$ is also dense in X for all $s \geq 0$.*

Proof. Assume there exists $s_0 \geq 0$ such that $A = \{cT(t)x \mid t \geq s_0,\ c \in \mathbb{R}\}$ is not dense in X. Hence there exists a bounded open set U such that $U \cap \overline{A} = \phi$. Therefore we have

$$U \subset \overline{\{cT(t)x \mid 0 \leq t \leq s_0,\ c \in \mathbb{R}\}}$$

by using the relation
$X = \overline{\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}} = \overline{\{cT(t)x \mid t \geq s_0,\ c \in \mathbb{R}\}} \cup \overline{\{cT(t)x \mid 0 \leq t \leq s_0,\ c \in \mathbb{R}\}}$. By the definition of semigroup, if there exists $t_0 > 0$ such that $T(t_0)x = 0$ then $T(t)x = 0$ for all $t \geq t_0$. So we have $T(t)x \neq 0$ for all $t \geq 0$ since the set $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ is dense in X. Since $T(t)x$ is continuous with t and $T(t)x \neq 0$ for all $t \geq 0$, there exists $m_1, m_2 \in \mathbb{R}$ such that $0 < m_1 \leq ||T(t)x|| \leq m_2$ for $0 \leq t \leq s_0$. There exists $M \geq 0$ such that $||y|| \leq M$ for any $y \in U$ because U is bounded. So we have $U \subset \overline{\left\{cT(t)x \mid 0 \leq t \leq s_0, |c| \leq \frac{M}{m_1}\right\}}$, which means $\overline{U}$ is compact. Hence X is finite dimensional, which contradicts that X is infinite dimensional. □

Lemma 2. *Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous semigroup on a separable Banach space X. Then the following are equivalent:*

(1) *$\{T(t)\}$ is supercyclic;*

(2) *for all $y, z \in X$ and all $\varepsilon > 0$, there exists $v \in X$, $t > 0$ and $c \in \mathbb{R}$ such that $||y - v|| < \varepsilon$ and $||z - cT(t)v|| < \varepsilon$;*

(3) *for all $y, z \in X$, all $\varepsilon > 0$ and for all $l \geq 0$, there exists $v \in X$, $t > l$ and $c \in \mathbb{R}$ such that $||y - v|| < \varepsilon$ and $||z - cT(t)v|| < \varepsilon$.*

Proof. (1) implies (3): Let $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ be dense in X. For any $y, z \in X$ and any $l \geq 0$, there exists $s > 0$ and $c_1 \in \mathbb{R}$ such that $||y - c_1 T(s)x|| < \varepsilon$, and there exists $u > s + l$ and $c_2 \in \mathbb{R}$ such that $||z - c_2 T(u)x|| < \varepsilon$ by Lemma 1. Put $v = c_1 T(s)x$. Then we have the first inequality. Put $t = u - s > l$ and $c = \frac{c_2}{c_1}$. Then we have the second inequality.
(3) implies (2): It is obvious.
(2) implies (1): The proof is similar to the proof in case of hypercyclic in [1]. □

3.2 Supercyclic translation semigroups on a separable Banach space, L^p_ρ and $C_{0,\rho}$

In this section we shall consider a translation semigroup in $L^p_\rho(I)$ and $C_{0,\rho}(I)$. At first we shall quote the lemma which is needed later.

Lemma 3. [1] *Let I be the interval $(-\infty,\infty)$ or $[0,\infty)$ and let ρ be an admissible weight function on I, that is, there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(\tau+t)$ for all $\tau \in I$ and $t > 0$. For $l > 0$, put $M_l = Me^{\omega l}$ for $\omega > 0$ and $M_l = M$ for $\omega \leq 0$. Then $M_l \geq 1$ and the inequality*

$$\frac{1}{M_l}\rho(\sigma) \leq \rho(\tau) \leq M_l\rho(\sigma+l) \tag{1a}$$

holds for any $\sigma \in I$ and any $\tau \in [\sigma, \sigma+l]$.

By using the lemma, we give a necessary and sufficient condition for a translation semigroup to be supercyclic.

Theorem 1. *Let X be the space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ and ρ be an admissible weight function. Let $\{T(t)\}$ be a translation semigroup on X. Then the following assertions hold:*

(1) *if $I = [0,\infty)$, then $\{T(t)\}$ is supercyclic;*

(2) *if $I = (-\infty,\infty)$, then $\{T(t)\}$ is supercyclic if and only if there exists a sequence $\{t_j\}_{j=1}^\infty$ ($t_j \to \infty$ as $j \to \infty$) such that $\lim_{j\to\infty}\rho(t_j+\theta)\rho(-t_j+\theta) = 0$ for each $\theta \in \mathbb{R}$.*

Proof. (1) Let X_0 be the set of all $x \in X$ such that the support of x is compact. For any $y, z \in X$ and any $\varepsilon > 0$, there exists $y_0 \in X_0$ such that $||y - y_0|| < \frac{\varepsilon}{2}$ since X_0 is dense in X. There exists $t_1 > 0$ such that $T(s)y_0 = 0$ for any $s \geq t_1$ since $y_0 \in X_0$. Put

$$\omega'(\tau) = \begin{cases} z(\tau - t_1) & t_1 \leq \tau \\ \frac{z(0)}{\varepsilon}\tau + z(0)(1 - \frac{t_1}{\varepsilon}) & t_1 - \varepsilon \leq \tau \leq t_1 \\ 0 & 0 \leq \tau \leq t_1 - \varepsilon. \end{cases}$$

Then $T(t_1)\omega' = z$ holds. Put $\omega = \frac{\varepsilon\omega'}{2||\omega'||}$, $c = \frac{2||\omega'||}{\varepsilon}$ and $v = y_0 + \omega$. Then we have $v \in X$, $||y - v|| = ||y - y_0 - \omega|| \leq ||y - y_0|| + ||\omega|| < \varepsilon$ and $||z - cT(t_1)v|| \leq ||z - cT(t_1)\omega|| + ||cT(t_1)\omega - cT(t_1)v|| = ||z - T(t_1)\omega'|| + ||cT(t_1)y_0|| = 0$. By Lemma 2 (2), $\{T(t)\}$ is supercyclic.
(2) We shall show the proof for the case $X = L^p_\rho(I, \mathbb{C})$ ($p \geq 1$), since it is similar to that in the case $X = C_{0,\rho}(I, \mathbb{C})$.
($\Rightarrow$) Let $\{T(t)\}$ be supercyclic. We will show $\lim_{j\to\infty}\rho(t_j+\theta)\rho(-t_j+\theta) = 0$.
Fix any $\theta \in \mathbb{R}$. Let $y, z \in X$ be functions with compact support $\subset [\theta, \theta+l]$ ($l > 0$), $y \geq 0, z \leq 0$, and $||y|| = ||z|| = 1$. By Lemma 2 (3), for any $\varepsilon > 0$ there exists $v_\varepsilon \in X$, $t_\varepsilon > l$ and $c_\varepsilon > 0$ such that $||c_\varepsilon T(t_\varepsilon)v_\varepsilon - z|| < \varepsilon$ and $||v_\varepsilon - y|| < \varepsilon$. Put $\omega_1 = v^+_{\varepsilon|[\theta,\theta+l]}$ and $\omega_2 = v^-_{\varepsilon|[\theta+t_\varepsilon,\theta+l+t_\varepsilon]}$.
Then we have the following: $\omega_1 \geq 0$, $\mathrm{supp}(\omega_1) \subset [\theta, \theta+l]$, $\mathrm{supp}(T(t_\varepsilon)\omega_1) \subset [\theta - t_\varepsilon, \theta + l - t_\varepsilon]$, $\omega_2 \leq 0$, $\mathrm{supp}(\omega_2) \subset [\theta + t_\varepsilon, \theta + l + t_\varepsilon]$, $\mathrm{supp}(T(t_\varepsilon)\omega_2) \subset [\theta, \theta + l]$. Then the following holds:

$$||c_\varepsilon T(t_\varepsilon)\omega_1|| < \varepsilon \tag{2a}$$
$$||y|| - ||\omega_1|| < \varepsilon \tag{2b}$$
$$||\omega_2|| < \varepsilon \tag{2c}$$
$$||z|| - ||c_\varepsilon T(t_\varepsilon)\omega_2|| < \varepsilon. \tag{2d}$$

By Lemma 3, there exists $M_l \geq 1$ satisfying (1a). Then the following holds:

$$\begin{aligned} ||c_\varepsilon T(t_\varepsilon)\omega_1||^p &= \int_{\theta-t_\varepsilon}^{\theta+l-t_\varepsilon} \rho(\tau)|c_\varepsilon \omega_1(\tau+t_\varepsilon)|^p d\tau \\ &\geq \frac{1}{M_l}\rho(\theta-t_\varepsilon)|c_\varepsilon|^p \int_{\theta-t_\varepsilon}^{\theta+l-t_\varepsilon} |\omega_1(\tau+t_\varepsilon)|^p d\tau \\ &= \frac{|c_\varepsilon|^p}{M_l}\rho(\theta-t_\varepsilon)\int_\theta^{\theta+l} |\omega_1(\tau)|^p d\tau, \end{aligned}$$

$$\begin{aligned} ||\omega_1||^p &= \int_\theta^{\theta+l} \rho(\tau)|\omega_1(\tau)|^p d\tau \\ &\leq M_l\rho(\theta+l)\int_\theta^{\theta+l} |\omega_1(\tau)|^p d\tau. \end{aligned}$$

So we have the inequality:

$$\frac{||\omega_1||^p}{M_l\rho(\theta+l)} \leq \frac{M_l||c_\varepsilon T(t_\varepsilon\omega_1)||^p}{|c_\varepsilon|^p\rho(\theta-t_\varepsilon)}. \tag{3a}$$

Similarly we have the following:

$$\begin{aligned} ||\omega_2||^p &= \int_{\theta+t_\varepsilon}^{\theta+l+t_\varepsilon} \rho(\tau)|\omega_2(\tau)|^p d\tau \\ &\geq \frac{1}{M_l}\rho(\theta+t_\varepsilon)\int_{\theta+t_\varepsilon}^{\theta+l+t_\varepsilon} |\omega_2(\tau)|^p d\tau, \end{aligned}$$

$$\begin{aligned} ||c_\varepsilon T(t_\varepsilon)\omega_2||^p &= \int_\theta^{\theta+l} \rho(\tau)|c_\varepsilon\omega_2(\tau+t_\varepsilon)|^p d\tau \\ &\leq M_l\rho(\theta+l)|c_\varepsilon|^p \int_{\theta+t_\varepsilon}^{\theta+l+t_\varepsilon} |\omega_2(\tau)|^p d\tau. \end{aligned}$$

So we have the inequality:

$$\frac{M_l||\omega_2||^p}{\rho(\theta+t_\varepsilon)} \geq \frac{||c_\varepsilon T(t_\varepsilon\omega_2)||^p}{M_l|c_\varepsilon|^p\rho(\theta+l)}. \tag{3b}$$

By the inequalities (2a), (3a), and (2b),

$$\begin{aligned} \varepsilon^p &> ||c_\varepsilon T(t_\varepsilon)\omega_1||^p \\ &\geq \frac{|c_\varepsilon|^p\rho(\theta-t_\varepsilon)||\omega_1||^p}{M_l^2\rho(\theta+l)} \\ &> \frac{|c_\varepsilon|^p\rho(\theta-t_\varepsilon)(1-\varepsilon)^p}{M_l^2\rho(\theta+l)} \end{aligned} \tag{3c}$$

holds. Similarly by (2c), (3b), and (2d),

$$\begin{aligned} \varepsilon^p &> ||\omega_2||^p \\ &\geq \frac{\rho(\theta+t_\varepsilon)||c_\varepsilon T(t_\varepsilon)\omega_2||^p}{M_l^2|c_\varepsilon|^p\rho(\theta+l)} \\ &> \frac{\rho(\theta+t_\varepsilon)(1-\varepsilon)^p}{M_l^2|c_\varepsilon|^p\rho(\theta+l)} \end{aligned} \tag{3d}$$

holds. By (3c) and (3d), we can verify that $\varepsilon^{2^p} > (\frac{(1-\varepsilon)^p}{M_l^2\rho(\theta+l)})^2\rho(\theta+t_\varepsilon)\rho(\theta-t_\varepsilon) \geq 0$ holds. If ε tends to 0, then $\rho(\theta-t_\varepsilon)\rho(\theta+t_\varepsilon)$ tends to 0.

($\Leftarrow$) Assume for each $\theta \in \mathbb{R}$, there exists a sequence $\{t_j\} \subset \mathbb{R}_+$ such that $\lim_{j\to\infty} \rho(t_j+\theta)\rho(-t_j+\theta) = 0$. Let y and z be any nonzero functions with compact support $[\theta-l,\theta]$ ($\theta \in \mathbb{R}$, $l > 0$). For $l > 0$, there exists M_l satisfying (1a) by Lemma 3. By the assumption, for any $\varepsilon > 0$, there exists $t_j > l$ such that

$$\rho(t_j+\theta)\rho(-t_j+\theta) < \frac{(\rho(\theta-l)\varepsilon)^2}{M_l^4||z||^p||y||^p}$$

holds. Put

$$v_j(\tau) = \begin{cases} y(\tau) & \tau \in [\theta-l,\theta] \\ \frac{1}{c_j}\cdot z(\tau-t_j) & \tau \in [t_j+\theta-l, t_j+\theta] \\ 0 & \text{otherwise} \end{cases}$$

with $c_j = (\frac{||z||^p M_l^2\rho(t_j+\theta)}{\varepsilon\rho(\theta-l)})^{\frac{1}{p}}$.

By Lemma 3 and the above inequality, we have

$$||v_j - y||^p = \int_{t_j+\theta-l}^{t_j+\theta} |\frac{1}{c_j}\cdot y(\tau+t_j)|^p\rho(\tau)\,d\tau \leq \frac{1}{c_j^p}\cdot\frac{M_l^2\rho(t_j+\theta)}{\rho(\theta-l)}||z||^p = \varepsilon$$

and

$$||c_jT(t_j)v_j - z||^p = \int_{\theta-l-t_j}^{\theta-t_j} |\frac{1}{c_j}\cdot z(\tau-t_j)|^p\rho(\tau)\,d\tau \leq c_j^p\frac{M_l^2\rho(\theta-t_j)}{\rho(\theta-l)}||y||^p < \varepsilon.$$

Therefore $\{T(t)\}$ is supercyclic by Lemma 2. □

Remark. It is possible that the supercyclicity is proved by showing the existence of the special function $x \in X$ such that $\{cT(t)x \mid t \geq 0, c \in \mathbb{R}\}$ is dense in X. We shall show that in the case of $X = C_{0,\rho}([0,\infty))$ the translation semigroup on X is supercyclic from the definition directly.

Let $C_{cpt}([0,\infty))$ be the space of continuous functions on $[0,\infty)$ with compact support. Then $C_{cpt}([0,\infty))$ is a dense subset of X. Let $C^0_{cpt,1}([0,\infty))$ be the set $\{f \in C_{cpt}([0,\infty)) \mid ||f||_\infty \leq 1, f(0) = 0\}$. Put $s(f) = \sup\{\tau \in [0,\infty) \mid f(\tau) \neq 0\}$ for any $f \in C_{cpt}([0,\infty))$.

Let $F = \{f_k\}_{k=1}^\infty$ be a countable subset of $C^0_{cpt,1}([0,\infty])$ such that for any $g \in C^0_{cpt,1}([0,\infty])$ and for any $\varepsilon > 0$, there exists $f \in F$ satisfying $||f-g||_\infty < \varepsilon$ and $|s(f)-s(g)| < 1$. Let $F' = \{f_1, f_2, f_1, f_3, f_2, f_1, \ f_4, f_3, f_2, f_1, \cdots\} = \{h_1, h_2, h_3, \cdots\}$. For $k \in \mathbb{N}$, put $L_k = s(h_k)+1$, $K_{k+1} = \Sigma_{i=1}^k L_i$ and $\alpha_k = \sup_{\tau\in[0,K_{k+1}+1]}\rho(\tau)$. Then α_k is finite by the definition of an admissible weight function ρ. Put $K_1 = 0$, $\beta_1 = \max\{\alpha_1, 1\}$ and

$$\beta_k = \max\{k\alpha_k, k\beta_1\alpha_k, \cdots, k\beta_{k-1}\alpha_k\}$$

for $k \geq 2$. Put

$$x(\tau) = \begin{cases} \frac{1}{\beta_1} h_1(\tau) & K_1 \leq \tau \leq K_2 \\ \frac{1}{\beta_2} h_2(\tau - K_2) & K_2 \leq \tau \leq K_3 \\ \cdots\cdots & \\ \frac{1}{\beta_k} h_k(\tau - K_k) & K_k \leq \tau \leq K_{k+1} \\ \cdots\cdots & \\ \cdots\cdots & \end{cases}$$

Then x is continuous on $[0, \infty)$, since $h_j \in C^0_{cpt,1}([0,\infty))$. So x belongs to X by the following relation:

$$\begin{aligned} \lim_{\tau\to\infty} |x(\tau)\rho(\tau)| &\leq \lim_{k\to\infty} \sup_{\tau\in[K_k,K_{k+1}]} \frac{1}{\beta_k} |h_k(\tau - K_k)|\rho(\tau) \\ &\leq \lim_{k\to\infty} \frac{1}{\beta_k} \cdot \alpha_k \leq \lim_{k\to\infty} \frac{1}{k} = 0. \end{aligned}$$

We shall show that for any $f \in X$ and any $\varepsilon > 0$, there exist $c \in \mathbb{R}$ and $t \geq 0$ such that $||f - cT(t)x|| < \varepsilon$. Since $C_{cpt}([0,\infty))$ is dense in X, there exists $f_0 \in C_{cpt}([0,\infty))$ $(||f_0||_\infty \neq 0)$ such that $\sup_{\tau\in[0,\infty)} |(f(\tau) - f_0(\tau))\rho(\tau)| < \frac{\varepsilon}{2}$.

Put $K = \sup_{\tau\in[0,s(f_0)+2]} \rho(\tau)$. There exists $h \in F$ such that $\sup_{\tau\in[0,\infty)} |\frac{f_0(\tau)}{||f_0||_\infty} - h(\tau+1)| < \frac{\varepsilon}{2K||f_0||_\infty}$ and $|s(f_0) - (s(h) - 1)| < 1$. By the way of construction of F', there exists countable numbers $m(1) < m(2) < \cdots < m(j) < \cdots$ such that $h = h_{m(j)} \in F'$. For any $j \in \mathbb{N}$, put $t_j = K_{m(j)} + 1$ and $c_j = \beta_{m(j)}||f_0||_\infty$. Then for $\tau \in [0, s(h)]$ we have $h_{m(j)}(\tau+1) = \beta_{m(j)} x(\tau + t_j)$. So by using the relations $s(h) \leq s(f_0) + 2$ and $s(f_0) \leq s(h)$, we have

$$\begin{aligned} &\sup_{\tau\in[0,s(h)]} |f_0(\tau) - c_j T(t_j)x(\tau)|\rho(\tau) = \sup_{\tau\in[0,s(h)]} ||f_0||_\infty |\frac{f_0(\tau)}{||f_0||_\infty} - h(\tau+1)|\rho(\tau) \\ &< ||f_0||_\infty \cdot \frac{\varepsilon}{2K||f_0||_\infty} \cdot K < \frac{\varepsilon}{2}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\tau\in[s(h),\infty)} |f_0(\tau) - c_j T(t_j)x(\tau)|\rho(\tau) = \sup_{\tau\in[s(h),\infty)} |c_j T(t_j)x(\tau)|\rho(\tau) \\ = &\sup_{l\geq j} \sup_{k\in\{1,2,\cdots,m(l+1)-m(l)\}} \sup_{\tau\in[K_{m(l)+k}+1, K_{m(l)+k+1}+1]} | \, ||f_0||_\infty \beta_{m(j)} \frac{h_{m(j)+k}(\tau - K_{m(l)+k})}{\beta_{m(l)+k}} |\rho(\tau - t_j) \\ \leq &\sup_{l\geq j} \sup_k ||f_0||_\infty \beta_{m(j)} \cdot \frac{\alpha_{m(l)+k}}{\beta_{m(l)+k}} \leq ||f_0||_\infty \cdot \frac{1}{m(j)}. \end{aligned}$$

Since $\lim_{j\to\infty} m(j) = \infty$, we have

$$\begin{aligned} ||f_0 - c_j T(t_j)x|| &\leq \max\left\{ \sup_{\tau\in[0,L]} |f_0(\tau) - c_j T(t_j)x(\tau)|\rho(\tau), \sup_{\tau\in[L,\infty)} |f_0(\tau) - c_j T(t_j)x(\tau)|\rho(\tau) \right\} \\ &< \frac{\varepsilon}{2}, \end{aligned}$$

for sufficiently large j. By the inequality $||f - c_j T(t_j)x|| \leq ||f - f_0|| + ||f_0 - c_j T(t_j)x|| < \varepsilon$, we get the conclusion.

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