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ON $\mathcal{L}$-STARCOMPACT SPACES

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ABSTRACT. A space X is *$\mathcal{L}$-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a Lindelöf subset L of X such that $St(L,\mathcal{U})=X$. We clarify the relations between $\mathcal{L}$-starcompact spaces and other related spaces and investigate topological properties of $\mathcal{L}$-starcompact spaces. A question of Hiremath [3] is answered.

1. INTRODUCTION

By a space, we mean a topological space. Let us recall [6] that a space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset B of X such that $St(B,\mathcal{U})=X$, where $St(B,\mathcal{U})=\bigcup\{U\in\mathcal{U}:U\cap B\neq\emptyset\}$. It is clear that every separable space is star-Lindelöf. Also, it is not difficult to see that every T_1-space with countable extent is star-Lindelöf. Therefore, every countably compact T_1-space is star-Lindelöf as well as every Lindelöf space. As generalities of star-Lindelöfness, the following classes of spaces are given (see [6]):

Definition 1.1. A space X is *$\mathcal{L}$-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a Lindelöf subset L of X such that $St(L,\mathcal{U})=X$.

Definition 1.2. A space X is *$1\frac{1}{2}$-starLindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U})=X$.

In [3], $\mathcal{L}$-starcompactness is called sLc property, and in [1], a $1\frac{1}{2}$-starLinde-löf space is called a star-Lindelöf space and a star-Lindelöf space is called a stronglystar-Lindelöf space.

From the above definitions, we have the following diagram:

$$\text{star-Lindelöf}\longrightarrow\mathcal{L}\text{-starcompact}\longrightarrow 1\tfrac{1}{2}\text{-starLindelöf}.$$

In the following section, we give examples showing that the converses in the above Diagram do not hold.

The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual,

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a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [2].

2. $\mathcal{L}$-STARCOMPACT SPACES AND RELATED SPACES

In [3], Hiremath asked if the product of two countably compact spaces is $\mathcal{L}$-starcompact. However it is not difficult to see that the following well-known example gives a negative answer to the above question (see [8, Theorem 2.7]), we shall give the proof roughly for the sake of completeness. The symbol $\beta(X)$ means the Čech-Stone compactification of a Tychonoff space X.

Example 2.1. *There exist two countably compact spaces X and Y such that $X \times Y$ is not $\mathcal{L}$-starcompact.*

Proof. Let D be a discrete space of the cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$, $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
(3) every infinite subset of E_α (resp. F_α) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

Those sets E_α and F_α are well-defined since every infinite closed set in $\beta(D)$ has the cardinality $2^{\mathfrak{c}}$ (see [5]). Then, $X \times Y$ is not $\mathcal{L}$-starcompact, because the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$ and $\mathcal{L}$-starcompactness is preserved by open and closed subsets. □

We end this section by giving examples which show the converses in the above diagram in §1 do not hold.

Example 2.2. *There exists an $\mathcal{L}$-starcompact Tychonoff space which is not star-Lindelöf.*

Proof. Let D be a discrete space of the cardinality $\mathfrak{c}$. Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then, X is $\mathcal{L}$-starcompact, since $\beta(D) \times \omega$ is a Lindelöf dense subset of X.

Next, we shall show that X is not star-Lindelöf. Let us consider the open cover

$$\mathcal{U} = \{\{d\} \times (\omega + 1) : d \in D\} \cup \{\beta(D) \times \{n\} : n \in \omega\}$$

of X. Let B be a countable subset of X. Then, there exists a $d^* \in D$ such that $B \cap (\{d^*\} \times (\omega + 1)) = \emptyset$. This means that $U = \{d^*\} \times (\omega + 1)$ is the only element of $\mathcal{U}$ containing the point $\langle d^*, \omega\rangle$, and hence $\langle d^*, \omega\rangle \notin St(B, \mathcal{V})$. □

Example 2.3. *There exists a $1\frac{1}{2}$-starLindelöf Tychonoff space which is not $\mathcal{L}$-starcompact.*

Proof. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Define

$$X = \mathcal{R} \cup (\mathfrak{c} \times \omega).$$

We topologize X as follows: $\mathfrak{c} \times \omega$ has the usual product topology and is an open subspace of X. On the other hand a basic neighbourhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}$ and a finite subset K of ω. To show that X is $1\frac{1}{2}$-starLindelöf, let $\mathcal{U}$ be an open cover of X. Let

$$M = \{n \in \omega : (\exists U \in \mathcal{U})(\exists \beta < \mathfrak{c})((\beta, \mathfrak{c}) \times \{n\} \subseteq U)\}.$$

For each $n \in M$, there exist $U_n \in \mathcal{U}$ and $\beta_n < \mathfrak{c}$ such that $(\beta_n, \mathfrak{c}) \times \{n\} \subseteq U_n$. If we put $\mathcal{V}' = \{U_n : n \in M\}$, then

$$\mathcal{R} \subseteq St(\cup\mathcal{V}', \mathcal{U}).$$

On the other hand, for each $n < \omega$, since $\mathfrak{c} \times \{n\}$ is countably compact, we can find a finite subfamily $\mathcal{V}_n$ of $\mathcal{U}$ such that

$$\mathfrak{c} \times \{n\} \subseteq St(\cup\mathcal{V}_n, \mathcal{U}).$$

Consequently, if we put $\mathcal{V} = \mathcal{V}' \cup \cup\{\mathcal{V}_n : n < \omega\}$, Then, $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$ and $X = St(\cup\mathcal{V}, \mathcal{U})$. Hence, X is $1\frac{1}{2}$-starLindelöf.

Next, we shall show that X is not $\mathcal{L}$-starcompact. Since $|\mathcal{R}| = \mathfrak{c}$, enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{r_\alpha\} \cup ((\alpha, \mathfrak{c}) \times r_\alpha)$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\mathfrak{c} \times \omega\}$$

of X and let L be a Lindelöf subset of X. Since $\mathcal{R}$ is discrete closed in X, $L \cap \mathcal{R}$ is countable. Hence, there exists $\beta' < \mathfrak{c}$ such that

$$L \cap \{r_\alpha : \alpha > \beta'\} = \emptyset. \tag{1}$$

On the other hand, $L \cap (\mathfrak{c} \times \{n\})$ is bounded in $\mathfrak{c} \times \{n\}$ for each $n < \omega$. Thus, there exists $\beta_n < \mathfrak{c}$ such that $\beta_n > \sup\{\alpha < \mathfrak{c} : \langle \alpha, n \rangle \in L\}$. Pick $\beta'' < \mathfrak{c}$ such that $\beta'' > \beta_n$ for each $n \in \omega$. Then,

$$((\beta'', \mathfrak{c}) \times \omega) \cap L = \emptyset. \tag{2}$$

Choose $\gamma < \mathfrak{c}$ such that $\gamma > \max\{\beta', \beta''\}$. Then, U_γ is the only element of $\mathcal{U}$ containing the point r_γ and $U_\gamma \cap L = \emptyset$ by (1) and (2). It follows that $r_\gamma \notin St(L, \mathcal{U})$, and which shows that X is not $\mathcal{L}$-starcompact. □

Remark 1. The author does not know if each arrow in the above diagram can be reversed in the realm of normal spaces.

3. Properties of $\mathcal{L}$-starcompact spaces

Topological behavior of $\mathcal{L}$-starcompact spaces are extensively studied by Hiremath [3] and Ikenaga [4]. The purpose of this section is to prove some results which supply their investigation. In [3, Example 3.6], Hiremath proved that a closed subspace of an $\mathcal{L}$-starcompact space need not be $\mathcal{L}$-starcompact. The following example shows that a regular closed subspace of an $\mathcal{L}$-starcompact space need not be $\mathcal{L}$-starcompact.

Example 3.1. *There exists a star-Lindelöf (hence, an $\mathcal{L}$-starcompact) Tychonoff space having a regular-closed subset which is not $\mathcal{L}$-starcompact.*

Proof. Let $S_1 = (\mathfrak{c} \times \omega) \cup \mathcal{R}$ be the same space as the space X in Example 2.3. As we prove above, S_1 is not $\mathcal{L}$-starcompact. Let $S_2 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then, S_2 is $\mathcal{L}$-starcompact because it is separable.

Assume $S_1 \cap S_2 = \emptyset$ and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ identifying the subspace $\mathcal{R}$ of S_1 with the subspace $\mathcal{R}$ of S_2. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then, $\varphi[S_1]$ is a regular-closed subspace of X which is not $\mathcal{L}$-starcompact.

We shall show that X is star-Lindelöf. Let $\mathcal{U}$ be an open cover of X. For each $n \in \omega$, since $\varphi[\mathfrak{c} \times \{n\}]$ is countably compact, there exists a finite subset $F_n \subseteq \varphi[\mathfrak{c} \times \{n\}]$ such that $\varphi[\mathfrak{c} \times \{n\}] \subseteq St(F_n, \mathcal{U})$. Thus, if we put $B' = \cup\{F_n : n \in \omega\}$, then

$$\varphi[\mathfrak{c} \times \omega] \subseteq St(B', \mathcal{U}).$$

On the other hand, since $\varphi[S_2]$ is separable, there exists a countable subset B'' of $\varphi[S_2]$ such that $\varphi[S_2] \subseteq St(B'', \mathcal{U})$. Consequently, we can show that $St(B' \cup B'', \mathcal{U}) = X$, and which shows that X is star-Lindelöf. □

Theorem 3.2. *An open F_δ-subset of an $\mathcal{L}$-starcompact space is $\mathcal{L}$-starcompact.*

Proof. Let X be an $\mathcal{L}$-starcompact space and let $Y = \cup\{H_n : n \in \omega\}$ be an open F_δ-subset of X, where the set H_n is closed in X for each $n \in \omega$. To show that Y is $\mathcal{L}$-starcompact, let $\mathcal{U}$ be an open cover of Y. we have to find a Lindelöf subset L of Y such that $St(L, \mathcal{U}) = Y$. For each $n \in \omega$, consider the open cover

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

of X. Since X is $\mathcal{L}$-starcompact, there exists a Lindelöf subset L_n of X such that $St(L_n, \mathcal{U}_n) = X$. Let $M_n = L_n \cap Y$. Since Y is a F_δ-set, M_n is Lindelöf, and clearly $H_n \subseteq St(M_n, \mathcal{U})$. Thus, if we put $L = \cup\{M_n : n \in \omega\}$, then L is a Lindelöf subset of Y and $St(L, \mathcal{U}) = Y$. Hence, Y is $\mathcal{L}$-starcompact. □

A *cozero-set* in a space X is a set of the form $f^{-1}(R \setminus \{0\})$ for some real-valued continuous function f on X. Since a cozero-set is an open F_σ-set, we have the following corollary:

Corollary 3.3. *A cozero-set of an $\mathcal{L}$-starcompact space is $\mathcal{L}$-starcompact.*

Let τ be an infinite cardinal. Recall that a space X is *Lindelöf-τ-bounded* if every subset of X of cardinality $\leq \tau$ is contained in a Lindelöf subset of X ([6]).

Theorem 3.4. *Every Lindelöf-ω_1-bounded space is star-Lindelöf.*

Proof. Let X be a Lindelöf-ω_1-bounded space . Suppose that X is not star-Lindelöf. Then, there exists an open cover $\mathcal{U}$ of X such that $St(B,\mathcal{U}) \neq X$ for every countable subset B of X. By induction, we can define a sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of X such that

$$x_\alpha \notin St(\{x_\beta : \beta < \alpha\},\mathcal{U}) \text{ for each } \alpha < \omega_1.$$

Since X be Lindelöf-ω_1-bounded, the set $\{x_\alpha : \alpha < \omega_1\}$ is contained in a Lindelöf subspace $L \subseteq X$. Thus, there exists a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ which covers L. Then at least one element of $\mathcal{V}$ contains uncountably many points x_α, which is a contradiction to the definition of the sequence $\{x_\alpha : \alpha < \omega_1\}$. Hence, X is star-Lindelöf. □

For a space X, let $l(X)$ be the *Lindelöf number* of X, i.e., the smallest cardinal λ such that every open cover of X has an open refinement $\mathcal{V}$ with $|\mathcal{V}| \leq \lambda$.

Theorem 3.5. *Let $\tau \geq \omega_1$. Let $X = Y \cup Z$, where Y is dense in X, Y is Lindelöf-τ-bounded and $l(Z) \leq \tau$. Then, X is $\mathcal{L}$-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since Y is Lindelöf-τ-bounded, from Theorem 3.4, there exists a countable subset B of Y such that $Y \subseteq St(B,\mathcal{U})$. So it remains to find a Lindelöf subset $L' \subseteq Y$ such that $Z \subseteq St(L',\mathcal{U})$. Since $l(Z) \leq \tau$, there is a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \tau$ and $Z \subseteq \cup\mathcal{V}$. Pick $x_V \in V \cap Y$ for each $V \in \mathcal{V}$. Since Y is Lindelöf-τ-bounded, the subset $\{x_V : V \in \mathcal{V}\}$ of Y is included in some Lindelöf subspace $L' \subseteq Y$. Hence, $Z \subseteq St(L',\mathcal{U})$. Let $L = L' \cup B$. Then, L is a Lindelöf subspace of X and $X = St(L,\mathcal{U})$, which completes the proof. □

In [3], Hiremath proved that a continuous image of an $\mathcal{L}$-startcompact space is $\mathcal{L}$-startcompact. By contrast, he also showed a perfect preimage of an $\mathcal{L}$-startcompact space need not be $\mathcal{L}$-startcompact. Now we give a positive result:

Theorem 3.6. *Let f be an open perfect map from a space X to an $\mathcal{L}$-starcompact space Y. Then, X is $\mathcal{L}$-starcompact.*

Proof. Since $f[X]$ is open and closed in Y, we may assume that $f[X] = Y$. Let $\mathcal{U}$ be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \cup\mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Pick an open neighbourhood V_y of y in Y such that $f^{-1}[V_y] \subseteq \cup\{U : U \in \mathcal{U}_y\}$, and we can assume that

$$V_y \subseteq \cap\{f[U] : U \in \mathcal{U}_y\}, \tag{1}$$

because f is open. Taking such open set V_y for each $y \in Y$, we have an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y. Let L be a Lindelöf subset of the $\mathcal{L}$-starcompact space Y such that $St(L,\mathcal{V}) = Y$. Since f is perfect, the set $f^{-1}(L)$ is a Lindelöf subset of X. To show that $St(f^{-1}(L),\mathcal{V}) = X$, let $x \in X$. Then, there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap L \neq \emptyset$. Since

$$x \in f^{-1}[V_y] \subseteq \cup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f[U]$ by (1), and hence $U \cap f^{-1}[L] \neq \emptyset$. Therefore, $x \in St(f^{-1}[L],\mathcal{U})$. Consequently , we have that $St(f^{-1}(L),\mathcal{U}) = X$. □

Corollary 3.7. (*Hiremath* [3]) *Let X be an $\mathcal{L}$-starcompact space and Y a compact space. Then, $X \times Y$ is $\mathcal{L}$-starcompact.*

The following theorem is a generalization of Corollary 3.7.

Theorem 3.8. *Let X be an $\mathcal{L}$-starcompact space and Y a locally compact, Lindelöf space. Then, $X \times Y$ is $\mathcal{L}$-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $y \in Y$, there exists an open neighbourhood V_y of y in Y such that $\mathrm{cl}_Y V_y$ is compact. By the Corollary 3.7, the subspace $X \times \mathrm{cl}_Y V_y$ is $\mathcal{L}$-starcompact. Thus, there exists a Lindelöf subset $L_y \subseteq X \times \mathrm{cl}_Y V_y$ such that

$$X \times \mathrm{cl}_Y V_y \subseteq St(L_y, \mathcal{U}).$$

Since Y is Lindelöf, there exists a countable cover $\{V_{y_i} : i \in \omega\}$ of Y. Let $L = \cup\{L_{y_i} : i \in \omega\}$. Then, L is a Lindelöf subset of $X \times Y$ such that $St(L, \mathcal{U}) = X \times Y$. Hence, $X \times Y$ is $\mathcal{L}$-starcompact. □

Hiremath [3] showed that the product of two Lindelöf spaces need not be $\mathcal{L}$-starcompact. In [1, Example 3.3.3], van Douwen-Reed-Roscoe-Tree also gave an example of a countably compact (and hence, starcompact) space X and a Lindelöf space Y such that $X \times Y$ is not star-Lindelöf. Now, we shall show that the product $X \times Y$ is not $\mathcal{L}$-starcompact:

Example 3.3.9. *There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not $\mathcal{L}$-starcompact.*

Proof. Let $X = \omega_1$ with the usual order topology. $Y = \omega_1 + 1$ with the following topology. Each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then, X is countably compact and Y is Lindelöf. Now, we show that $X \times Y$ is not $\mathcal{L}$-starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$, and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$$

of $X \times Y$ and let L be a Lindelöf subset of $X \times Y$. Then, $\pi_X[L]$ is a Lindelöf subset of X, where $\pi_X : X \times Y \to X$ is the projection. Thus, there exists $\beta < \omega_1$ such that $L \cap ((\beta, \omega_1) \times Y) = \emptyset$. Pick α with $\alpha > \beta$. Then, $\langle \alpha, \beta \rangle \notin St(L, \mathcal{U})$ since V_β is the only element of $\mathcal{U}$ containing $\langle \alpha, \beta \rangle$. Hence, $X \times Y$ is not $\mathcal{L}$-starcompact. which completes the proof. □

Remark. In [4, Example 2], Ikenaga gave an example of a Lindelöf space X and a separable space Y such that $X \times Y$ is not star-Lindelöf. By contrast, as far as the author knows, it is open whether the product of an $\mathcal{L}$-starcompact space and a separable space is $\mathcal{L}$-starcompact.

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