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On shape theory and its applications

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1 Introduction

Shape theory is a homotopy theory for general topological spaces and has been proved to be very effective especially for spaces that have bad local behavior, but the process to build up the theory itself is important in various areas of mathematics. The most userful tool in the shape theory is an inverse system, and in this approach "bad" objects are represented as an inverse system of "good" objects. Using inverse systems allows us to work categorically and hence provides a systematic and user friendly approach to crack the "bad" objects.

Based on the inverse system approach, in this paper we present applications of shape theory to various areas of geometric topology. Since shape theory deals with general topological spaces, the significant differences from the usual homotopy theory are the possibility of more applications and the possibility that things that are not possible in the homotopy category of polyhedra may become possible if the category is extended from the homotopy category of polyhedra to the shape category. More precisely, in the first part of the paper we present a generalization over shape category of the well-known result Kan-Thurston theorem in algebraic topology and as an application generalize the well-known theorems of Dranishnikov [4] and Edwards [1] in cohomological dimension. In the second part we introduce the generalized stable shape theory and a duality in that category and as an application present a Vietoris-Begle theorem for pro-homology groups that are induced by CW spectra.

2 Kan-Thurston theorem in shape theory

All spaces in this section are assumed to have base points. First recall

Theorem 2.1 (Kan and Thurston [9]) *For each path-connected space X, there exist a space TX and a map $t : TX \to X$, natural for maps on X, with the following properties:*

(KT1) $t_* : H_*(TX; t^*A) \to H_*(X; A)$ *and* $t^* : H^*(X; A) \to H^*(TX; t^*A)$ *are isomorphisms of singular homologies and cohomologies with local coefficients; and*

(KT2) $t_* : \pi_1(TX) \to \pi_1(X)$ *is onto, and* $\pi_i(TX) \cong 0$ *for* $i \neq 0$.

Maunder gave a simpler proof to the theorem and obtained the following variation:

Theorem 2.2 (Maunder [12]) *For each finite connected simplicial complex K, there exist a finite simplicial complex TK of the same dimension, and a map $t_K : TK \to K$, natural for simplicial maps on K, with properties (KT1) and (KT2).*

A compactum X is said to be *approximately aspherical* if every map of X into a polyhedron factors up to homotopy through a finite aspherical CW complex. Note that our definition is slightly stronger than the original definition of shape asphericity of Dydak and Yokoi [7] by requiring the finiteness of the factoring CW complex. Asphericity of compacta in the study of cell-like maps was first considered by Daverman [2] and continued by Daverman and Dranishnikov [3]. The following is a characterization of an approximately aspherical compactum:

Theorem 2.3 *For every compactum X, the following are equivalent:*
i) *X is an approximately aspherical compactum;*
ii) *X admits an expansion of X, $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ such that each X_i is a fintie aspherical polyhedron (here, the expansion is in the sense of [11, p. 19]) ; and*
iii) *Every polyhedral expansion of X, $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ has the property that every i admits $i' \geq i$ such that $p_{ii'}$ factors through a finite aspherical polyhedron.*

The following is the Kan-Thurston theorem in the shape theory:

Theorem 2.4 (Miyata [14]) *For each continuum X (resp., continuum with $\dim X < \infty$), there exist an approximately aspherical compactum Y (resp., approximately aspherical compactum Y with $\dim Y = \dim X$) and a surjective map $\varphi : Y \to X$ with the following properties:*

(S1) *φ induces isomorphisms of Čech homologies and cohomologies;*

(S2) $\varphi_* : \text{pro-}\pi_1(Y) \to \text{pro-}\pi_1(X)$ *is an epimorphism; and*

(S3) *For each connected closed subset A of X, $\varphi^{-1}(A)$ is an aproximately aspherical compactum, and $\varphi|\varphi^{-1}(A) : \varphi^{-1}(A) \to A$ satisfies properties (S1) and (S2).*

For any compactum X let $\mathrm{sd}\, X$ denote the shape dimension of X. There is another version of the Kan-Thurston theorem in the shape theory:

Theorem 2.5 (Miyata [14]) *For each continuum X of $\mathrm{sd}\, X < \infty$, there exist an approximately aspherical compactum Y of $\dim Y = \mathrm{sd}\, X$ and a shape morphism $\varphi : Y \to X$ with properties (S1) and (S2).*

The following is the Kan-Thurston theorem in the generalized stable shape theory:

Theorem 2.6 (Miyata [14]) i) *Every continuum has the weak stable shape type of an approximately aspherical compactum.*
ii) *Every continuum X of $\mathrm{sd}\, X < \infty$ has the stable shape type of an approximately aspherical compactum Y of $\dim Y = \mathrm{sd}\, X$.*

3 An application of Kan-Thurston theorem to cohomological dimension

For each compactum X and abelian group G, the *cohomological dimension* $\mathrm{cdim}_G\, X \leq n$ if $X \tau K(G, n)$, where for any ANR P, $X \tau P$ denotes the property that every map of any closed subset of X into P extends over X. For each compactum X, $\dim X$ denotes the covering dimension of X. Recall the following well-known results:

Theorem 3.1 (Edwards [1, 19]) *For each compactum X, $\mathrm{cdim}_{\mathbb{Z}}\, X \leq n$ if and only if there exists a cell-like map $f : Y \to X$ from a compactum Y of $\dim Y \leq n$.*

and

Theorem 3.2 (Dranishnikov [4]) *For each compactum X and for each prime p, $\mathrm{cdim}_{\mathbb{Z}/p}\, X \leq n$ if and only if there exists a surjective map $f : Y \to X$ from a compactum Y of $\dim Y \leq n$ such that each fibre is acyclic modulo p.*

A question rises: Can we choose a more specific compactum for Y in each of the above theorems? Using the notion of approximately asphericity, we can generalize those theorems as follows:

Theorem 3.3 (Miyata[14]) *For each continuum X and for each prime p,* $\mathrm{cdim}_{\mathbb{Z}} X \le n$ *(resp.,* $\mathrm{cdim}_{\mathbb{Z}/p} X \le n$ *) if and only if there exist an approximately aspherical compactum Y with* $\dim Y \le n$ *and a surjective map $f : Y \to X$ such that each fibre is acyclic (resp., acyclic modulo p).*

4 Generalized stable shape and duality

In this section, we briefly recall the construction of the generalized stable shape theory and present a duality in this category. For more details, see [15, 16]. All spaces in this section are assumed to have base points. Let **HCW** denote the homotopy category of CW complexes, and let $\mathbf{HCW}_{\mathrm{spec}}$ denote the homotopy category of CW spectra.

Let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an **HCW**-expansion of a space X in the sense of [11], and let $E(\boldsymbol{X}) = (E(X_\lambda), E(p_{\lambda\lambda'}), \Lambda)$ be the inverse system in $\mathbf{HCW}_{\mathrm{spec}}$ induced by the inverse system $\boldsymbol{X}$ in **HCW**. A morphism $\boldsymbol{e} : E(\boldsymbol{X}) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$ is said to be a *generalized expansion* of X in $\mathbf{HCW}_{\mathrm{spec}}$ provided whenever $\boldsymbol{f} : E(\boldsymbol{X}) \to \boldsymbol{F}$ is a morphism in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$, then there exists a unique morphism $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$ such that $\boldsymbol{f} = \boldsymbol{g}\boldsymbol{e}$. For any two generalized expansions $\boldsymbol{e} : E(\boldsymbol{X}) \to \boldsymbol{E}$ and $\boldsymbol{e}' : E(\boldsymbol{X}) \to \boldsymbol{E}'$ in $\mathbf{HCW}_{\mathrm{spec}}$ there exists the natural isomorphism $\boldsymbol{i} : \boldsymbol{E} \to \boldsymbol{E}'$ in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$ such that $\boldsymbol{i}\boldsymbol{e} = \boldsymbol{e}'$.

We define the *generalized stable shape category* $\mathbf{Sh}_{\mathrm{spec}}$ for spaces as follows: Let $\mathrm{ob}\,\mathbf{Sh}_{\mathrm{spec}}$ be the set of all spaces and CW-spectra. For any $X, Y \in \mathrm{ob}\,\mathbf{Sh}_{\mathrm{spec}}$, let $\mathcal{E}_{(X,Y)}$ be the set of all morphisms $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$ where $\boldsymbol{E}$ is either a rudimentary system (X) (if X is a CW-spectrum) or an inverse system of CW-spectra such that $\boldsymbol{e} : E(\boldsymbol{X}) \to \boldsymbol{E}$ is a generalized expansion of X in $\mathbf{HCW}_{\mathrm{spec}}$ (if X is a space), and similarly for $\boldsymbol{F}$. We define an equivalence relation $\sim$ on $\mathcal{E}_{(X,Y)}$ as follows: for $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ and $\boldsymbol{g}' : \boldsymbol{E}' \to \boldsymbol{F}'$ in $\mathcal{E}_{(X,Y)}$, $\boldsymbol{g} \sim \boldsymbol{g}'$ if and only if $\boldsymbol{j}\boldsymbol{g} = \boldsymbol{g}'\boldsymbol{i}$ in $\mathbf{pro\text{-}HCW}_{\mathrm{spec}}$, where $\boldsymbol{i} : \boldsymbol{E} \to \boldsymbol{E}'$ and $\boldsymbol{j} : \boldsymbol{F} \to \boldsymbol{F}'$ are the natural isomorphisms. We then define $\mathbf{Sh}_{\mathrm{spec}}(X, Y) = \mathcal{E}_{(X,Y)}/\sim$. The stable shape category for compact spaces defined by

Henn [8] is embedded in $\mathbf{Sh}_{\mathsf{spec}}$. There is also a functor from the shape category $\mathbf{Sh}$ to $\mathbf{Sh}_{\mathsf{spec}}$. Then, for any spaces X and Y, if $\Sigma^k X$ and $\Sigma^k Y$ are equivalent in $\mathbf{Sh}$ for some $k \geq 0$ then X and Y are equivalent in $\mathbf{Sh}_{\mathsf{spec}}$. The converse holds for any compact Hausdorff spaces X and Y with finite shape dimension.

For each CW-spectrum E, let E_* and E^* denote the homology and cohomology theories on $\mathbf{HCW}_{\mathsf{spec}}$ associated with E, respectively. So, for each CW-spectrum X and for each $q \in \mathbb{Z}$, $E_q(X) = [\Sigma^q S^0, E \wedge X]$ and $E^q(X) = [X, \Sigma^q E]$. Then we can define the covariant and contravariant functors $E_* : \mathbf{HCW}_{\mathsf{spec}} \to \mathbf{Ab}_*$ and $E^* : \mathbf{HCW}_{\mathsf{spec}} \to \mathbf{Ab}_*$, where $\mathbf{Ab}_*$ is the category of graded abelian groups and homomorphisms. These functors naturally extend to the functors pro-$E_* : \mathbf{Sh}_{\mathsf{spec}} \to$ **pro-Ab**$_*$ and pro-$E^* : \mathbf{Sh}_{\mathsf{spec}} \to$ **pro-Ab**$_*$, and, taking limits, the functors $\check{E}_* : \mathbf{Sh}_{\mathsf{spec}} \to \mathbf{Ab}_*$ and $\check{E}^* : \mathbf{Sh}_{\mathsf{spec}} \to \mathbf{Ab}_*$.

We have a duality in $\mathbf{Sh}_{\mathsf{spec}}$ as follows:

Theorem 4.1 (Miyata [13]) i) *For each compactum X, there exist a CW-spectrum X^* and a natural isomorphism*

$$\tau : \mathbf{Sh}_{\mathsf{spec}}(Y \wedge X, E) \to \mathbf{Sh}_{\mathsf{spec}}(Y, X^* \wedge E)$$

for any compact Hausdorff space Y and CW-spectrum E. Moreover, such X^* is unique up to homotopy.

ii) *For each $\varphi \in \mathbf{Sh}_{\mathsf{spec}}(X, X')$ where X and X' are compact metric spaces , there exists a map $\varphi^* : X'^* \to X^*$ such that the following diagram commutes for any compact Hausdorff space Y and CW-spectrum E:*

$$\begin{array}{ccc} \mathbf{Sh}_{\mathsf{spec}}(Y \wedge X, E) & \xrightarrow{\tau} & \mathbf{Sh}_{\mathsf{spec}}(Y, X^* \wedge E) \\ {\scriptstyle \mathbf{Sh}_{\mathsf{spec}}(1_Y \wedge \varphi, 1_E)} \uparrow & & \uparrow {\scriptstyle \mathbf{Sh}_{\mathsf{spec}}(1_Y, \varphi^* \wedge 1_E)} \\ \mathbf{Sh}_{\mathsf{spec}}(Y \wedge X', E) & \xrightarrow{\tau} & \mathbf{Sh}_{\mathsf{spec}}(Y, X'^* \wedge E) \end{array}$$

Moreover, such φ^* is unique up to weak homotopy.

There is also a dual notion of the generalized stable shape, which is called the coshape, and the coshape category is denoted by $\mathbf{coSh}_{\mathsf{spec}}$. Then we have the following duality between $\mathbf{Sh}_{\mathsf{spec}}$ and $\mathbf{coSh}_{\mathsf{spec}}$:

Theorem 4.2 *For any compacta X and Y, there is an isomorphism*

$$D : \mathbf{Sh}_{\mathsf{spec}}(X, Y) \to \mathbf{coSh}_{\mathsf{spec}}(Y^*, X^*).$$

There are also dualities between homology and cohomology groups induced by CW spectra:

Theorem 4.3 *For each CW spectrum and for each compactum X, there exist natural isomorphisms*

$$\check{E}^n(X) \cong E_{-n}(X^*) \text{ and } \check{E}_n(X) \cong \hat{E}^{-n}(X^*).$$

5 An application of duality: Vietoris-Begle theorem

Using the duality in the previous section, we can prove a version of Vietoris-Begle theorem for pro-homology groups induced by CW spectra. First, let us recall the following versions of Vietoris-Begle theorem:

Theorem 5.1 ([18, p. 344]) *Let G be any abelian group, and let $f : X \to Y$ be a closed surjective map between paracompact Hausdorff spaces. Suppose $\tilde{\overline{H}}^q(f^{-1}(y); G) \cong 0$ for each $y \in Y$ and for each $q = 0, 1, \ldots, n$. Then the induced homomorphism $\overline{H}^q(f; G) : \overline{H}^q(Y; G) \to \overline{H}^q(X; G)$ is an isomorphism for each $q = 0, 1, \ldots, n$ and a monomorphism for $q = n + 1$. Here $\tilde{\overline{H}}^*$ denotes the reduced Alexander cohomology theory.*

Theorem 5.2 (Volovikov and Ngyen [20]) *Let G be any abelian group, and let $f : X \to Y$ be a surjective map between compacta. Suppose $\tilde{\overline{H}}_q(f^{-1}(y); G) \cong 0$ for each $y \in Y$ and for each $q = 0, 1, \ldots, n$. Then the induced homomorphism $\overline{H}_q(f; G) : \overline{H}_q(X; G) \to \overline{H}_q(Y; G)$ is an isomorphism for each $q = 0, 1, \ldots, n$ and an epimorphism for $q = n+1$.*

Theorem 5.3 (Dydak [5]) *Let X and Y be compacta, let $f : X \to Y$ be a surjective map, and let R be a principal ideal domain. Suppose* pro-$\tilde{\overline{H}}_q(f^{-1}(y); R) \cong 0$ *for each $y \in Y$ and for each $q = 0, 1, \ldots, n$. Then the induced morphism* pro-$\overline{H}_q(f; R)$: pro-$\overline{H}_q(X; R) \to$ pro-$\overline{H}_q(Y; R)$ *is an isomorphism for each $q = 0, 1, \ldots, n$ and an epimorphism for $q = n + 1$.*

Generalized versions of Theorems 5.1 can be found in Lawson [10] and Dydak [5], and those of Theorem 5.2 in Dydak [5].

For the rest of this section, all spaces are regarded as pointed spaces with distinct base point $+$.

Theorem 5.4 (Dydak and Kozlowski [6]) *Let E be a CW spectrum, and let $f : X \to Y$ be a closed surjective map between paracompact Hausdorff spaces such that* $\operatorname{Ind} Y = m < \infty$. *If $f|f^{-1}(y) : f^{-1}(y) \to \{y\}$ induces an isomorphism $\check{E}^k(y) \to \check{E}^k(f^{-1}(y))$ for each $y \in Y$ and $k = m_0, m_0{+}1, \ldots, m_0{+}m$, then $\check{E}^k(f) : \check{E}^k(Y) \to \check{E}^k(X)$ is an isomorphism for $k = m_0 + m$ and a monomorphism for $k = m_0 + m + 1$. Here* $\operatorname{Ind} Y$ *denotes the large inductive dimension of Y.*

As an application of the duality in the generalized stable shape, we have the following form of Vietoris-Begle theorem:

Theorem 5.5 (Miyata and Watanabe [17]) *Let E be a ring spectrum, and let $f : X \to Y$ be a surjective map from a compact metric space X to a compact metric space Y with a finite covering dimension such that for each $y \in Y$, $f^{-1}(y)$ has a finite stable shape dimension. If $f|f^{-1}(y) : f^{-1}(y) \to \{y\}$ induces an isomorphism* $\text{pro-}E_*(f^{-1}(y)) \to \text{pro-}E_*(y)$ *for each $y \in Y$, then the induced morphism* $\text{pro-}E_*(f) : \text{pro-}E_*(X) \to \text{pro-}E_*(Y)$ *is an isomorphsim.*

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