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Fuzzy Stopping in Continuous-Time Systems with Randomness and Fuzziness

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1. Introduction

This paper extends fuzzy stopping times in the discrete-time models to continuous-time ones, and presents a fuzzy stopping model in a continuous-time 'fuzzy stochastic systems' which is constructed from fuzzy random variables. In Section 2, the notations and definitions of fuzzy random variables are given and a continuous-time fuzzy stochastic system is formulated. Next, in Section 3, fuzzy stopping times are introduced for continuous-time fuzzy stochastic systems, and a stopping model by stopping stopping times is presented. In Section 4, in associated stopping model for fuzzy stochastic systems, an optimal fuzzy stopping time is constructed under a regularity assumption regading stopping rules. In Section 5, it is shown that the optimal fuzzy reward is a unique solution of an optimality equation under a differentiability condition.

2. Fuzzy stochastic systems

First, we introduce some notations of fuzzy random variables. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. Let $\mathbb{R}$ be the set of all real numbers. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

We use a metric δ_∞ on $\mathcal{R}$ defined by

$$\delta_\infty(\tilde{a}, \tilde{b}) := \sup_{\alpha \in [0,1]} \delta(\tilde{a}_\alpha, \tilde{b}_\alpha) \quad \text{for } \tilde{a}, \tilde{b} \in \mathcal{R}, \tag{2.1}$$

where δ is the Hausdorff metric on $\mathbb{R}$. A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if

$$\{(\omega, x) \in \Omega \times \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1], \tag{2.2}$$

where $\mathcal{B}$ is the Borel σ-field of $\mathbb{R}$. We can find some equivalent conditions in general cases ([4]), however, in this paper, we adopt a simple equivalent condition in the following lemma.

Lemma 2.1 (Wang and Zhang [7, Theorems 2.1 and 2.2]). *For a map $\tilde{X} : \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:*

(i) *$\tilde{X}$ is a fuzzy random variable.*

(ii) *The maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$.*

Now we introduce expectations of fuzzy random variables for the description of stopping models in fuzzy stochastic systems. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$. For an integrably bounded fuzzy random variables $\tilde{X}$, we put closed intervals

$$E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)\right], \quad \alpha \in [0,1]. \tag{2.3}$$

Then, the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number ([2, Lemma 3],[8]):

$$E(\tilde{X})(x) := \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\right\} \quad \text{for } x \in \mathbb{R}, \tag{2.4}$$

where 1_D is the classical indicator function of a set D.

Next, we formulate fuzzy stochastic systems. Let $[0,\infty)$ be the time space, and let $\{\tilde{X}_t\}_{t\geq 0}$ be a process of integrably bounded fuzzy random variables such that $E(\sup_{t\geq 0} \tilde{X}_{t,0}^+) < \infty$, where $\tilde{X}_{t,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$ for $t \geq 0$. We assume that the map $t \mapsto \tilde{X}_t(\omega) (\in \mathcal{R})$ is continuous on $[0,\infty)$ for almost all $\omega \in \Omega$. $\{\mathcal{M}_t\}_{t\geq 0}$ is a family of nondecreasing sub-σ-fields of $\mathcal{M}$ which is right continuous, i.e. $\mathcal{M}_t = \bigcap_{r:r>t} \mathcal{M}_r$ for all $t \geq 0$, and fuzzy random variables $\tilde{X}_t$ are $\mathcal{M}_t$-adapted, i.e. random variables $\tilde{X}_{r,\alpha}^-$ and $\tilde{X}_{r,\alpha}^+$ $(0 \leq r \leq t; \alpha \in [0,1])$ are $\mathcal{M}_t$-measurable. And $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{t\geq 0} \mathcal{M}_t$. Then $(\tilde{X}_t, \mathcal{M}_t)_{t\geq 0}$ is called a continuous-time 'fuzzy stochastic system'. A map $\tau : \Omega \mapsto [0,\infty]$ is said to be a stopping time if

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{M}_t \quad \text{for all } t \geq 0. \tag{2.5}$$

Then we have the following lemma.

Lemma 2.2. *Let τ be a finite stopping time. Define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_{\tau(\omega)}(\omega) \quad \textit{for } \omega \in \Omega. \tag{2.6}$$

Then, $\tilde{X}_\tau$ is a fuzzy random variable.

3. A fuzzy stopping model

In this section, we introduce a 'fuzzy stopping time' in accordance with the continuous-time fuzzy stochastic system $(\tilde{X}_t, \mathcal{M}_t)_{t\geq 0}$ defined in Section 2, and we discuss a stopping problem by using fuzzy stopping times. Let $\mathcal{I}$ be the set of all bounded closed sub-intervals of $\mathbb{R}$ and let $g : \mathcal{I} \mapsto \mathbb{R}$ be a continuous σ-additively homogeneous map, that is, g satisfies (3.1) and (3.2):

$$g\left(\sum_{n=0}^{\infty} c_n\right) = \sum_{n=0}^{\infty} g(c_n) \tag{3.1}$$

for bounded closed intervals $\{c_n\}_{n=0}^{\infty} \subset \mathcal{I}$ such that $\sum_{n=0}^{\infty} c_n \in \mathcal{I}$ and

$$g(\lambda c) = \lambda g(c) \tag{3.2}$$

for bounded closed intervals $c \in \mathcal{I}$ and real numbers $\lambda \geq 0$, where the operation on closed intervals is defined ordinary as $\sum_{n=0}^{\infty} c_n := \mathrm{cl}\{\sum_{n=0}^{\infty} x_n \mid x_n \in c_n, n = 0, 1, 2, \cdots\}$ and $\lambda c := \{\lambda x \mid x \in c\}$. We call this scalarization satisfying (3.1) and (3.2) a 'linear ranking function', and it is used for the evaluation of fuzzy numbers (Fortemps and Roubens [5]). Now we introduce an evaluation of the fuzzy random variable $\tilde{X}_\tau$ provided that τ is a finite stopping time. Let $\omega \in \Omega$. From (2.6), the α-cut of the fuzzy number $\tilde{X}_\tau(\omega)$ is a closed interval $\tilde{X}_{\tau(\omega),\alpha}(\omega)$, and the expectation is given by the closed interval $E(\tilde{X}_{\tau,\alpha})$ from the definition (2.3). Using the linear ranking function g, we estimate it by $g(E(\tilde{X}_{\tau,\alpha}))$. Therefore, the evaluation of the fuzzy random variable $\tilde{X}_\tau$ is given by the integral

$$\int_0^1 g(E(\tilde{X}_{\tau,\alpha}))\, d\alpha. \tag{3.3}$$

Then we have the following lemma regarding (3.3).

Lemma 3.1. *For a finite stopping time τ, it holds that*

$$\int_0^1 g(E(\tilde{X}_{\tau,\alpha}))\, d\alpha = \int_0^1 E(g(\tilde{X}_{\tau,\alpha}))\, d\alpha = E\left(\int_0^1 g(\tilde{X}_{\tau,\alpha}(\cdot))\, d\alpha\right). \tag{3.4}$$

Now we introduce fuzzy stopping times, which is a fuzzification of classical stopping times and is a continuous-time extension of fuzzy stopping times in [8].

Definition 3.1. A map $\tilde{\tau} : [0, \infty) \times \Omega \mapsto [0, 1]$ is called a fuzzy stopping time if it satisfies the following (i) – (iii):

(i) For each $t \geq 0$, the map $\omega \mapsto \tilde{\tau}(t, \omega)$ is $\mathcal{M}_t$-measurable.

(ii) For almost all $\omega \in \Omega$, the map $t \mapsto \tilde{\tau}(t, \omega)$ is non-increasing and right continuous and has left-hand limits on $[0, \infty)$.

(iii) For almost all $\omega \in \Omega$, there exists $t_0 \geq 0$ such that $\tilde{\tau}(t, \omega) = 0$ for all $t \geq t_0$.

Definition 3.1 is the similar idea to fuzzy stopping times given in dynamic fuzzy systems by Kurano et al. [3]. Regarding the membership grade of fuzzy stopping times, $\tilde{\tau}(t, \omega) = 0$ means 'to stop at time t' and $\tilde{\tau}(t, \omega) = 1$ means 'to continue at time t' respectively. We have the following lemma regarding the properties of fuzzy stopping times.

Lemma 3.2.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto [0, \infty)$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{t \geq 0 \mid \tilde{\tau}(t, \omega) < \alpha\}, \quad \omega \in \Omega \quad \text{for } \alpha \in (0, 1], \tag{3.5}$$

where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{\omega \mid \tilde{\tau}_\alpha(\omega) \leq t\} \in \mathcal{M}_t$ *for* $t \geq 0$;

(b) $\tilde{\tau}_\alpha(\omega) \leq \tilde{\tau}_{\alpha'}(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha \geq \alpha'$;

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ *a.a.* $\omega \in \Omega$.

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha \in [0,1]}$ be maps $\tilde{\tau}_\alpha : \Omega \mapsto [0, \infty)$ satisfying the above (a) (b) and (d). Define a map $\tilde{\tau} : [0, \infty) \times \Omega \mapsto [0, 1]$ by*

$$\tilde{\tau}(t, \omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\tau}_\alpha > t\}}(\omega)\} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{3.6}$$

Then $\tilde{\tau}$ is a fuzzy stopping time.

We consider the estimation of the fuzzy stochastic system stopped at a fuzzy stopping time $\tilde{\tau}$. Let $\omega \in \Omega$. A fuzzy stopping time $\tilde{\tau}$ is called finite if $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ for almost all $\omega \in \Omega$. Let $\tilde{\tau}$ be a finite fuzzy stopping time. From Lemma 3.2(i), its α-cut is $\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega) := \tilde{X}_{\tilde{\tau}_\alpha(\omega), \alpha}(\omega)$, where $\tilde{\tau}_\alpha(\omega)$ is a 'classical' stopping time given by (3.5). Therefore, from the evaluation method in (3.3), we define a random variable

$$G_{\tilde{\tau}}(\omega) := \int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega))\, d\alpha, \quad \omega \in \Omega. \tag{3.7}$$

The expectation $E(G_{\tilde{\tau}})$ is the evaluation of the fuzzy random variable $\tilde{X}_{\tilde{\tau}}$. In this paper, we discuss the following problem.

Problem 1. Find a fuzzy stopping time $\tilde{\tau}^*$ such that $E(G_{\tilde{\tau}^*}) \geq E(G_{\tilde{\tau}})$ for all fuzzy stopping times $\tilde{\tau}$.

In Problem 1, $\tilde{\tau}^*$ is called an 'optimal fuzzy stopping time'. By Lemma 3.1, we have

$$E(G_{\tilde{\tau}}) := E\left(\int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\cdot))\, d\alpha\right) = \int_0^1 E(g(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}))\, d\alpha \tag{3.8}$$

for fuzzy stopping times $\tilde{\tau}$. In order to analyze Problem 1, in the next section we need to discuss the following subproblem induced from (3.8).

Problem 2. Let $\alpha \in [0,1]$. Find a stopping time τ^* such that $E(g(\tilde{X}_{\tau^*,\alpha})) \geq E(g(\tilde{X}_{\tau,\alpha}))$ for all stopping times τ.

In Problem 2, τ^* is called an 'α-optimal stopping time'.

4. An optimal fuzzy stopping time

In this section, we consider a method to construct an optimal fuzzy stopping time. In order to characterize α-optimal stopping times, we let

$$U_t^\alpha := \operatorname*{ess\,sup}_{\tau:\text{ stopping times}, \tau \geq t} E(g(\tilde{X}_{\tau,\alpha})|\mathcal{M}_t) \quad \text{for } t \geq 0. \tag{4.1}$$

Then we have that U_t^α are right continuous with respect to $t \geq 0$ since $\tilde{X}_{t,\alpha}$ and $\mathcal{M}_t$ are right continuous with respect to $t \geq 0$ and g is continuous. We define a stopping time $\sigma_\alpha^* : \Omega \mapsto [0,\infty)$ by

$$\sigma_\alpha^*(\omega) := \inf \left\{ t \geq 0 \mid U_t^\alpha(\omega) = g(\tilde{X}_{t,\alpha}(\omega)) \right\} \tag{4.2}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the next theorem is obtained by the classical stopping problems ([1] and [6, Theorem 3 in Sect.3.3.3]).

Theorem 4.1. *Let $\alpha \in [0,1]$. If σ_α^* is finite almost surely, then σ_α^* is α-optimal and $E(U_0^\alpha) = E(g(\tilde{X}_{\sigma_\alpha^*,\alpha}))$.*

In order to construct an optimal fuzzy stopping time from the α-optimal stopping times $\{\sigma_\alpha^*\}_{\alpha\in[0,1]}$, we need the following regularity condition.

Assumption A (Regularity). The map $\alpha \mapsto \sigma_\alpha^*(\omega)$ is non-increasing for almost all $\omega \in \Omega$.

Under Assumption A, we can define a map $\tilde{\sigma}^* : [0,\infty) \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^*(t,\omega) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{\{\sigma_\alpha^* > t\}}(\omega)\} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{4.3}$$

For a fuzzy stopping time $\tilde{\sigma}^*(t,\omega)$, we denote its α-cut in (3.5) of by $\tilde{\sigma}_\alpha^*(\omega)$. Then we note that $\tilde{\sigma}_\alpha^*(\omega)$ and $\sigma_\alpha^*(\omega)$ are equal except at most countable many $\alpha \in (0,1]$.

Theorem 4.2 (Optimal fuzzy stopping time). *Suppose Assumption A holds. If $P(\tilde{\sigma}_0^* < \infty) = 1$, then $\tilde{\sigma}^*$ is an optimal fuzzy stopping time for Problem 1. Further it holds that*

$$\tilde{\sigma}_\alpha^*(\omega) := \min\{t \geq 0 \mid \tilde{\sigma}^*(t,\omega) < \alpha\}, \quad \omega \in \Omega \quad \text{for } \alpha \in (0,1]. \tag{4.4}$$

The following result implies a comparison between the optimal values of the 'classical' stopping model and the 'fuzzy' stopping model (Problem 1). Then we find that the fuzzy stopping model is more better than the classical one. This fact has been explicitly shown in the discrete-time model by [8].

Corollary 4.1. *It holds that, under the same assumptions as Theorem 4.2,*

$$E(G_{\tau^*}) \leq E(G_{\tilde{\sigma}^*}), \tag{4.5}$$

where $\tilde{\sigma}^*$ is the optimal fuzzy stopping time and τ^* is an optimal stopping time in the class of classical stopping times.

5. Optimality equations

In this section, we consider the optimality conditions for the optimal rewards $\{U_t^\alpha\}_{t\geq 0}$. The following theorem shows their optimality characterization.

Theorem 5.1. *For $\alpha \in [0,1]$ and $t \geq 0$, the following (i) – (iii) hold:*

(i) *For almost all $\omega \in \Omega$, it holds that*

$$U_t^\alpha(\omega) \geq g(\tilde{X}_{t,\alpha}(\omega)).$$

(ii) *For almost all $\omega \in \Omega$, it holds that*

$$U_t^\alpha(\omega) \geq E(U_r^\alpha|\mathcal{M}_t)(\omega), \quad r \in [t,\infty).$$

(iii) *For almost all $\omega \in \Omega$ satisfying $U_t^\alpha(\omega) > g(\tilde{X}_{t,\alpha}(\omega))$, there exists $\varepsilon > 0$ such that*

$$U_t^\alpha(\omega) = E(U_r^\alpha|\mathcal{M}_t)(\omega), \quad r \in [t,\varepsilon).$$

In the rest of this section we discuss the optimality equations for the optimal reward process $\{U_t^\alpha\}_{t\geq 0}$. Let $L^2([0,\infty))$ be the space of continuous functions $u_. : [0,\infty) \mapsto \mathbb{R}$ satisfying $\int_0^\infty (u_r)^2\,\mathrm{d}r < \infty$ and $\lim_{t\to\infty} u_t = 0$. Let $\mathcal{L}$ be the space of functions $u_. \in L^2([0,\infty))$ such that $u_.$ is differentiable on $[0,\infty)$ and $\mathrm{d}u_t/\mathrm{d}t \in L^2([0,\infty))$. Then we write $Au_t := -\mathrm{d}u_t/\mathrm{d}t$. For $t \geq 0$, we put a bilinear form on $\mathcal{L} \times \mathcal{L}$ by

$$\langle u_., v_. \rangle_t = \int_t^\infty u_r v_r\,\mathrm{d}r \quad \text{for } u_., v_. \in \mathcal{L}. \tag{5.1}$$

For a stochastic process $\{Y_t\}_{t\geq 0}$, we define the differential AY_t by a stochastic process:

$$AY_t(\omega) := \lim_{s\downarrow 0} \frac{Y_t(\omega) - Y_{t+s}(\omega)}{s} \tag{5.2}$$

if the limit exists. The following theorem gives an optimality equation of the optimal fuzzy reward process $\{U_t^\alpha\}_{t\geq 0}$ by the differential.

Assumption B. It holds that $U_\cdot^\alpha(\omega) \in \mathcal{L}$ and $g(\tilde{X}_{\cdot,\alpha}(\omega)) \in \mathcal{L}$ for almost all $\omega \in \Omega$ and all $\alpha \in (0,1]$.

Theorem 5.2 (Optimality equation). *Suppose Assumption B hold. Let $\alpha \in (0,1]$. The optimal reward process $\{U_t^\alpha\}_{t\geq 0}$ is a unique solution satisfying the following three inequalities (5.3) – (5.5): For almost all $\omega \in \Omega$,*

$$U_t^\alpha(\omega) \geq g(\tilde{X}_{t,\alpha}(\omega)) \quad \text{for all } t \geq 0; \tag{5.3}$$

$$AU_t^\alpha(\omega) \geq 0 \quad \text{for all } t \geq 0; \tag{5.4}$$

$$\Big\langle AU_\cdot^\alpha(\omega), U_\cdot^\alpha(\omega) - g(\tilde{X}_{\cdot,\alpha}(\omega))\Big\rangle_t = 0 \quad \text{for all } t \geq 0. \tag{5.5}$$

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