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Sufficient conditions for Carathéodory functions

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Abstract. For Carathéodory functions $p(z)$ which are analytic in the open unit disk U with $p(0)=1$, S.S.Miller(Bull.Amer.Math.Soc.81(1975),79-81) has shown some sufficient conditions applying the differential inequalities.The object of the present paper is to derive some improvements of results by S.S.Miller.

1 Introduction

Let A be the class of functions $p(z)$ of the form

$$p(z)=1+p_1z+p_2z^2+\cdots \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. If $p(z)$ in A satisfies $\operatorname{Re}p(z)>0$ for $z\in U$, then we say that $p(z)$ is the Carathéodory function. For Carathéodory functions, Miller [1] has given

Theorem A. *Let $p(z)$ be in the class A.*

(i) *If* $\operatorname{Re}\{p(z)^2+zp'(z)\}>0 \quad (z\in U)$, *then* $\operatorname{Re}p(z)>0 \quad (z\in U)$.

(ii) *If* $\operatorname{Re}\{p(z)+\alpha zp'(z)\}>0 \quad (z\in U)$ *for some* $\alpha \quad (\alpha\geqq 0)$, *then* $\operatorname{Re}p(z)>0$ $(z\in U)$,

(iii) *If* $p(z)\neq 0 \quad (z\in U)$ *and* $\operatorname{Re}\left\{p(z)-\dfrac{zp'(z)}{p(z)^2}\right\}>0 \quad (z\in U)$, *then* $\operatorname{Re}p(z)>0 \quad (z\in U)$.

Let $f(z)$ and $g(z)$ be analytic in U. If there exists an analytic function $w(z)$ with $w(0)=0$ and $|w(z)|<1 \quad (z\in U)$ such that $f(z)=g(w(z))$, then $f(z)$ is said to be subordinate to $g(z)$ in U.

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We denote this subordination by $f(z) \prec g(z)$.We note that the subordination $f(z) \prec g(z)$ implies that $f(U) \subset g(U)$. Applying the subordination principles, we improve Theorem A by Miller [1]. To prove our results for Carathéodory functions, we have to recall here the following lemma due to Nunokawa [3] (also due to Miller and Mocanu [2]).

Lemma. *Let $p(z) \in A$ and suppose that there exists a point $z_0 \in U$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$ with $p(z_0) \neq 0$. Then we have*

$$z_0 p'(z_0) \leqq -\frac{1}{2}(1 + a^2), \tag{1.2}$$

where $p(z_0) = ia \quad (a \neq 0)$.

2 Subordination theorems for Carathéodory functions

Our first result for Carathéodory functions is contained in

Theorem 1. *Let $p(z) \in A$ and $w(z)$ be analytic in U with $w(0) = \alpha$ and $w(z) \neq k \quad (k \in \mathbb{R}, z \in U)$. If*

$$\alpha p(z)^2 + \beta z p'(z) \prec w(z), \tag{2.1}$$

then $\operatorname{Re} p(z) > 0 \quad (z \in U)$, where $\beta > 0$, $\alpha \geqq -\frac{\beta}{2}$, and $k \leqq -\frac{\beta}{2}$.

Proof. Let us suppose that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad for \quad |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0 \quad (p(z_0) \neq 0).$$

Then Lemma gives that $p(z_0) = ia \quad (a \neq 0)$ and $z_0 p'(z_0) \leqq -\frac{1}{2}(1 + a^2)$. It follows that

$$\begin{aligned} \alpha p(z_0)^2 + \beta z_0 p'(z_0) &= -\alpha a^2 + \beta z_0 p'(z_0) \\ &\leqq -\frac{1}{2}\left\{\beta + (2\alpha + \beta)a^2\right\} \\ &\leqq -\frac{\beta}{2}. \end{aligned} \tag{2.2}$$

Since $w(0) = \alpha$ and $w(e^{i\theta}) \leqq -\frac{\beta}{2}$, the inequality (2.2) contradicts our condition (2.1). Therefore $\operatorname{Re} p(z) > 0$ for all $z \in U$. □

Remark 1. Theorem 1 is the improvement of (i) of Theorem A by Miller [1].

Corollary 1. *If* $p(z) \in A$ *satisfies*

$$\alpha p(z)^2 + \beta z p'(z) \prec \frac{2\alpha+\beta}{2}\left(\frac{1+z}{1-z}\right)^2 - \frac{\beta}{2}, \tag{2.3}$$

where $\beta > 0$ *and* $\alpha \geqq -\frac{\beta}{2}$, *then* $\operatorname{Re} p(z) > 0$ $(z \in U)$.

Proof. Taking

$$w(z) = \frac{2\alpha+\beta}{2}\left(\frac{1+z}{1-z}\right)^2 - \frac{\beta}{2} \tag{2.4}$$

in Theorem 1, we see that $w(z)$ is analytic in U, $w(0) = \alpha$ and

$$w(e^{i\theta}) = \frac{2\alpha+\beta}{2}\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 - \frac{\beta}{2} \leqq -\frac{\beta}{2}. \tag{2.5}$$

Thus $w(z)$ satisfies the conditions in Theorem 1. □

Theorem 2. *Let* $p(z) \in A$ *and* $w(z)$ *be analytic in* U *with* $w(0) = \alpha$ *and* $w(z) \neq ik$ $(k \in \mathbb{R}, z \in U)$. *If*

$$\alpha p(z) + \beta \frac{zp'(z)}{p(z)} \prec w(z), \tag{2.6}$$

then $\operatorname{Re} p(z) > 0$ $(z \in U)$, *where* $\alpha > 0$, $\beta > 0$, *and* $k^2 \geqq \beta(2\alpha+\beta)$.

Proof. From the subordination (2.6), we have $p(z) \neq 0$ in U, because if $p(z)$ has a zero of order l at $z = z_0 \in U$, then we have $p(z) = (z - z_0)^l q(z)$, where $q(z)$ is analytic in U, $q(z_0) \neq 0$, and l is a positive integer.
Letting $z \to z_0$ such that

$$\arg(z - z_0) = \arg(z_0) - \frac{\pi}{2},$$

we have

$$\lim_{z \to z_0} \operatorname{Im}\left(\alpha p(z) + \beta \frac{zp'(z)}{p(z)}\right) = \lim_{z \to z_0} \operatorname{Im}\left(\alpha p(z) + \frac{\beta z (l q(z) + (z - z_0) q'(z))}{(z - z_0) q(z)}\right)$$

$$= +\infty.$$

This contradicts (2.6) and so we conclude that $p(z) \neq 0$ for all $z \in U$.
We assume that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad for \quad |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0.$$

Then using Lemma, we have

$$\begin{aligned} \alpha p(z_0) + \beta \frac{z_0 p'(z_0)}{p(z_0)} &= i\alpha a + \frac{\beta}{ia} z_0 p'(z_0) \\ &= i\left(\alpha a - \frac{\beta}{a} z_0 p'(z_0)\right) \\ &= iv, \end{aligned} \tag{2.7}$$

where v is real, because $z_0 p'(z_0) \leqq -\frac{1}{2}(1+a^2)$. Furthermore, we have, if $a > 0$, then

$$\begin{aligned} v &\geqq \alpha a + \frac{\beta}{2a}(1+a^2) \\ &\geqq \sqrt{\beta(2\alpha+\beta)}, \end{aligned} \tag{2.8}$$

and if $a < 0$, then

$$\begin{aligned} v &\leqq -\alpha b - \frac{\beta}{2b}(1+a^2) \quad (b = -a > 0) \\ &\leqq -\sqrt{\beta(2\alpha+\beta)}. \end{aligned} \tag{2.9}$$

This contradicts our condition that $w(e^{i\theta}) = ik$ $(|k| \geqq \sqrt{\beta(2\alpha+\beta)})$.Thus we conclude that $\operatorname{Re} p(z) > 0$ for all $z \in U$. □

Using Theorem 2, we have the following corollary.

Corollary 2. *If* $p(z) \in A$ *satisfies*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+4z+z^2}{1-z^2}, \tag{2.10}$$

then $\operatorname{Re} p(z) > 0$ $(z \in U)$.

Proof. Let us consider the case of $\alpha = \beta = 1$ in Theorem 2. Defining the function $w(z)$ by

$$w(z) = \frac{1+4z+z^2}{1-z^2}, \tag{2.11}$$

we know that $w(z)$ is analytic in U, $w(0)=1$, and

$$w(e^{i\theta}) = \frac{2+\cos\theta}{\sin\theta} i. \tag{2.12}$$

Letting

$$g(\theta) = \left(\frac{2+\cos\theta}{\sin\theta}\right)^2 \quad (0 \leqq \theta \leqq 2\pi), \tag{2.13}$$

we have $g'(\theta) = 0$ when $\cos\theta = -\frac{1}{2}$.

If follows from the above that $g(\theta) \geqq 3$, that is, that $w(z) \neq ik$ $(|k| \geqq \sqrt{3})$. □

Next, we derive

Theorem 3. *If $p(z) \in A$ satisfies*

$$\mathrm{Re}\left\{\alpha p(z) - \beta\frac{zp'(z)}{p(z)^2}\right\} > -\frac{\beta}{2} \quad (z \in U) \tag{2.14}$$

for some $\alpha \geqq 0$ and $\beta > 0$, then $\mathrm{Re}\, p(z) > 0 \quad (z \in U)$.

Proof. Applying the same method as the proof of Theorem 2, the condition (2.14) gives us that $p(z) \neq 0$ in U, because if $p(z)$ has a zero of order l at a point $z = z_0 \in U$, then we have $p(z) = (z-z_0)^l q(z)$, where $q(z)$ is analytic in U, $q(z_0) \neq 0$ and l is a positive integer. Letting $z \to z_0$ such that

$$\arg(z - z_0) = \frac{\arg(z_0) - \arg(q(z_0))}{l+1},$$

we see that

$$\lim_{z\to z_0}\left(\alpha p(z) - \beta\frac{zp'(z)}{p(z)^2}\right) = \lim_{z\to z_0}\left(\alpha p(z) - \beta\frac{lzq(z) + (z-z_0)zq'(z)}{(z-z_0)^{l+1}q(z)^2}\right)$$
$$= -\infty.$$

This contradicts our condition (2.14) and so we have $p(z) \neq 0$ in U.
By means of Lemma, if there exists a point $z_0 \in U$ such that

$$\mathrm{Re}\, p(z) > 0 \quad for \quad |z| < |z_0|$$

and

$$\mathrm{Re}\, p(z_0) = 0,$$

then $p(z_0) = ia \quad (a \neq 0)$ and $z_0 p'(z_0) \leqq -\frac{1}{2}(1+a^2)$.

This implies that

$$\mathrm{Re}\left\{\alpha p(z_0) - \beta\frac{z_0 p'(z_0)}{p(z_0)^2}\right\} \leqq -\frac{\beta}{2a^2}(1+a^2) \leqq -\frac{\beta}{2} \tag{2.15}$$

which contradicts our condition (2.14). Thus $\mathrm{Re}\, p(z) > 0$ for all $z \in U$. □

Remark 2. Theorem 3 is the improvement of (iii) of Theorem A by Miller [1].

Finally we have

Corollary 3. *If $p(z) \in A$ satisfies*

$$\alpha p(z) - \beta \frac{zp'(z)}{p(z)^2} \prec \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2} \tag{2.16}$$

for some $\alpha \geqq 0$ and $\beta > 0$, then $\mathrm{Re}\, p(z) > 0$ $(z \in U)$.

Proof. Since the function

$$w(z) = \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2} \tag{2.17}$$

maps the open unit disk U onto the complex domain which has the slit

$$\delta = \left\{ w : \mathrm{Re}\,(w) < -\frac{\beta}{2} \right\},$$

the proof of Corollary 3 follows from the above. □

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