数理解析研究所講究録
1197巻 2001年 207-226

球対称なポテンシャルを含む半線形楕円型方程式の正値解について

On Positive Solutions for Semilinear Elliptic Equations with Radially Symmetric Potential

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1. INTRODUCTION AND MAIN RESULTS

In this paper we consider positive solutions of the following nonlinear elliptic equation with a harmonic potential term $|x|^2u$

$$-\Delta u + \left(\lambda + |x|^2\right) u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and $p > 1$. This problem arises in the study of standing wave solutions

$$\psi(t,x) = \exp\left(i\lambda t\right) u(x)$$

for the nonlinear Schrödinger equation with a harmonic potential

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + |x|^2\psi - |\psi|^{p-1}\psi, \quad (t,x) \in \mathbb{R}^{1+n}, \tag{1.2}$$

which is a model equation to describe the Bose-Einstein condensate with attractive interparticle interactions under a magnetic trap (see, e.g., [16]). We see that $\psi(t,x) = \exp\left(i\lambda t\right) u(x)$

*Supported in part by JSPS Research Fellowships for Japanese Young Scientists and Grant-in-Aid for JSPS Research Fellowships (11-8942).

is a solution of (1.2) if $u(x)$ satisfies (1.1). Since (1.2) has two conserved quantities, the energy and the particle number

$$E(\psi) := \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}|x|^2|\psi|^2 - \frac{1}{p+1}|\psi|^{p+1} \right) dx, \quad N(\psi) := \int_{\mathbb{R}^n} |\psi|^2 dx,$$

it is natural to study the solutions of (1.1) and (1.2) in the energy space

$$\Sigma := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(|u|^2 + |\nabla u|^2 + |xu|^2 \right) dx < \infty \right\}.$$

Since the embedding $\Sigma \hookrightarrow L^q(\mathbb{R}^n)$ is compact for $2 \le q < 2n/(n-2)^+$, by the standard variational method, we can prove that there exists at least one solution of

$$\begin{cases} -\Delta u + \left(\lambda + |x|^2\right) u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n, \\ u \in \Sigma, \quad u(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

if $n \ge 1$, $\lambda > -n$ and $1 < p < (n+2)/(n-2)^+$ (see [5] for details). Here, we note that the condition $\lambda > -n$ appears naturally to show the existence of solutions for (1.3), because the first eigenvalue of $-\Delta + |x|^2$ on Σ is equal to n and the corresponding eigenfunctions are $C\exp(-|x|^2/2)$. Recently, the stability of standing waves for (1.2) has been studied by [5, 21]. For studying the stability of standing waves, it is important and fundamental to investigate the structure of solutions to (1.3) (see, e.g., [2, 3, 6, 13, 14, 17]).

In [8], we have proved the uniqueness of solution for (1.3) when $n \ge 3$ as follows.

Theorem 1.1. (See [8].) *Assume $n \ge 3$, $\lambda > -n$ and $1 < p < (n+2)/(n-2)$. Then* (1.3) *has a unique solution.*

In this paper, we will show the uniqueness of solution for (1.3) in case $n = 2$. For related uniqueness results, we refer the reader to [9, 10, 18] and the references cited therein.

By a bootstrap argument using the fact that

$$(-\Delta + |x|^2 + 1)^{-1} L^q(\mathbb{R}^n) = \{ v \in W^{2,q}(\mathbb{R}^n) : \ |x|^2 v \in L^q(\mathbb{R}^n) \}$$

if $n \geq 1$ and $1 < q < \infty$ (see, e.g., [12, Theorem 2.5]), it is shown in a similar way as in [2, Theorem 8.1.1] that all solutions of (1.3) belong to $C^2(\mathbb{R}^n)$ and satisfy $\lim_{|x|\to\infty} u(x) = 0$ (see [5] for details). Moreover, by [11, Theorem 2], we see that all solutions of (1.3) are radially symmetric about the origin. Therefore, the problem for solutions of (1.3) is reduced to that for radial solutions of (1.3). Since we are interested in radial solutions ($u = u(r)$ with $r = |x|$) of (1.3), we study the initial value problem

$$\begin{cases} u'' + \dfrac{n-1}{r}u' - \left(\lambda + r^2\right)u + |u|^{p-1}u = 0, \quad r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0, \end{cases} \tag{1.4}$$

where the prime denotes the differentiation with respect to r. In Section 2, it will be shown that (1.4) has a unique global solution $u(r) \in C^2([0,\infty))$, which is denoted by $u(r;\alpha)$. We classify $u(r;\alpha)$ as follows:

(i) $u(r;\alpha)$ is a *crossing solution* if $u(r;\alpha)$ has a zero in $(0,\infty)$, i.e., there exists some $z \in (0,\infty)$ such that $u(z;\alpha) = 0$.

(ii) $u(r;\alpha)$ is an *entirely positive solution* if $u(r;\alpha) > 0$ for all $r \in [0,\infty)$.

Moreover, we define

$$\Sigma_{rad} := \left\{ u \in C^1([0,\infty)) : \int_0^\infty \left(|u|^2 + |u'|^2 + |ru|^2\right) r^{n-1} dr < \infty \right\}.$$

Then our main result is the following.

Theorem 1.2. *If $n = 2$, $\lambda > -n$ and $1 < p < \infty$, then there exists a unique positive number α_0 such that the structure of positive solutions to (1.4) is as follows.*

(a) *For every $\alpha \in (\alpha_0, \infty)$, $u(r;\alpha)$ is a crossing solution.*

(b) *If $\alpha = \alpha_0$, then $u(r;\alpha)$ is an entirely positive solution with $u(r;\alpha) \in \Sigma_{rad}$ and satisfies*

$$\lim_{r\to\infty} r^{(n+\lambda)/2} \exp\left(r^2/2\right) u(r;\alpha) \in (0,\infty). \tag{1.5}$$

(c) *For every $\alpha \in (0, \alpha_0)$, $u(r;\alpha)$ is an entirely positive solution with $u(r;\alpha) \notin \Sigma_{rad}$.*

Since, as stated above, all solutions of (1.3) belong to $C^2(\mathbb{R}^n)$ and are radially symmetric about the origin, as a corollary of Theorem 1.2, we have

Theorem 1.3. *If $n = 2$, $\lambda > -n$ and $1 < p < \infty$, then* (1.3) *has a unique solution.*

Remark 1.1. The uniqueness question for solutions of (1.3) seems to be open for $n = 1$.

In order to prove Theorem 1.2, we apply the classification theorem by Yanagida and Yotsutani [19, 20]. Let $\varphi(r)$ be a solution of

$$\begin{cases} \varphi'' + \left(\dfrac{n-1}{r} + 2r\right)\varphi' + (n-\lambda)\varphi = 0, \quad r > 0, \\ \varphi(0) = 1, \quad \varphi'(0) = 0. \end{cases} \tag{1.6}$$

For a solution $u(r)$ of (1.4), if we put

$$u(r) = \exp\left(r^2/2\right)\varphi(r)v(r), \tag{1.7}$$

then we see that $v(r)$ satisfies

$$\begin{cases} (g(r)v')' + g(r)K(r)|v|^{p-1}v = 0, \quad r > 0, \\ v(0) = \alpha > 0, \quad v'(0) = 0, \end{cases} \tag{1.8}$$

where

$$g(r) := r^{n-1}\exp\left(r^2\right)\varphi(r)^2, \quad K(r) := \exp\left(\frac{p-1}{2}r^2\right)|\varphi(r)|^{p-1}. \tag{1.9}$$

We should note that $\varphi(r) > 0$ on $[0, \infty)$ if $\lambda > -n$ by (i) of Proposition 2.2 in Section 2. To see whether $u(r)$ has a zero or not, we have only to check this property for $v(r)$. For this purpose, we employ the classification theorem by Yanagida and Yotsutani [20], which is stated as follows. Let $g(r)$ and $K(r)$ satisfy

$$\begin{cases} g(r) \in C^2([0,\infty)); \\ g(r) > 0 \quad \text{on} \quad (0,\infty); \\ 1/g(r) \notin L^1(0,1); \\ 1/g(r) \in L^1(1,\infty), \end{cases} \tag{g}$$

and

$$\begin{cases} K(r) \in C(0,\infty); \\ K(r) \geq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0,\infty); \\ h(r)K(r) \in L^1(0,1); \\ g(r)\left(h(r)/g(r)\right)^p K(r) \in L^1(1,\infty), \end{cases} \tag{K}$$

where

$$h(r) := g(r)\int_r^\infty g(s)^{-1}ds.$$

Moreover, define

$$G(r) := \frac{2}{p+1}g(r)h(r)K(r) - \int_0^r g(s)K(s)ds, \tag{1.10}$$

$$H(r) := \frac{2}{p+1}h(r)^2\left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s)\left(\frac{h(s)}{g(s)}\right)^p K(s)ds, \tag{1.11}$$

and

$$r_G := \inf\{r \in (0,\infty) : \ G(r) < 0\}, \quad r_H := \sup\{r \in (0,\infty) : \ H(r) < 0\}.$$

Theorem A (Yanagida and Yotsutani [19, 20]). *Assume that $g(r)$ and $K(r)$ satisfy the conditions (g) and (K). Let $v(r;\alpha)$ be a solution of*

$$\begin{cases} (g(r)v')' + g(r)K(r)(v^+)^p = 0, \quad r > 0, \\ v(0) = \alpha > 0, \quad v'(0) = 0, \end{cases} \tag{1.12}$$

where $v^+ := \max\{v, 0\}$, and suppose that $G(r) \not\equiv 0$ on $(0,\infty)$.

(i) *If*

$$0 < r_H \leq r_G < \infty, \tag{1.13}$$

then there exists a unique positive number α_0 such that the structure of solutions to (1.12) *is as follows.*

(a) *For every $\alpha \in (\alpha_0, \infty)$, $v(r;\alpha)$ has a zero in $(0,\infty)$.*

(b) *If* $\alpha = \alpha_0$, *then* $v(r;\alpha) > 0$ *on* $[0,\infty)$ *and*

$$0 < \lim_{r\to\infty} \left(\int_r^\infty g(s)^{-1} ds \right)^{-1} v(r;\alpha) < \infty. \tag{1.14}$$

(c) *For every* $\alpha \in (0,\alpha_0)$, $v(r;\alpha) > 0$ *on* $[0,\infty)$ *and*

$$\lim_{r\to\infty} \left(\int_r^\infty g(s)^{-1} ds \right)^{-1} v(r;\alpha) = \infty. \tag{1.15}$$

(ii) *If* $r_G < \infty$ *and* $r_H = 0$ (*i.e.*, $H(r) \geq 0$ *on* $[0,\infty)$), *then* $v(r;\alpha)$ *is positive on* $[0,\infty)$ *and satisfies* (1.15) *for every* $\alpha > 0$.

(iii) *If* $r_G = \infty$ (*i.e.*, $G(r) \geq 0$ *on* $[0,\infty)$), *then* $v(r;\alpha)$ *has a zero in* $(0,\infty)$ *for every* $\alpha > 0$.

Remark 1.2. Note that if $v(r;\alpha)$ is positive on $[0,\infty)$, then $v(r;\alpha)$ satisfies either (1.14) or (1.15), because $\left(\int_r^\infty g(s)^{-1}ds\right)^{-1} v(r;\alpha)$ is non-decreasing on $(0,\infty)$.

In Section 3, noting that $g(r)$ and $K(r)$ given by (1.9) satisfy the assumptions (g) and (K), we will prove the following proposition.

Proposition 1.1. *If* $n = 2$, $\lambda > -n$ *and* $1 < p < \infty$, *then condition* (1.13) *holds for* (1.8) *with* (1.9).

By the proposition above, we see that $u(r;\alpha)$ is an entirely positive solution for $\alpha \in (0,\alpha_0]$ and a crossing solution for $\alpha \in (\alpha_0,\infty)$. Moreover, for the asymptotic behaviour of entirely positive solutions, we will prove the following proposition in Section 5.

Proposition 1.2. *Let* $u(r;\alpha)$ *and* $v(r;\alpha)$ *be entirely positive solutions of* (1.4) *and* (1.8) *with* (1.9), *respectively. Then, the following three conditions are equivalent* :

(i) $u(r;\alpha) \in \Sigma_{rad}$, (ii) $u(r;\alpha)$ *satisfies* (1.5), (iii) $v(r;\alpha)$ *satisfies* (1.14).

Theorem 1.2 follows from Theorem A and Propositions 1.1 and 1.2.

2. PRELIMINARIES

In this section, we prepare some results which have been shown in [8]. First, we give the following proposition.

Proposition 2.1. *The initial value problem* (1.4) *has a unique global solution* $u(r;\alpha)$ *in* $C^2([0,\infty))$ *for every* $\alpha > 0$.

Next, we study the properties of solutions to the initial value problem (1.6). It is shown that there exists a unique solution $\varphi(r) \in C^2([0,\infty))$ of (1.6) by a standard way. Moreover, we obtain the following proposition, which plays an important role in what follows.

Proposition 2.2. *Assume* $\lambda > -n$ *and let* $\varphi(r) \in C^2([0,\infty))$ *be the unique solution of* (1.6). *Then we have*

(i) $\varphi(r) > 0$ *on* $[0,\infty)$. *Moreover,* $\varphi'(r) < 0$ *on* $[0,\infty)$ *if* $-n < \lambda < n$ *and* $\varphi'(r) > 0$ *on* $[0,\infty)$ *if* $\lambda > n$. *Especially,* $\varphi(r) \equiv 1$ *on* $[0,\infty)$ *for* $\lambda = n$.

(ii) $m := \sup_{r\geq 0} \left| \dfrac{r\varphi'(r)}{\varphi(r)} \right|$ *is finite, and* $r^m\varphi(r)$ *is non-decreasing on* $[0,\infty)$.

(iii) $\dfrac{r\varphi'(r)}{\varphi(r)} = \dfrac{\lambda - n}{2} + O\left(\dfrac{1}{r^2}\right)$ *as* $r \to \infty$.

(iv) *The limit* $L := \lim_{r\to\infty} r^{(n-\lambda)/2}\varphi(r)$ *exists in* $(0,\infty)$.

(v) *There exist positive constants* C *and* R *such that*

$$\left| r^n \exp\left(r^2\right) \varphi(r)^2 \int_r^\infty s^{1-n} \exp\left(-s^2\right) \varphi(s)^{-2} ds - \left(\frac{1}{2} - \frac{\lambda}{4} r^{-2}\right) \right| \leq C r^{-4}$$

holds for all $r \geq R$.

(vi) $\dfrac{r\varphi'(r)}{\varphi(r)} = \dfrac{\lambda - n}{n} r^2 + o\left(r^2\right)$ *as* $r \to 0$.

3. PROOF OF PROPOSITION 1.1

In this section, we give the proof of Proposition 1.1 by applying Theorem A. In order to apply Theorem A, we first check the conditions imposed on the coefficients of (1.12).

Lemma 3.1. *If $n = 2$ and $\lambda > -n$, then $g(r)$ and $K(r)$ given by (1.9) satisfy the assumptions (g) and (K).*

Therefore, $g(r)$ and $K(r)$ given by (1.9) are admissible. Inserting their definition (1.9) into (1.10) and (1.11), we obtain

$$\begin{aligned} G(r) = {} & \frac{2}{p+1} r^{2n-2} \exp\left(\frac{p+3}{2} r^2\right) \varphi(r)^{p+3} \int_r^\infty s^{1-n} \exp\left(-s^2\right) \varphi(s)^{-2} ds \\ & - \int_0^r s^{n-1} \exp\left(\frac{p+1}{2} s^2\right) \varphi(s)^{p+1} ds, \end{aligned} \tag{3.1}$$

$$\begin{aligned} H(r) = {} & \frac{2}{p+1} r^{2n-2} \exp\left(\frac{p+3}{2} r^2\right) \varphi(r)^{p+3} \left(\int_r^\infty s^{1-n} \exp\left(-s^2\right) \varphi(s)^{-2} ds\right)^{p+2} \\ & - \int_r^\infty s^{n-1} \exp\left(\frac{p+1}{2} s^2\right) \varphi(s)^{p+1} \left(\int_s^\infty t^{1-n} \exp\left(-t^2\right) \varphi(t)^{-2} dt\right)^{p+1} ds. \end{aligned}$$

In order to show (1.13), we investigate the profiles of $G(r)$ and $H(r)$. First, we study the increase and decrease. Differentiating (1.10) and (1.11), we obtain

$$G'(r) = \left(\int_r^\infty g(s)^{-1} ds\right)^{-p-1} H'(r) = \frac{2}{p+1} g(r) K(r) \left(\Phi(r) - \frac{p+3}{2}\right), \tag{3.2}$$

where

$$\begin{aligned} \Phi(r) &:= \left(2g'(r) + \frac{g(r)K'(r)}{K(r)}\right) \int_r^\infty g(s)^{-1} ds \\ &= r^{n-2} \exp\left(r^2\right) \varphi(r)^2 \left\{(p+3)\left(r^2 + \frac{r\varphi'(r)}{\varphi(r)}\right) + 2(n-1)\right\} \\ &\quad \times \int_r^\infty s^{1-n} \exp\left(-s^2\right) \varphi(s)^{-2} ds. \end{aligned} \tag{3.3}$$

In view of (3.2), $G(r)$ and $H(r)$ have the same extremal points, namely those $r > 0$ which satisfy $\Phi(r) = (p+3)/2$. So, to know the sign of $G'(r)$ and $H'(r)$, we need to study the relation between $\Phi(r)$ and $(p+3)/2$. We first study the behaviour of $\Phi(r)$ near $r = 0$ and $r = \infty$ by using Proposition 2.2.

Lemma 3.2. *If $n = 2$ and $\lambda > -n$, then $\lim_{r \to 0} \Phi(r) = +\infty$.*

Lemma 3.3. *If $\lambda > -n$, then*

$$\Phi(r) = \frac{p+3}{2} - \frac{n(p-1)+4}{4} r^{-2} + O\left(r^{-4}\right) \quad \text{as} \quad r \to \infty.$$

Noting that $\Phi(r)$ is continuous in $[0, \infty)$, we see that there exists at least one crossing point of $y = \Phi(r)$ and $y = (p+3)/2$ in (r, y)-plane by Lemmas 3.2 and 3.3. The following lemma will be proved in Section 4.

Lemma 3.4. *If $n = 2$, $\lambda > -n$ and $1 < p < \infty$, then there exists a unique number $r_* \in (0, \infty)$ satisfying $\Phi(r_*) = (p+3)/2$ such that*

$$\begin{cases} \Phi(r) > \dfrac{p+3}{2} & \text{on} \quad [0, r_*); \\ \Phi(r) < \dfrac{p+3}{2} & \text{on} \quad (r_*, \infty). \end{cases} \tag{3.4}$$

Therefore, from (3.2) and Lemma 3.4, we have

Lemma 3.5. *If $n = 2$, $\lambda > -n$ and $1 < p < \infty$, then there exists a unique number $r_* \in (0, \infty)$ such that $G(r)$ and $H(r)$ are increasing on $[0, r_*)$ and decreasing on (r_*, ∞).*

Moreover, in order to locate r_G and r_H, we need to investigate the behaviour of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$ by using Proposition 2.2.

Lemma 3.6. *Assume $n = 2$, $\lambda > -n$ and $1 < p < \infty$. Then we have*

(i) $\lim_{r \to \infty} G(r) = -\infty$, (ii) $\lim_{r \to 0} G(r) = 0$,

(iii) $\lim_{r \to \infty} H(r) = 0$, (iv) $\lim_{r \to 0} H(r) \in [-\infty, 0)$.

Now we prove Proposition 1.1.

Proof of Proposition 1.1. As is already seen in Lemma 3.5, both $G(r)$ and $H(r)$ have exactly one local maximum at $r_* \in (0, \infty)$. Moreover, in view of Lemma 3.6, $H(r)$ is negative near $r = 0$ and positive for large r. Thus $H(r_*) > 0$ and $0 < r_H < r_*$. Furthermore,

we obtain $G(r_*) > 0$ from $G(0) = 0$, and the negativity of $G(r)$ for large r yields $0 < r_* < r_G < \infty$; so we conclude that condition (1.13) holds. ∎

4. PROOF OF LEMMA 3.4

For the case $\lambda \geq 0$, we can show Lemma 3.4 in this paper by using same method as the proof of Lemma 3.4 in [8]. So we will consider the case $-n < \lambda < 0$ in the following.

First, we define the following functions which we need in this section:

$$
\begin{aligned}
V(r) &:= r^2 + \lambda, \\
X(r) &:= r^2 + \frac{r\varphi'(r)}{\varphi(r)}, \\
F(r) &:= (p+3)^2 r^2 V(r) - 4, \\
J(r) &:= (p+3)X(r) + 2, \\
L(r) &:= (p+3)X(r)^2 + 4X(r) + (p+3)r^2 V(r) \equiv \frac{J(r)^2 + F(r)}{p+3}, \\
P(r) &:= \frac{J(r)^2}{L(r)}, \\
Q(r) &:= 4F(r)X(r) + rF'(r), \\
R(r) &:= 16q^2 r^4 V(r)^3 - 4q(q+32)r^2 V(r)^2 - 64(q-4)V(r) \\
&\qquad + 4q^2 r^4 V(r)V''(r) - 80qrV'(r) - 16qr^2 V''(r) - 5q^2 r^4 V'(r)^2,
\end{aligned}
$$

where $q := (p+3)^2$. By these definitions, we can rewrite $\Phi(r)$ as

$$
\Phi(r) = J(r) \exp\left(r^2\right) \varphi(r)^2 \int_r^\infty s^{-1} \exp\left(-s^2\right) \varphi(s)^{-2} ds. \tag{4.1}
$$

Differentiating $\Phi(r)$, we have

$$
r\Phi'(r) = L(r) \exp\left(r^2\right) \varphi(r)^2 \int_r^\infty s^{-1} \exp\left(-s^2\right) \varphi(s)^{-2} ds - J(r). \tag{4.2}
$$

(Here, we use the equality $rX'(r) = r^2 V(r) - X(r)^2$.) In order to evaluate the sign of $\Phi'(r)$, we will investigate the behaviour of $J(r)$ and $L(r)$ for $r \in [0, \infty)$.

First, we study the profile of $J(r)$. Note that the following lemma holds.

Lemma 4.1. *There exists a unique positive number β satisfying $\beta > \sqrt{-\lambda}$, $F(\beta) = 0$ and $F'(\beta) > 0$ such that $F(r) < 0$ on $[0, \beta)$ and $F(r) > 0$ on (β, ∞).*

Proof. Trivial. ∎

It follows from $\varphi'(0) = 0$ and (iii) of Proposition 2.2 that

$$J(0) = 2 \quad \text{and} \quad \lim_{r \to \infty} J(r) = +\infty.$$

Moreover, we obtain the following lemma.

Lemma 4.2. *Function $J(r)$ satisfies one of the following conditions:*

(J1) *$J(r) > 0$ for all $r \geq 0$.*

(J2) *There exist two positive numbers r_1 and r_2 satisfying $r_1 < \beta < r_2$ and $J(r_1) = J(r_2) = 0$ such that $J(r) > 0$ on $(0, r_1) \cup (r_2, \infty)$ and $J(r) < 0$ on (r_1, r_2).*

(J3) *$J(\beta) = J'(\beta) = 0$ and $J(r) > 0$ on $(0, \infty) \setminus \beta$.*

Proof. We have the following two equalities:

$$\begin{aligned} J'(r)|_{J(r)=0} &= \frac{F(r)}{(p+3)r}, \\ J''(r)|_{J(r)=J'(r)=0} &= \frac{2\{(p+3)^2 r^4 + 4\}}{(p+3)r^2} (> 0). \end{aligned}$$

Therefore, noting Lemma 4.1, we can see that if $J(r)$ has a zero, then $J(r)$ satisfies (J2) or (J3). Thus we conclude this lemma. ∎

Next, we study the profile of $L(r)$. Using (iii) and (vi) of Proposition 2.2, we have

$$L(r) = \lambda(p+5)r^2 + o(r^2) \quad \text{as} \quad r \to 0 \quad \text{and} \quad \lim_{r \to \infty} L(r) = +\infty.$$

Therefore, $L(r)$ is negative for sufficiently small $r > 0$. Moreover, we can show the following lemma.

Lemma 4.3. *Function $L(r)$ satisfies one of the following conditions:*

(L1) *There exists a unique positive number r_3 satisfying $L(r_3) = 0$ such that $L(r) < 0$ on $(0, r_3)$ and $L(r) > 0$ on (r_3, ∞).*

(L2) *There exist two positive numbers r_4 and r_5 satisfying $L(r_4) = L'(r_4) = L(r_5) = 0$ such that $L(r) < 0$ on $(0, r_5) \setminus r_4$ and $L(r) > 0$ on (r_5, ∞).*

Proof. Noting the equality $(p+3)L(r) \equiv J(r)^2 + F(r)$, we have $L(\beta) > 0$ if $J(r)$ satisfies the condition (J1) or (J2), and $L(\beta) = 0$ if $J(r)$ satisfies the condition (J3) from Lemmas 4.1 and 4.2. Moreover, we can see $L(\beta) > 0$ for all $r > \beta$ from Lemma 4.1. So it is sufficient to evaluate $L(r)$ for $r < \beta$. Note that the following equality holds:

$$L'(r)|_{L(r)=0} = \frac{Q(r)}{(p+3)r}.$$

Concerning the profile of $Q(r)$, we have the following lemma whose proof will be given below.

Lemma 4.4. *There exists a unique number $r_6 \in (0, \beta)$ satisfying $Q(r_6) = 0$ such that*

$$Q(r) < 0 \quad on \quad (0, r_6) \quad and \quad Q(r) > 0 \quad on \quad (r_6, \beta].$$

Therefore, similarly to the proof of Lemma 4.2, we can decide the location of zeros for $L(r)$. ∎

Remark 4.1. Set $\hat{r} := \sup\{r \in (0, \infty) \ : \ L(r) < 0\}$. Then we can put $\hat{r} = r_3$ or $\hat{r} = r_5$ if $L(r)$ satisfies the condition (L1) or (L2), respectively. Moreover, $\hat{r} \equiv \beta$ if $J(r)$ satisfies the condition (J3).

Proof of Lemma 4.4. Using (vi) of Proposition 2.2, we obtain

$$Q(r) = 2\lambda(q-4)r^2 + o(r^2) \quad \text{as} \quad r \to 0 \quad \text{and} \quad Q(\beta) = \beta F'(\beta) > 0. \tag{4.3}$$

Therefore, there exists at least one zero in $(0, \beta)$ by noting $2\lambda(q-4) < 0$. So we will evaluate the sign of $Q'(r)|_{Q(r)=0}$. Since the equality

$$Q'(r)|_{Q(r)=0} = \frac{rR(r)}{4F(r)}$$

holds and $F(r) < 0$ on $[0, \beta)$, it is sufficient to investigate the profile of $R(r)$ on $[0, \beta)$. Noting that $V(r)$, $V'(r)$ and $V''(r)$ are positive for $r \in \left(\sqrt{-\lambda}, \beta\right)$, we have

$$\begin{aligned} R(r) &= 16qr^2V(r)^2F(r) - 4q(q+16)r^2V(r)^2 - 64(q-4)V(r) \\ &\quad + 4qr^2V''(r)F(r) - 80qrV'(r) - 5q^2r^4V'(r)^2 \\ &< 0 \quad \text{for} \quad r \in \left[\sqrt{-\lambda}, \beta\right). \end{aligned}$$

Therefore, there exists at least one zero on $\left(0, \sqrt{-\lambda}\right)$ in view of $R(0) = -64\lambda(q-4) > 0$. Differentiating $R(r)$, we obtain

$$\begin{aligned} R'(r) &= 64q^2r^3V(r)^3 + 48q^2r^4V(r)^2V'(r) - 8q(q+32)rV(r)^2 - 8q(q+32)r^2V(r)V'(r) \\ &\quad - 16(9q-16)V'(r) + 16q^2r^3V(r)V''(r) - 6q^2r^4V'(r)V''(r) \\ &\quad - 112qrV''(r) + 4q^2r^4V(r)V'''(r) - 16qr^2V'''(r) - 20q^2r^3V'(r)^2. \end{aligned}$$

Then it follows from $R(r) = 0$, which implies

$$\begin{aligned} -8 \cdot 32qr^2V(r)V'(r) &= 16q^3r^6V(r)^3V'(r) - 4q^2(q+32)r^4V(r)^2V'(r) \\ &\quad - 64q^2r^2V(r)V'(r) + 4q^3r^6V(r)V'(r)V''(r) \\ &\quad - 80q^2r^3V'(r)^2 - 16q^2r^4V'(r)V''(r) - 5q^3r^6V'(r)^3, \end{aligned}$$

and $V(r) < 0$, $V'(r) > 0$ and $V''(r) > 0$ for $r \in \left(0, \sqrt{-\lambda}\right)$ that

$$\begin{aligned} R'(r)|_{R(r)=0} &= 64q^2r^3V(r)^3 - 80q^2r^4V(r)^2V'(r) - 256qrV(r)^2 + 16q^3r^6V(r)^3V'(r) \\ &\quad + 4q^3r^4V(r)V'(r)V''(r) - 22q^2r^4V'(r)V''(r) - 5q^3r^6V'(r)^3 \\ &\quad + 16(16-q)V'(r) + 16q^2r^3V(r)V''(r) - 112qrV''(r) \\ &\quad + 4q^2r^4V(r)V'''(r) - 16qr^2V'''(r) - 4qV'(r)\left\{(qr^2V(r)+4)^2 + 16\right\} \\ &\quad - 8q^2r\left\{\left(V(r) + \frac{5}{2}rV'(r)\right)^2 + \frac{25}{4}r^2V'(r)^2\right\} \\ &< 0 \quad \text{for} \quad r \in \left(0, \sqrt{-\lambda}\right). \end{aligned}$$

(Note that $V'''(r) = 0$.) Thus there exists a unique positive number $r_7 \in (0, \beta)$ such that

$$\begin{cases} R(r) > 0 \quad \text{on} \quad [0, r_7), \quad \text{i.e.,} \quad Q'(r) < 0 \quad \text{if} \quad Q(r) \quad \text{has a zero in} \quad [0, r_7), \\ R(r_7) = 0 \quad \text{and} \quad R'(r_7) < 0, \quad \text{i.e.,} \quad Q'(r) = 0 \quad \text{if} \quad Q(r) \quad \text{has a zero at} \quad r = r_7, \\ R(r) < 0 \quad \text{on} \quad (r_7, \beta), \quad \text{i.e.,} \quad Q'(r) > 0 \quad \text{if} \quad Q(r) \quad \text{has a zero in} \quad (r_7, \beta). \end{cases}$$

Define $\xi := \inf\{r \in (0, \infty) \ : \ Q(r) = 0\}$. Then $Q'(\xi) \geq 0$ from (4.3), which implies $\xi \geq r_7$. Moreover, we have $\xi \neq r_7$. In fact, if $\xi = r_7$, then $Q(r_7) = Q'(r_7) = 0$ and $Q''(r_7)$ must be non-positive from (4.3). However, it is impossible by the following equality

$$Q''(r_7) = \frac{2R(r_7) + r_7R'(r_7)}{4F(r_7)} \, (> 0).$$

Therefore, there exists a unique number $r_6 \in (r_7, \beta)$ satisfying $Q(r_6) = 0$ such that $Q(r) < 0$ on $(0, r_6)$ and $Q(r) > 0$ on $(r_6, \beta]$. ∎

Now we obtain the following lemma about the profile of $\Phi(r)$ in view of (4.1), (4.2) and Lemmas 4.2 and 4.3.

Lemma 4.5. *Function $\Phi(r)$ satisfies the following conditions:*

(i) *If $J(r)$ satisfies the condition* (J1), *then $\Phi(r)$ is decreasing on $(0, \hat{r}]$.*

(ii) *If* $J(r)$ *satisfies the condition* (J2), *then* $\hat{r}$ *satisfies* $r_1 < \hat{r} < r_2$. *Moreover,* $\Phi(r)$ *is decreasing on* $(0, r_1)$ *and increasing on* $(\hat{r}, r_2)$. *Especially,* $\Phi(r_1) = \Phi(r_2) = 0$ *and* $\Phi(r) < 0$ *on* (r_1, r_2).

(iii) *If* $J(r)$ *satisfies the condition* (J3), *then* $\Phi(r)$ *is decreasing on* $(0, \beta)$ *and* $\Phi(\beta) = 0$ *with* $\Phi'(\beta) = 0$.

Proof. If $J(r)$ satisfies the condition (J2), then we have $L(r_1) = J(r_1)^2 + F(r_1) < 0$ and $L(r_2) = J(r_2)^2 + F(r_2) > 0$ because $J(r_1) = J(r_2) = 0$ and $r_1 < \beta < r_2$. Thus we get $r_1 < \hat{r} < r_2$ in view of Lemma 4.3. Moreover, noting Remark 4.1, we can show the increase and decrease of $\Phi(r)$ for each cases automatically from Lemmas 4.2 and 4.3. ∎

It remains to consider the behaviour of $\Phi(r)$ for

$$r > \hat{r}, \quad r > r_2 \quad \text{or} \quad r > \beta \tag{4.4}$$

when $J(r)$ satisfies the condition (J1), (J2) or (J3), respectively. Our strategy is to evaluate the critical values of $\Phi(r)$. Namely, we will investigate the value of $\Phi(r^*)$ for r^* satisfying $\Phi'(r^*) = 0$. Combining (4.1) and (4.2) with $r = r^*$, we obtain

$$\Phi(r^*) = P(r^*) \equiv \frac{J(r^*)^2}{L(r^*)}. \tag{4.5}$$

Now we study the profile of $P(r)$ for (4.4). First, it is easily seen that

$$P(r) \quad \text{converges to} \quad \frac{p+3}{2} \quad \text{as} \quad r \to \infty \quad \text{with increasing.} \tag{4.6}$$

Moreover, we can show the following lemma.

Lemma 4.6. *Function* $P(r)$ *satisfies the following conditions*:

(i) *If* $J(r)$ *satisfies the condition* (J1), *then there exists a unique positive number* $\bar{r}$ *satisfying* $P(\bar{r}) = (p+3)/2$ *such that*

$$P(r) > \frac{p+3}{2} \quad \text{on} \quad (\hat{r}, \bar{r}) \quad \text{and} \quad 0 < P(r) < \frac{p+3}{2} \quad \text{on} \quad (\bar{r}, \infty).$$

(ii) *If $J(r)$ satisfies the condition* (J2), *then*

$$0 < P(r) < \frac{p+3}{2} \quad on \quad (r_2, \infty).$$

(iii) *If $J(r)$ satisfies the condition* (J3), *then*

$$0 < P(r) < \frac{p+3}{2} \quad on \quad (\beta, \infty).$$

Proof. Assume $J(r)$ satisfies the condition (J1). It follows from Lemmas 4.2 and 4.3 that

$$\lim_{r \to \hat{r}+0} P(r) = +\infty. \tag{4.7}$$

Moreover, for $r > \hat{r}$ there exists at least one crossing point of $y = P(r)$ and $y = (p+3)/2$ in (r, y)-plane from (4.6) and (4.7). For $\bar{r}$ satisfying $P(\bar{r}) = (p+3)/2$, we have $F(\bar{r}) = J(\bar{r})^2 > 0$; so it must be that $\bar{r} > \beta$ holds true. Moreover, we have the following equality

$$P'(\bar{r}) = \frac{F(\bar{r}) f(\bar{r})}{\bar{r}\left(J(\bar{r})^2 + F(\bar{r})\right)^2},$$

where

$$\begin{aligned} f(r) &:= 8F(r) - (p+3) r F'(r) \\ &= -2(p-1)(p+3)^2 r^2 V(r) - (p+3)^3 r^3 V'(r) - 32. \end{aligned}$$

Therefore, we can see $f(r) < 0$ for all $r > \beta$; so we obtain $P'(\bar{r}) < 0$ which implies the uniqueness of crossing point from (4.6) and (4.7).

If $J(r)$ satisfies the condition (J2), then $P(r_2) = 0$ holds. Therefore, from (4.6) and $P'(r)|_{P(r)=(p+3)/2} < 0$ on (β, ∞), $y = P(r)$ cannot cross $y = (p+3)/2$ on (r_2, ∞).

Moreover, if $J(r)$ satisfies the condition (J3), then we have $\lim_{r \to \beta} P(r) = 0$ by l'Hospital's theorem. So it is impossible that $y = P(r)$ crosses $y = (p+3)/2$ on (β, ∞) by the same reason stated above. Thus we finish the proof. ∎

Now we will prove Lemma 3.4.

Proof of Lemma 3.4. If $J(r)$ satisfies the condition (J1), then $\Phi(r) > 0$ for all $r \geq 0$ in view of (4.1). As already seen, $\Phi(r)$ is decreasing on $(0, \hat{r}]$. If $y = \Phi(r)$ first crosses $y = (p+3)/2$ at $r = r_* \in (0, \bar{r}]$, then $\Phi(r)$ is decreasing on $(r_*, \bar{r}]$ because $P(r) > (p+3)/2$ on $(\hat{r}, \bar{r})$, and $\Phi(r)$ has a local minimum at some point in $(\bar{r}, \infty)$. Moreover, noting that $P(r) < (p+3)/2$ on $(\bar{r}, \infty)$ and Lemma 3.3, we can see that it is impossible that $\Phi(r) \geq (p+3)/2$ at some point in $(\bar{r}, \infty)$. On the other hand, if $y = \Phi(r)$ first crosses $y = (p+3)/2$ at $r = r_* \in (\bar{r}, \infty)$, then $\Phi(r) < (p+3)/2$ holds in (r_*, ∞) by the same reason stated above. Thus we can see that $y = \Phi(r)$ crosses $y = (p+3)/2$ only once on $(0, \infty)$.

If $J(r)$ satisfies the condition (J2), then $0 < \Phi(r) < (p+3)/2$ on (r_2, ∞) holds true by noting $\Phi(r_2) = 0$, $P(r) < (p+3)/2$ and $J(r) > 0$ on (r_2, ∞) and Lemma 3.3. Thus $y = \Phi(r)$ crosses $y = (p+3)/2$ once at some point in $(0, r_1)$ from Lemma 4.5.

Finally, if $J(r)$ satisfies the condition (J3), then $0 < \Phi(r) < (p+3)/2$ on (β, ∞) by the same reason as the case (J2). Therefore, there exists only one crossing point of $y = \Phi(r)$ and $y = (p+3)/2$ in $(0, \beta)$ from Lemma 4.5. Thus we can conclude this lemma. ∎

5. PROOF OF PROPOSITION 1.2

In this section, we give the proof of Proposition 1.2.

First, we show the equivalence between (ii) and (iii). For functions $f_1(r)$ and $f_2(r)$, we denote $f_1(r) \sim f_2(r)$ if the limit $\lim_{r\to\infty} f_1(r)/f_2(r)$ exists in $(0, \infty)$. From (1.9) and (iv) and (v) of Proposition 2.2, we have

$$\int_r^\infty g(s)^{-1} ds \sim r^{-n} \exp\left(-r^2\right) \varphi(r)^{-2} \sim r^{-\lambda} \exp\left(-r^2\right),$$

from which together with (1.7) and (iv) of Proposition 2.2, the equivalence between (ii) and (iii) follows.

Next, we show that (ii) implies (i). From (1.4), we see that $u'(r)$ satisfies

$$\left(r^{n-1}u'(r)\right)' = r^{n-1}\left\{\left(\lambda + r^2\right)u(r) - |u(r)|^{p-1}u(r)\right\}. \tag{5.1}$$

Since $u(r)$ decays exponentially as $r \to \infty$, integrating (5.1) on (r_1, r) and letting $r \to \infty$, we see that $r^{n-1}u'(r)$ has a limit as $r \to \infty$ and this limit must be zero by (1.5). Thus, integrating (5.1) on (r, ∞), we see that $u'(r)$ decays exponentially as $r \to \infty$, from which together with (1.5), we obtain that $u(r) \in \Sigma_{rad}$.

Finally, we show that (i) implies (iii). We first note that $u(r)$ and $u'(r)$ decay exponentially as $r \to \infty$, because $u(r)$ is a solution of (1.4) in Σ_{rad} (see [1, Lemma 2] and [15]). Set $U(r) := (1+r)^{(n+\lambda)/2}u(r)$ and

$$V(r) := \eta(r)\left(U'(r) + rU(r)\right), \quad \eta(r) := r^{n-1}(1+r)^{-(n+\lambda)}\exp\left(-r^2/2\right),$$

then $V(r)$ satisfies

$$V'(r) = \eta(r)U(r)\left\{\frac{n+\lambda}{1+r} + \frac{(n-1)(n+\lambda)}{2r(1+r)} - \frac{(n+\lambda)(n+\lambda+2)}{4(1+r)^2} - |u(r)|^{p-1}\right\}.$$

From the assumption $\lambda > -n$ and the exponential decay of $u(r)$ at infinity, there exists $r_2 \in (1, \infty)$ such that

$$\frac{n+\lambda}{1+r} + \frac{(n-1)(n+\lambda)}{2r(1+r)} - \frac{(n+\lambda)(n+\lambda+2)}{4(1+r)^2} - |u(r)|^{p-1} > 0 \quad \text{for} \quad r \geq r_2.$$

Thus, noting that $U(r) > 0$, we see that $V(r)$ is non-decreasing on $[r_2, \infty)$. If there exists $r_3 > r_2$ such that $V(r_3) > 0$, then $V(r) \geq V(r_3) > 0$ for all $r \geq r_3$. This implies that

$$U'(r) + rU(r) \geq V(r_3)/\eta(r), \quad r \geq r_3.$$

However, this is a contradiction, because the left hand side converges to 0 from the definition of $U(r)$ and the exponential decay of $u(r)$ and $u'(r)$ at infinity, while the right hand side goes to infinity as $r \to \infty$ from the definition of $\eta(r)$. Therefore, we have $V(r) \leq 0$ for all

$r \geq r_2$. This shows that

$$\begin{aligned}\left(\exp\left(r^2/2\right)U(r)\right)' &= \exp\left(r^2/2\right)\left(U'(r)+rU(r)\right)\\ &= r^{1-n}(1+r)^{n+\lambda}\exp\left(r^2\right)V(r) \leq 0, \quad r \geq r_2,\end{aligned}$$

which implies that $U(r) \leq C_1 \exp\left(-r^2/2\right)$ for $r \geq r_2$, where $C_1 = \exp\left(r_2{}^2/2\right)U(r_2)$. From the definition of $U(r)$, we obtain that

$$u(r) \leq C_2 r^{-(n+\lambda)/2}\exp\left(-r^2/2\right), \quad r \geq r_4 \tag{5.2}$$

for some $C_2 > 0$ and $r_4 > 0$. As in the proof for the equivalence between (ii) and (iii) above, from (5.2), (1.7), (1.9) and (iv) and (v) of Proposition 2.2, we see that $v(r)$ satisfies

$$v(r) \leq C_3 \int_r^\infty g(s)^{-1} ds, \quad r \geq r_5 \tag{5.3}$$

for some $C_3 > 0$ and $r_5 > 0$. Since $v(r)$ is an entirely positive solution of (1.8) with (1.9), from (5.3) and Theorem A, we see that $v(r)$ satisfies (1.14). This completes the proof.

References

1. H. Berestycki and P. L. Lions, Nonlinear scalar field equation I, Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345.
2. T. Cazenave, "An introduction to nonlinear Schrödinger equations," Textos de Métods Matemáticos 26, IM-UFRJ, Rio de Janeiro 1993.
3. T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.*, **85** (1982), 549–561.
4. C. Dohmen and M. Hirose, Structure of positive radial solutions to the Haraux-Weissler equation, *Nonlinear Anal.*, **33** (1998), 51-69.
5. R. Fukuizumi, "Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential," Master Thesis, University of Tokyo, 2000.
6. M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry I, *J. Funct. Anal.*, **74** (1987), 160–197.

7. M. Hirose, Structure of positive radial solutions to the Haraux-Weissler equation II, *Adv. Math. Sci. Appl.*, **9** (1999), 473–497.
8. M. Hirose and M. Ohta, Structure of positive radial solutions to scalar field equations with harmonic potential, *J. Diff. Eq.* (to appear).
9. Y. Kabeya and K. Tanaka, Uniqueness of positive radial solutions of semilinear elliptic equations in $\mathbb{R}^n$ and Séré's non-degeneracy condition, *Comm. Partial Differential Equations*, **24** (1999), 563–598.
10. M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.*, **105** (1989), 234–266.
11. Y. Li and W. -M. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Comm. Partial Differential Equations*, **18** (1993), 1043–1054.
12. N. Okazawa, An L^p theory for Schrödinger operators with nonnegative potentials, *J. Math. Soc. Japan*, **36** (1984), 675–688.
13. H. A. Rose and M. I. Weinstein, On the bound states of the nonlinear Schrödinger equation with a linear potential, *Physica D*, **30** (1988), 207–218.
14. J. Shatah and W. Strauss, Instability of nonlinear bound states, *Comm. Math. Phys.*, **100** (1985) 173–190.
15. W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, **55** (1977) 149–162.
16. T. Tsurumi and M. Wadati, Collapses of wavefunctions in multi-dimensional nonlinear Schrödinger equations under harmonic potential, *J. Phys. Soc. Japan*, **66** (1997), 3031–3034.
17. M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, **39** (1986) 51–68.
18. E. Yanagida, Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.*, **115** (1991), 257–274.
19. E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.*, **124** (1993), 239–259.
20. E. Yanagida and S. Yotsutani, Pohozaev identity and its applications, *Sūrikaisekikenkyūsho Kōkyūroku, Kyoto University*, **834** (1993), 80–90.
21. J. Zhang, Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials, *Z. Angew. Math. Phys.* (to appear).