数理解析研究所講究録
1197 巻 2001 年 73-83

Hamilton-Jacobi equations in Hilbert spaces with applications to Navier-Stokes equations

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We consider the following stationary Hamilton-Jacobi equation

$$(E) \qquad \lambda u+\langle\partial\varphi(x)+\mathcal{B}(x), Du\rangle+F(x, Du)=0 \quad \text{in} \quad S.$$

Here $\varphi : H\to[0, \infty]$ is a proper lower semicontinuous convex function, λ is a positive constant, $\partial\varphi$ denotes the subdifferential of φ, S is a closed subset of H which satisfies $S\subset\overline{D(\partial\varphi)}$, and moreover, $S\cap D(\partial\varphi)$ is dense in S. $u(x)$ represents a real unknown function on S, and Du denotes the Fréchet derivative in H of u. F is a given real function on $S\times H$. $\mathcal{B}$ is a nonlinear (multivalued) operator from $D(\mathcal{B})\subset H$ into H such that $D(\partial\varphi)\subset D(\mathcal{B})$.

When $\mathcal{B}=0$ and $S=\overline{D(\partial\varphi)}$, the existence and comparison theorems for (E) were proved in Ishii[3], Tataru[7] and Crandall-Lions[2].

We introduce the existence and comparison theorem for (E) in the case that $\mathcal{B}$ is an unbounded (multivalued) operator.

We give the assumptions on $\partial\varphi$, $\mathcal{B}$ and F. Let $\nu\in(0,2]$.

$(\mathrm{A}1)_{\nu}$ There exist positive constants C_{1} and $0<k<1$ such that

$$\langle b_{1}, \xi_{1}\rangle\geq-k|(\partial\varphi)^{0}(x_{1})|^{2}-C_{1},$$

$$|b_1| \leq C_1(|(\partial\varphi)^0(x_1)|^\nu + 1)$$

and

$$\langle (b_1 + \xi_1) - (b_2 + \xi_2), x_1 - x_2 \rangle \geq -C_1|x_1 - x_2|^2$$

for all $x_1, x_2 \in S \cap D(\partial\varphi)$, $\xi_1 \in \partial\varphi(x_1)$, $\xi_2 \in \partial\varphi(x_2)$, $b_1 \in \mathcal{B}(x_1)$, $b_2 \in \mathcal{B}(x_2)$.

(A2) $F \in C(S \times H)$. Moreover for each $R > 0$,

$$\lim_{r \searrow 0} \sup\{F(x,p) - F(x,q) \mid x \in S, \\ p, q \in B(0,R),\ |x| \leq R,\ |p - q| \leq r\} \leq 0.$$

(A3) There is a constant $C_2 > 0$ such that

$$|F(x,p)| \leq C_2(|p|^2 + 1) \qquad \text{for all} \quad (x,p) \in S \times H.$$

(A4) There is a continuous function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega(0) = 0$ for which

$$F(y, \alpha(x-y)) - F(x, \alpha(x-y)) \leq \omega\left(\alpha|x-y|^2 + \frac{1}{\alpha}\right)$$

for all x, $y \in S \cap D(\partial\varphi)$ and $\alpha > 0$.

(A5) If $\theta > 0$, $\psi \in C^1(S)$, and $\theta\varphi + \psi$ attains a minimum value at $\hat{x} \in S$, then there are sequences $\{x_n\}$, $\{\xi_n\}$ and $\{\zeta_n\} \subset H$ such that the following three properties are satisfied:

(i) $x_n \to \hat{x}$ in H as $n \to \infty$.

(ii) $x_n \in S \cap D(\partial\varphi)$, $\xi_n \in \partial\varphi(x_n)$, and $\zeta_n \in \mathcal{B}(x_n)$ for $n \in \mathbf{N}$.

(iii) The inequality

$$\langle \xi_n + \zeta_n, p + D\psi(x_n) + \theta\xi_n \rangle + F(x_n, p + D\psi(x_n) + \theta\xi_n) \\ \leq \langle \xi_n + \zeta_n, p \rangle + F(x_n, p)$$

holds for all $p \in H$.

(A6) $F(x, 0)$ is bounded on S.

Our main result in this paper is stated as follows.

Theorem 1. *Assume that* $(\mathrm{A}1)_\nu$-(A6) *hold for some* $\nu \in (0, 2)$. *Then there is a unique viscosity solution* $u \in BUC(S)$ *of* (E).

The above theorem can be applied to characterizing the value functions associated with optimal control of systems governed by partial differential equations of parabolic type. In particular, it is important to be able to deal with nonstationary Navier-Stokes equations in bounded domains.

2 Viscosity solutions and comparison result

Let $\varphi : H \to [0, \infty]$ be a proper lower semicontinuous convex function.

Notation. We use the standard notation :

$$\partial\varphi(x) = \{p \in H | \ \varphi(y) \geq \varphi(x) + \langle p, y - x\rangle \ \text{ for all } y \in H\},$$

$$D(\varphi) = \{x \in H | \ \varphi(x) < \infty\},$$

$$D(\partial\varphi) = \{x \in H | \ \partial\varphi(x) \neq \emptyset\}.$$

Then it is well known that $\partial\varphi$ is a (multivalued) maximal monotone operator on H, i.e., the following (i) and (ii) hold.

(i) (Monotonicity) For $x, y \in D(\partial\varphi)$ and $p \in \partial\varphi(x)$, $q \in \partial\varphi(y)$,

$$\langle p - q, x - y\rangle \geq 0.$$

(ii) (Solvability) $R(I + \partial\varphi) = H$.

We refer the reader to Brezis[1] for the proof of maximal monotonicity of $\partial\varphi$. By the maximal monotonicity we see that $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$.

$(\partial\varphi)^0$ denotes the minimal section of $\partial\varphi$, i.e., $(\partial\varphi)^0(x)$ is defined as the unique element of least norm in $\partial\varphi(x)$.

Definition 1. Let $\mathcal{O}$ be a relatively open subset of S. A function $u \in C^1(\mathcal{O})$ is called a classical subsolution (resp., supersolution and solution) of (E) in $\mathcal{O}$ if

$$\lambda u(x) + \langle \xi + \zeta, Du(x)\rangle + F(x, Du(x)) \leq 0 \quad (\text{resp.,} \ \geq 0 \text{ and } = 0)$$

for all $x \in \mathcal{O}$, $\xi \in \partial\varphi(x)$ and $\zeta \in \mathcal{B}(x)$.

Before we give the difinition of a viscosity solution, we introduce the notation. $B(x,r)$ denotes the closed ball of radius r with center x in H. Let $\mathcal{O}$ be a relatively open subset of S. For $u : \mathcal{O} \to \mathbf{R}$ we define

$$u^*(x) = \lim_{r \searrow 0} \sup\{u(y) \mid |y - x| \leq r, y \in \mathcal{O}\},$$

$$u_*(x) = \lim_{r \searrow 0} \inf\{u(y) \mid |y - x| \leq r, y \in \mathcal{O}\}.$$

It is clear that $u_* \leq u \leq u^*$ and $u_* = -(-u)^*$. Also, if u is locally bounded in $\mathcal{O}$, then

$$u_* \in LSC(\mathcal{O}) \qquad \text{and} \qquad u^* \in USC(\mathcal{O}).$$

Here we write

$$USC(\mathcal{O}) = \{f : \mathcal{O} \to \mathbf{R} \cup \{\pm\infty\} \text{ is upper semicontinuous}\},$$

$$LSC(\mathcal{O}) = \{f : \mathcal{O} \to \mathbf{R} \cup \{\pm\infty\} \text{ is lower semicontinuous}\}.$$

Definition 2. Let $\mathcal{O}$ be a relatively open subset of S. A function $u : \mathcal{O} \to \mathbf{R}$ is called a *viscosity subsolution* (resp., *viscosity supersolution*) of (E) in $\mathcal{O}$ if it is locally bounded in $\mathcal{O}$ and if there is a $\theta_0 > 0$ such that whenever $\psi \in C^1(\mathcal{O})$, $0 < \theta < \theta_0$, and $(u - \theta\varphi)^* - \psi$ (resp., $(u + \theta\varphi)_* - \psi$) attains a maximum (resp., minimum) value

m at a point $\hat{x} \in \mathcal{O}$, then

$$\begin{aligned}&\lambda(m+\psi(\hat{x})+\theta\varphi(\hat{x}))\\&\qquad+\liminf_{r\searrow 0}\{\langle \xi+\zeta, D\psi(x)+\theta\xi\rangle + F(x, D\psi(x)+\theta\xi) \mid\\&\qquad x \in B(\hat{x},r)\cap S\cap D(\partial\varphi), \xi\in\partial\varphi(x), \zeta\in\mathcal{B}(x)\} \le 0\end{aligned}$$

(resp.,

$$\begin{aligned}&\lambda(m+\psi(\hat{x})-\theta\varphi(\hat{x}))\\&\qquad+\limsup_{r\searrow 0}\{\langle \xi+\zeta, D\psi(x)-\theta\xi\rangle + F(x, D\psi(x)-\theta\xi) \mid\\&\qquad x \in B(\hat{x},r)\cap S\cap D(\partial\varphi), \xi\in\partial\varphi(x), \zeta\in\mathcal{B}(x)\} \ge 0).\end{aligned}$$

A function $u : \mathcal{O} \to \mathbf{R}$ is called a *viscosity solution* of (E) if it is both a viscosity subsolution and a viscosity supersolution of (E) in $\mathcal{O}$.

A comparison result for (E) is the following.

Theorem 2. *Assume that* $(\mathrm{A1})_2$-(A4) *hold. Let u and v be a viscosity subsolution and a viscosity supersolution of* (E), *respectively. Assume that u and $-v$ are bounded above on $D(\varphi)\cap S$. Then $u \le v$ in $D(\varphi)\cap S$. Moreover the inequality*

$$\limsup_{r\searrow 0}\{u(x)-v(y) \mid x,y\in D(\varphi)\cap S, |x-y|\le r\} \le 0.$$

See [6] for the proof of Theorem 2.

3 Existence result

In this section, we introduce the existence theorem for viscosity solutions of (E). Before we state the existence theorem, we give the following proposition which is called Perron's method.

Proposition 1 (Perron's method). *Assume that* (A1)$_\nu$-(A5) *hold for some* $\nu \in (0,2)$. *Let* f *and* g *be a viscosity subsolution and a viscosity supersolution of* (E) *in* S, *respectively. Assume that* $f \leq g$ *in* S. *Define*

$$u(x) = \sup \{v(x) \mid v \textit{ is a viscosity subsolution of } (\mathrm{E}) \textit{ in } S \\ \text{and } f \leq v \leq g \textit{ in } S\} \quad \textit{for} \quad x \in S.$$

Then u *is a viscosity solution of* (E) *in* S.

We need the following two lemmas in order to prove Proposiotion 1. See e.g., [3], [6] for details.

Lemma 1. *Assume that* (A1)$_\nu$-(A4) *hold for some* $\nu \in (0,2)$. *Let* $\mathcal{F}$ *be a nonempty set of viscosity subsolutions of* (E) *in* S *and set* $u(x) = \sup \{v(x) \mid v \in \mathcal{F}\}$ *for* $(x) \in S$. *Then, if* u *is locally bounded in* S, *then* u *is a viscosity subsolution of* (E) *in* S.

See [3] for the proof of the above lemma.

Lemma 2. *Assume that* (A5) *holds. Let* $\delta \geq 0$ *and* $v \in C^1(S)$. *Set* $u(x) = v(x) - \delta\varphi(x)$ *and suppose that*

$$\lambda u(x) + \langle \xi + \zeta, Dv(x) - \delta\xi \rangle + F(x, Dv(x) - \delta\xi) \leq 0$$

holds for all $x \in S \cap D(\partial\varphi)$, $\xi \in \partial\varphi(x)$ *and* $\zeta \in \mathcal{B}(x)$. *Then* u *is a viscosity subsolution of* (E) *in* S. *Similarly, if* $u(x) = v(x) + \delta\varphi(x)$ *and the inequality*

$$\lambda u(x) + \langle \xi + \zeta, Dv(x) + \delta\xi \rangle + F(x, Dv(x) + \delta\xi) \geq 0$$

holds for all $x \in S \cap D(\partial\varphi)$, $\xi \in \partial\varphi(x)$ *and* $\zeta \in \mathcal{B}(x)$. *Then* u *is a viscosity supersolution of* (E) *in* S.

The proof of this lemma is easy and standard, and is left to the reader.

We obtain the following result by Theorem 2, Proposition 1 and Lemma 2. See [6] for the proof for Theorem 3.

Theorem 3. *Assume that* (A1)$_\nu$-(A6) *hold for some* $\nu \in (0,2)$. *There is a viscosity solution* $u \in BUC(S)$ *of (E).*

We obtain Theorem 1 by Theorem 2 and Theorem 3.

4 An optimal control problem of systems governed by the three-dimensional Navier-Stokes equations

Let Ω be an arbitrary bounded domain in $\mathbf{R}^3$ with smooth boundary. In the usual way we define $\mathbf{L}^2(\Omega)$ and the Sobolev space $\mathbf{H}^m(\Omega)$ composed of three-dimensional vector functions. We use the following spaces of solenoidal vector functions.

$$\mathbf{C}^\infty_{0,\sigma}(\Omega) = \{\mathbf{u} = (u^1, u^2, u^3) : u^i \in C_0^\infty(\Omega) \ (i = 1,2,3), \ \mathrm{div}\ \mathbf{u} = 0\},$$

$$\mathbf{H}_\sigma(\Omega) = \text{the completion of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ under the } \mathbf{L}^2(\Omega) - \text{norm},$$

$$\mathbf{H}^1_\sigma(\Omega) = \text{the completion of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ under the } \mathbf{H}^1(\Omega) - \text{norm}.$$

We denote by P the orthogonal projection operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}_\sigma(\Omega)$.

We define a proper lower semicontinuous convex function φ by

$$\varphi(\mathbf{u}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \mathbf{u}|^2\, dx & \text{if } \mathbf{u} \in \mathbf{H}^1_\sigma(\Omega), \\ +\infty & \text{if } \mathbf{u} \in \mathbf{H}_\sigma(\Omega) \setminus \mathbf{H}^1_\sigma(\Omega), \end{cases}$$

where

$$|\nabla \mathbf{u}|^2 = \sum_{i,j=1}^{3} \left| \frac{\partial u^j}{\partial x_i} \right|^2 .$$

Then it can be shown that $\partial\varphi$ coincides with the Stokes operator A defined by

$$\begin{cases} D(A) &= \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_\sigma(\Omega), \\ A(\mathbf{u}) &= -P\Delta\mathbf{u} \quad \text{for } \ \mathbf{u} \in D(A). \end{cases}$$

We define a nonlinear operator $\mathcal{B}$ in $\mathbf{H}_\sigma(\Omega)$ by

$$\begin{cases} D(\mathcal{B}) = D(\partial\varphi), \\ \mathcal{B}(\mathbf{u}) = P(\mathbf{u}\cdot\nabla)\mathbf{u} \ \text{ for } \ \mathbf{u} \in D(\mathcal{B}). \end{cases}$$

We consider the following initial value problem

$$\text{(IVP)} \quad \begin{cases} \dfrac{dX}{dt}(t) + \partial\varphi(X(t)) + \mathcal{B}(X(t)) \ni g(X(t),t), \quad t \in (0,\infty), \\ X(0) \ = \ x, \end{cases}$$

where $g : \mathbf{H}_\sigma(\Omega) \times [0,\infty) \to \mathbf{H}_\sigma(\Omega)$ is assumed to satisfy
for each $u \in \mathbf{H}_\sigma(\Omega)$, the function $g(u,t)$ of t is measurable;

there is a constant $M > 0$ such that

$$\|g(u,t) - g(v,t)\|_{\mathbf{L}^2(\Omega)} \le M\|u-v\|_{\mathbf{L}^2(\Omega)}$$

for all $u, \ v \in \mathbf{H}_\sigma(\Omega)$ and $0 \le t < \infty$.

Proposition 2. *Under the above assumptions, there exists a positive number δ_0 such that if*

$$\|x\|_{\mathbf{L}^2(\Omega)} + \varphi(x) \le \delta_0 \quad \text{and} \quad \|g\|_{L^\infty(\mathbf{H}_\sigma(\Omega)\times(0,\infty);\mathbf{H}_\sigma(\Omega))} \le \delta_0,$$

(IVP) *has a unique s-strong solution $X(t)$ in $[0,\infty)$, and X satisfies the following* (i) *and* (ii).

(i) *There exists a constant $C(\delta_0)$ such that*

$$C(\delta_0) \to 0 \qquad (\text{as } \ \delta_0 \to 0),$$

and

$$\|X\|_{L^\infty(0,\infty;\mathbf{H}_\sigma(\Omega))} + \sup_{t>0} \varphi(X(t)) \le C(\delta_0).$$

(ii) *Let* $Y(t)$ *be an s-strong solution of* (IVP) *in* $[0,\infty)$ *with* y *in place of* x. *If*

$$\|y\|_{\mathbf{L}^2(\Omega)} + \varphi(y) \le \delta_0,$$

then there is a positive constant m_0 *such that*

$$\|X(t) - Y(t)\|_{\mathbf{L}^2(\Omega)} \le e^{m_0 t}\|x - y\|_{\mathbf{L}^2(\Omega)} \quad \text{for} \quad t \ge 0.$$

See e.g., [4], [5] for the proof of Proposition 2.

Next we consider the following control system:
(NS)

$$\begin{cases} \dfrac{dX}{dt}(t) + \partial\varphi(X(t)) + \mathcal{B}(X(t)) \ni g(X(t),\alpha(t)), & t \in (0,\infty), \\ X(0) \ = \ x. & \end{cases}$$

Let functions $f : \mathbf{H}_\sigma(\Omega) \times \mathbf{H}_\sigma(\Omega) \to \mathbf{R}$ and $g : \mathbf{H}_\sigma(\Omega) \times \mathbf{H}_\sigma(\Omega) \to \mathbf{H}_\sigma(\Omega)$ be given and satisfy the following condition (A7).

(A7) $f \in C(\mathbf{H}_\sigma(\Omega) \times \mathbf{H}_\sigma(\Omega); \mathbf{R})$, $g \in C(\mathbf{H}_\sigma(\Omega) \times \mathbf{H}_\sigma(\Omega); \mathbf{H}_\sigma(\Omega))$ and there is a constant $K > 0$ such that

$$|f(x,z)| \le K, \qquad |f(x,z) - f(y,z)| \le K\|x - y\|_{\mathbf{L}^2(\Omega)}$$

$$\|g(x,z)\|_{\mathbf{L}^2(\Omega)} \le \delta_0, \qquad \|g(x,z) - g(y,z)\|_{\mathbf{L}^2(\Omega)} \le K\|x - y\|_{\mathbf{L}^2(\Omega)}$$

for all $x, y, z \in \mathbf{H}_\sigma(\Omega)$.

We define

$$X_\delta = \{x \in \mathbf{L}^2(\Omega) |\ \|x\|_{\mathbf{L}^2(\Omega)} + \varphi(x) \le \delta\} \quad (0 < \delta \le \delta_0),$$

$$\mathcal{C} = \{\alpha : [0,\infty) \to \mathbf{H}_\sigma(\Omega) |\ \ \alpha(\cdot) \text{ is measurable}\},$$

$$Y = Y(\delta) = \{X(t; x, \alpha) |\ t \ge 0, \quad x \in X_\delta, \quad \alpha \in \mathcal{C}\}.$$

Here $X(t; x, \alpha)$ is the s-strong solution of (NS). If δ is sufficiently small, then the set $\overline{Y}$ is invariant, i.e., the following property is satisfied:

$$X(t; x, \alpha) \in \overline{Y} \qquad \text{for all } x \in \overline{Y}, \alpha \in \mathcal{C} \text{ and } 0 \le t < \infty.$$

Let $x \in \overline{Y}$ and consider the problem of minimizing the cost functional

$$J(x,\alpha) = \int_0^\infty e^{-\lambda t} f(X(t),\alpha(t))\,dt,$$

over all $\alpha \in \mathcal{C}$, where $\lambda > 0$ is a given constant and X is the unique s-strong solution of (NS) in $[0,\infty)$. We define the value function V by

$$V(x) = \inf_{\alpha \in \mathcal{C}} J(x,\alpha) \quad \text{for } x \in \overline{Y}. \tag{1}$$

Proposition 3. *Assume that* (A7) *holds. Then the value function V defined in* (1) *is bounded and uniformly continuous in $\overline{Y}$.*

The proof of this lemma is left to the reader. Also, see, e.g., [3].

The following theorem is proved by the method of dynamic progamming (cf. [3]).

Theorem 4. *Assume that* (A7) *holds. Then the value function V defined in* (1) *is a unique viscosity solution of*

$$\lambda u + \langle \partial\varphi(x) + \mathcal{B}(x), Du\rangle_{\mathbf{L}^2(\Omega)} + F(x,Du) = 0 \quad \text{in } \overline{Y}, \tag{2}$$

where the function F is given by

$$F(x,p) = \sup_{z \in \mathbf{H}_\sigma(\Omega)} \{-\langle g(x,z),p\rangle_{\mathbf{L}^2(\Omega)} - f(x,z)\}. \tag{3}$$

Remark. In case of $S = \overline{Y}$, the above function F satisfies (A2),(A3) and (A4), and moreover, we see that $(\mathrm{A1})_2$ holds. By Theorem 2. it follows that V is the unique viscosity solution of (2).

We need the next lemma to prove Theorem 4.

Lemma 3. (Dynamic Programming Principle). *Assume that* (A7) *holds. For any $x \in \overline{Y}$ and $\tau > 0$, the equality*

$$V(x) = \inf_{\alpha \in \mathcal{C}} \Big\{ \int_0^\tau e^{-\lambda t} f(X(t),\alpha(t))\,dt + e^{-\lambda\tau} V(X(\tau)) \Big\} \tag{4}$$

holds, where $X(t)$ denotes the s-strong solution of (NS) *in* $[0,\infty)$.

The above discussion can be also applied to the two-dimensional Navier-Stokes equations.

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