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WHAT IS SUPERANALYSIS? IS IT NECESSARY? –WHAT IS DONE, WHAT IS LEFT OPEN.

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ABSTRACT. In standard theory of real analysis having relation with PDE(=Partial Differential Equation), we usually take as coefficient fields $\mathbf{R}$ or $\mathbf{C}$. In order to treat "boson" and "fermion" on equal footing, the so-called even (bosonic) and odd (ferminionic) variables are introduced formally in physics literature. To make rigorous such new variables, we introduced the new algebras called Fréchet-Grassmann algebras $\mathfrak{R}$ or $\mathfrak{C}$ which play the role of $\mathbf{R}$ or $\mathbf{C}$, respectively. Over this algebra, we construct elementary and real analysis. In this note, we explain not only the necessity of this new notion and its applications but also the reason why the analysis on the superspace based on the Banach-Grassmann algebras is not so preferable when we apply this analysis to treat the systems of PDE.

1. INTRODUCTION

In this note, I try to explain the necessity of new concept, called superanalysis. Which is started with the desire in physics world to treat photon and electron on the equal footing. Moreover, physicist's treatise of super symmetric quantum mechanics makes it clear the effect of introducing new Grassmann variables.

On the other hand, Manin [41] claimed the need of three directions in geometry of 2000's mathematics, which are, even, odd and arithmetic directions. Here, I explain the two directions of three are appeared very naturally when we are dealing with systems of PDEs without diagonalization procedure.

In §2, we recall the Feynman's problem which claims implicitly the need of the classical mechanics corresponding to the systems of PDE. By using the Chevalley's theorem that (a) every matrices are decomposd by Clifford algebras, and (b) the Clifford algebras have representations on Grassmann algebras, we may represent the systems of PDE as the sclar type one but with dependent and independent variables in non-commutative Fréchet-Grassmann algebras. Using this formulation, we give a partial answer to the Feynman's problem. We enumerate problems which may be studied in the same fashion; (i) WKB approximation of the Dirac equation, (ii) a trial to extend the Melin's inequality for positivity of systems of PDE by Sung, (iii) characterization of ellipticity for the systems of PDE, (iv) a generalization of Hopf-Cole transformation by Maslov, (v) whether the Euler equation is attackable by superanalysis?

§3 is devoted to the Witten's treatment of Morse thoery and etc. by using superanalysis. Aharonov and Casher's theorem, retreatise of Atiyah-Singer Index theorem by susy QM, are also proposed by superanalysis.

In §4, we apply this technique to Gaussian Random Matrices and get a precise asymptotic formula for the Wigner's semi-circle law. A beautiful formula given by physicist's are checked from a mathematician's point of view.

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In the final section §5, we recall the problem of Gelfand on dynamical theory and propose a candidate of its solution.

Unfamiliar notion from superanalysis will be seen, for example, in [25, 27, 29, 35].

2. Feynman's problem for spin

2.1. Feynman's path integral representation and his problem. Feynman [16] introduced the expression

$$(2.1)\qquad E(t,s:q,q') = \int_{C_{t,s;q,q'}} [d\gamma]\, e^{i\hbar^{-1}\int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))d\tau}, \quad L(t,\gamma,\dot\gamma) = \frac{1}{2}|\dot\gamma|^2 - V(t,\gamma)$$

$$\text{where}\qquad C_{t,s;q,q'} \sim \{\gamma(\cdot)\in C([s,t]:\mathbf{R}^m)\,|\,\gamma(s)=q',\ \gamma(t)=q\},$$

and rederived the Schrödinger equation, not by substituting $-i\hbar\partial_q$ into $H(t,q,p) = \frac{1}{2}|p|^2 + V(t,q)$. This expression contains the notorious Feynman measure $[d\gamma]$, but this derivation is efficiently used to construct a fundamental solution of the Schrödinger equation for suitable potentials. That is, a Fourier Integral Operator

$$U(t,s)u(q) = (2\pi\hbar)^{-m/2}\int_{\mathbf{R}^m} dq'\, D^{1/2}(t,s;q,q')e^{i\hbar^{-1}S(t,s;q,q')}u(q')$$

gives a "good parametrix" of the Schrödinger equation (shown by Fujiwara [18, 19]). Here, $S(t,s;q,q')$ satisfies the Hamilton-Jacobi equation and $D(t,s;q,q')$, the van Vleck determinant of $S(t,s;q,q')$, satisfies the continuity equation. (**"good parametrix"** means that not only it gives a parametrix but also its dependence on $\hbar$ and its relation to the "classical quantities" are explicit.)

This formula is reformulated (by Inoue [28]) in the Hamiltonian form as

$$U_H(t,s)\,u(q) = (2\pi\hbar)^{-m/2}\int_{\mathbf{R}^m} dp\, {D_H}^{1/2}(t,s;q,p)e^{i\hbar^{-1}S_H(t,s;q,p)}\hat u(p),$$

where, with $(*) = (t,\underline{t};q,p)$ and $(**) = (t,q,S_q(*))$,

$$(\mathrm{H-J})\ \begin{cases} S_t(*) + H(t,q,S_q(*)) = 0, \\ S(\underline{t},\underline{t};q,p) = qp, \end{cases} \quad \text{and} \quad (\mathrm{C})\ \begin{cases} \dfrac{\partial}{\partial t}D(*) + \dfrac{\partial}{\partial q}(D(*)H_p(**)) = 0, \\ D(\underline{t},\underline{t};q,p) = 1. \end{cases}$$

On the other hand, Feynman(-Hibbs) [17] posed the following problem:

> ...... path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

[Problem for system of PDE]: We regard Feynman's problem as calling a **new methodology of solving systems of PDE**. By the way, a system of PDE has **two non-commutativities**,

(i) one from $[\partial_q, q] = 1$ (Heisenberg relation),

(ii) the other from $[A,B]\neq 0$ (A,B: matrices).

Non-commutativity from Heisenberg relation is nicely controlled by using Fourier transformations (the theory of Ψ.D.Op.). Here, we want to give a **new method of treating non-commutativity** $[A,B]\neq 0$; after identifying matrix operations as differential operators and using Fourier transformations, we may develop a theory of Ψ.D.Op. for supersmooth functions on superspace $\mathfrak{R}^{m|n}$.

Dogmatic opinion. For a given system of PDE, if we may reduce that system to scalar PDEs by diagonalization, then we doubt whether it is truely necessary to use matrix representation. Therefore, if we need to represent some equations using matrices, we should try to treat system of PDE as it is, **without diagonalization**. (Remember the Witten model which is represented 2 independently looking equations but if they are treated as a system, that system has supersymmetry.)

Remark. We may consider the method employed here, as a trial to **extend the "method of characteristics" to PDE with matrix-valued coefficients.**

2.2. A partial solution for Feynman's problem. Now, we give a partial answer of this problem by taking the Weyl equation as the simplest model with spin. That is, we rederive the Weyl equation from the Hamiltonian mechanics on superspace (called pseudo classical mechanics). More precisely speaking, **introducing odd variables to decompose the matrix structure**, we define a Hamiltonian function on the superspace from which we construct solutions of the superspace version of the Hamilton-Jacobi and the continuity equations, respectively. (The even and odd variables are assumed to have the inner structure represented by a countable number of Grassmann generators with the Fréchet topology.) Defining a Fourier Integral Operator with phase and amplitude given by these solutions, we may define the good parametrix for the (super) Weyl equation. This means, back to the ordinary matrix-valued representation, that we rederive the Weyl equation and therefore we give a partial solution of Feynman's problem ("partial" because we have not yet constructed an explicit integral representation of the fundametal solution itself).

We reformulate the above problem in mathematical language as follows:

Problem: Find a **"good representation"** of $\psi(t,q) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^2$ satisfying

$$\text{(W)}\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q) = \mathbb{H}(t)\psi(t,q),\\ \psi(\underline{t},q) = \underline{\psi}(q). \end{cases}$$

Here, $\underline{t}$ is arbitrarily fixed and

$$\mathbb{H}(t) = \mathbb{H}\Big(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\Big) = \sum_{k=1}^{3} c\sigma_k\Big(\frac{\hbar}{i}\frac{\partial}{\partial q_k} - \frac{\varepsilon}{c}A_k(t,q)\Big) + \varepsilon A_0(t,q) \tag{2.2}$$

with the Pauli matrices $\{\sigma_j\}$.

In order to get a good parametrix, we transform the Weyl equation (W) on the Euclidian space $\mathbb{R}^3$ with value $\mathbb{C}^2$ to the super Weyl equation (SW) on the superspace $\mathfrak{R}^{3|2}$ with value $\mathfrak{C}$:

$$\text{(SW)}\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,x,\theta) = \mathcal{H}\Big(t,x,\dfrac{\hbar}{i}\dfrac{\partial}{\partial x},\theta,\dfrac{\partial}{\partial\theta}\Big)u(t,x,\theta),\\ u(\underline{t},x,\theta) = \underline{u}(x,\theta). \end{cases}$$

Remark. For example, the operators

$$\sigma_1(\theta,\frac{\partial}{\partial\theta}) = \theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2},\quad \sigma_2(\theta,\frac{\partial}{\partial\theta}) = i(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}),\quad \sigma_3(\theta,\frac{\partial}{\partial\theta}) = 1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2},$$

act on $u(\theta_1,\theta_2) = u_0 + u_1\theta_1\theta_2$ as $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&i\\-i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, respectively.

Theorem 2.1. *Let* $\{A_j(t,q)\}_{j=0}^3 \in C^\infty(\mathbb{R}\times\mathbb{R}^3 : \mathbb{R})$ *satisfy, for any* $k = 0,1,2,\cdots$,

$$|||A_j|||_{k,\infty} = \sup_{t,q,|\gamma|=k} |(1+|q|)^{|\gamma|-1}\partial_q^\gamma A_j(t,q)| < \infty \quad \text{for}\quad j = 0,\cdots,3. \tag{2.3}$$

We have a "good parametrix" for (SW) *represented by*

$$\mathcal{U}(t,\underline{t})\,u(x,\theta) = (2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}} d\xi\,d\pi\,\mathcal{D}^{1/2}(t,\underline{t};x,\theta,\xi,\pi)e^{i\hbar^{-1}\mathcal{S}(t,\underline{t};x,\theta,\xi,\pi)}\mathcal{F}\underline{u}(\xi,\pi).$$

Here, $\mathcal{S}(t,\underline{t};x,\theta,\xi,\pi)$ and $\mathcal{D}(t,\underline{t};x,\theta,\xi,\pi)$ satisfy the Hamilton-Jacobi equation and the continuity equation, respectively:

$$\text{(H-J)}\ \begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}+\mathcal{H}\Big(t,x,\dfrac{\partial \mathcal{S}}{\partial x},\theta,\dfrac{\partial \mathcal{S}}{\partial \theta}\Big)=0,\\ \mathcal{S}(\underline{t},\underline{t};x,\theta,\xi,\pi)=\langle x|\xi\rangle+\langle\theta|\pi\rangle,\end{cases} \quad \textit{and}\quad \text{(C)}\ \begin{cases}\dfrac{\partial}{\partial t}\mathcal{D}+\dfrac{\partial}{\partial x}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big)+\dfrac{\partial}{\partial\theta}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big)=0,\\ \mathcal{D}(\underline{t},\underline{t};x,\theta,\xi,\pi)=1.\end{cases}$$

Here, for $u(x,\theta)=u_0(x)+u_1(x)\theta_1\theta_2$, Fourier transformation $\mathcal{F}$ is defined by

$$\mathcal{F}u(\xi,\pi)=(2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}}dxd\theta\, e^{-i\hbar^{-1}(\langle x|\xi\rangle+\langle\theta|\pi\rangle)}u(x,\theta)=\hbar\hat{u}_1(\xi)+\hbar^{-1}\hat{u}_0(\xi)\pi_1\pi_2.$$

Using the identification maps

$$\sharp: L^2(\mathbb{R}^3:\mathbb{C}^2)\to \mathfrak{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})\quad\text{and}\quad \flat:\mathfrak{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})\to L^2(\mathbb{R}^3:\mathbb{C}^2),$$

$$\sharp\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(x,\theta)=u_0(x)+u_1(x)\theta_1\theta_2\quad\text{with}\quad u_j(x)=\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\partial_q^\alpha\psi_{j+1}(x_{\mathrm B})x_{\mathrm S}^\alpha\quad\text{for}\quad x=x_{\mathrm B}+x_{\mathrm S},\ j=0,1,$$

$$(\flat u)(q)=\begin{pmatrix}\psi_1(q)\\ \psi_2(q)\end{pmatrix}\quad\text{with}\quad \psi_1(q)=u(x,\theta)\big|_{\substack{x=q\\ \theta=0}},\ \psi_2(q)=\frac{\partial^2}{\partial\theta_2\partial\theta_1}u(x,\theta)\big|_{\substack{x=q\\ \theta=0}},$$

we get

Corollary 2.2. *Let $\{A_j(t,q)\}_{j=0}^3\in C^\infty(\mathbb{R}\times\mathbb{R}^3:\mathbb{R})$ satisfy* (2.3). *We have a good parametrix for* (W) *represented by*

$$\mathbf{U}(t,\underline{t})\underline{\psi}(q)=\flat(2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}}d\xi\, d\pi\,\mathcal{D}^{1/2}(t,\underline{t};x,\theta,\xi,\pi)e^{i\hbar^{-1}\mathcal{S}(t,\underline{t};x,\theta,\xi,\pi)}\mathcal{F}(\sharp\underline{\psi})(\xi,\pi)\Big|_{x_{\mathrm B}=q}.$$

An explicit solution: For $\varepsilon=0$, the above formula gives an exact solution for the free Weyl equation.

$$\mathcal{E}(t,0)\underline{u}(\bar{x},\bar{\theta})=(2\pi\hbar)^{-3/2}\hbar\int_{\mathbb{R}^{3|2}}d\underline{\xi}d\underline{\pi}\,\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})^{1/2}e^{i\hbar^{-1}\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})}\mathcal{F}\underline{u}(\underline{\xi},\underline{\pi}).\tag{2.4}$$

Here

$$\begin{aligned}\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})&=\langle\bar{x}|\underline{\xi}\rangle+\big[|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|)-i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)\big]^{-1}\\ &\quad\times\big[|\underline{\xi}|\langle\bar{\theta}|\underline{\pi}\rangle-\hbar\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1+i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2-\hbar^{-1}\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1-i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2\big],\end{aligned}$$

satisfies Hamilton-Jacobi equation, and

$$\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=|\underline{\xi}|^{-2}\big[|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|)-i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)\big]^2,$$

satisfies the continuity equation.

After integrating w.r.t $d\underline{\pi}$ in (2.4), we have

$$u(t,\bar{x},\bar{\theta})=\mathcal{E}(t,0)\underline{u}(\bar{x},\bar{\theta})=u_0(t,\bar{x})+u_1(t,\bar{x})\bar{\theta}_1\bar{\theta}_2$$

with

$$\begin{aligned}u_0(t,\bar{x})&=(2\pi\hbar)^{-3/2}\int_{\mathfrak{R}^{3|0}}d\underline{\xi}\,e^{i\hbar^{-1}\langle\bar{x}|\underline{\xi}\rangle}\{\cos(c\hbar^{-1}t|\underline{\xi}|)\hat{\underline{u}}_0(\underline{\xi})\\ &\qquad -i|\underline{\xi}|^{-1}\sin(c\hbar^{-1}t|\underline{\xi}|)[\underline{\xi}_3\hat{\underline{u}}_0(\underline{\xi})+(\underline{\xi}_1-i\underline{\xi}_2)\hat{\underline{u}}_1(\underline{\xi})]\}\\ u_1(t,\bar{x})&=(2\pi\hbar)^{-3/2}\int_{\mathfrak{R}^{3|0}}d\underline{\xi}\,e^{i\hbar^{-1}\langle\bar{x}|\underline{\xi}\rangle}\{\cos(c\hbar^{-1}t|\underline{\xi}|)\hat{\underline{u}}_1(\underline{\xi})\\ &\qquad -i|\underline{\xi}|^{-1}\sin(c\hbar^{-1}t|\underline{\xi}|)[(\underline{\xi}_1+i\underline{\xi}_2)\hat{\underline{u}}_0(\underline{\xi})-\underline{\xi}_3\hat{\underline{u}}_1(\underline{\xi})]\},\end{aligned}$$

which is equivalent to the following expression.

Proposition 2.3. *For any* $t \in \mathbb{R}$, $\underline{\psi} \in L^2(\mathbb{R}^3 : \mathbb{C})$,

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = (2\pi\hbar)^{-3/2}\int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}qp}e^{-i\hbar^{-1}t\hat{\mathbb{H}}}\hat{\underline{\psi}}(p) = \int_{\mathbb{R}^3} dq'\,\mathbb{E}(t,q,q')\underline{\psi}(q'), \tag{2.5}$$

with

$$\mathbb{E}(t,q,q') = (2\pi\hbar)^{-3}\int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}(q-q')p}\big[\cos(c\hbar^{-1}t|p|)\mathbb{I}_2 - ic^{-1}|p|^{-1}\sin(c\hbar^{-1}t|p|)\hat{\mathbb{H}}\big].$$

Here,

$$\hat{\mathbb{H}} = \hat{\mathbb{H}}(q,p) = \sum_{j=1}^{3} c\sigma_j p_j = c\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}.$$

Important Remark. The reason why we prefer the Fréchet-Grassmann algebra instead of the Banach-Grassmann algebra?

We need the precise estimate of a solution $(x(t),\xi(t),\theta(t),\pi(t))$ of the classical mechanics corresponding to $\mathcal{H}(x,\xi,\theta,\pi)$. For example, to know the dependence of $x(t)$ on the initial data $(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$, we need to prove the following:

Let $|t-\underline{t}| \le 1$. If $|a+b| = 2$ and $k = |\alpha+\beta| = 0,1,2,\cdots$, there exist constants $C_2^{(k)}$ independent of $(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ such that

$$|\pi_{\mathrm{B}}\partial_{\underline{x}}^{\alpha}\partial_{\underline{\xi}}^{\beta}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(x(t,\underline{t}\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) - \underline{x})| \le C_2^{(k)}|t-\underline{t}|^{1+(1/2)(1-(1-k)_+)}.$$

Such a estimate for the ℓ^1-norm for $\partial_{\underline{x}}^{\alpha}\partial_{\underline{\xi}}^{\beta}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(x(t,\underline{t}\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) - \underline{x}) \in \mathfrak{R}_{\mathrm{ev}}$ w.r.t. the Grassmann generators $\{\sigma^I\}_{I\in\mathcal{I}}$ seems extremely complicated.

2.3. Problems in systems of PDE.

2.3.1. *WKB approach to Dirac equation by Pauli, de Broglie, Rubinow & Keller.* The modified Dirac equation with an anomalous magnetic moment,may be written in the form

$$i\hbar\frac{\partial}{\partial t}\psi = [c\alpha_j(\frac{\hbar}{i}\frac{\partial}{\partial q_j} - \frac{e}{c}A_j) + e\Phi + \beta mc^2]\psi + g\frac{ie\hbar}{2mc}F_{kl}(\alpha^k\alpha^l - \alpha^l\alpha^k)\psi \tag{2.6}$$

where

$$F_{kl} = \frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k}.$$

Pauli tried to have a solution in the following form:

$$\psi \sim e^{i\hbar^{-1}S}\sum_{n=0}^{\infty}(-i\hbar)^n a_n,$$

where S is a scalar function, a_n are matrix-valued functions. Though Pauli didn't decide the all terms completely, his procedure yields the correct result in inhomogeneous field regions and fixed finite distances from them, but not at all distances of the order $\hbar^{-1}$ from them, so claimd in Rubinow and Keller [48].

Our problem is to apply our method to the superversion of (2.6) and to get the corresponding result mathematically.

For the case of the free Dirac equation, that is, when $A_j = \Phi = 0$, we have the result [27]: Given $\underline{\psi}(q)$, find a good representation of $\psi(t,q) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4$, satisfying

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,q) = \mathbb{H}\psi(t,q) \\ \psi(0,q) = \underline{\psi}(q) \end{cases} \tag{2.7}$$

with

$$\mathbb{H} = -i\hbar c\alpha_k\frac{\partial}{\partial q_k} + mc^2\beta.$$

Here, $\hbar$ is the Planck's constant, c, m are constants, $\psi(t,q) = {}^t(\psi_1(t,q), \psi_2(t,q), \psi_3(t,q), \psi_4(t,q$
summation with respect to $k = 1,2,3$ is abbrebiated, and the matrices $\{\alpha_k, \beta\}$ satisfy the Clifford re

(2.8) $$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbb{I}_4, \quad \alpha_k\beta + \beta\alpha_k = 0, \quad \beta^2 = \mathbb{I}_4 \quad j,k = 1,2,3.$$

In the following, we use the Dirac representation of matrices

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3.$$

Applying formally the Fourier transformation with respect to $q \in \mathbf{R}^3$ to (2.7), we get

(2.9) $$i\hbar\frac{\partial}{\partial t}\hat{\psi}(t,p) = \mathbb{H}(p)\hat{\psi}(t,p)$$

where

(2.10) $$\mathbb{H}(p) = c\alpha_j p_j + mc^2\beta = c\begin{pmatrix} mc & 0 & p_3 & p_1 - ip_2 \\ 0 & mc & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & -mc & 0 \\ p_1 + ip_2 & -p_3 & 0 & -mc \end{pmatrix}.$$

Remarking $\mathbb{H}^2(p) = c^2\|p\|^2\mathbb{I}_4$ with $\|p\| = \sqrt{m^2c^2 + |p|^2}$, we have,

(2.11) $$e^{-i\hbar^{-1}t\mathbb{H}(p)} = \cos(c\hbar^{-1}t\|p\|)\mathbb{I}_4 - \frac{i}{c\|p\|}\sin(c\hbar^{-1}t\|p\|)\mathbb{H}(p).$$

Therefore, we have readily

Proposition 2.4. *For any $t \in \mathbf{R}$ and $\underline{\psi} \in L^2(\mathbf{R}^3 : \mathbf{C})^4 = L^2(\mathbf{R}^3 : \mathbf{C}^4)$,*

(2.12) $$\psi(t,q) = e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = (2\pi\hbar)^{-3/2}\int_{\mathbf{R}^3} dp\, e^{i\hbar^{-1}qp}e^{-i\hbar^{-1}t\mathbb{H}(p)}\hat{\underline{\psi}}(p).$$

For $\underline{\psi} \in \mathcal{S}(\mathbf{R}^3 : \mathbf{C})^4$, we have formally

(2.13) $$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = \int_{\mathbf{R}^3} dq'\, \mathbb{E}(t, q - q')\underline{\psi}(q')$$

with

(2.14) $$\mathbb{E}(t,q) = (2\pi\hbar)^{-3}\int_{\mathbf{R}^3} dp\, e^{i\hbar^{-1}qp}\big[\cos(c\hbar^{-1}t\|p\|)\mathbb{I}_4 - \frac{i}{c\|p\|}\sin(c\hbar^{-1}t\|p\|)\mathbb{H}(p)\big] \in \mathcal{S}'(\mathbf{R}^3 : \mathbf{C})^4.$$

Applying our analysis on superspace, we have the following.

Theorem 2.5 (Path-integral representation of a solution for the free Dirac equation).

(2.15) $$\psi(t,q) = \mathrm{b}\Big((2\pi)^{-3/2}e^{\pi i/4}\iint_{\mathfrak{R}^{3|3}} d\underline{\xi}d\underline{\pi}\, \mathcal{D}^{1/2}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}(\sharp\psi)(\underline{\xi},\underline{\pi})\Big)\Big|_{\bar{x}_{\mathrm{B}}=q}.$$

Here, $\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathcal{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are given by

(2.16) $$\begin{aligned} &\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \langle\bar{x}|\underline{\xi}\rangle + \langle\bar{\theta}|\underline{\pi}\rangle + \bar{\mathcal{B}}(t)[2imc\bar{\theta}_3\underline{\pi}_3 + (\hbar\bar{\Theta} - i\underline{\Pi})(\bar{\theta}_3 + i\hbar^{-1}\underline{\pi}_3)], \\ &\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \bar{\delta}(t), \end{aligned}$$

where

$$\bar{\mathcal{B}}(t) = \bar{\mathcal{A}}(t)\bar{\delta}^{-1}(t), \quad \bar{\mathcal{A}}(t) = \mathrm{a}(t) - 2imc\mathrm{b}(t), \quad \bar{\delta}(t) = 1 - 2\mathrm{b}(t)|\underline{\xi}|^2 - 2imc\bar{\mathcal{A}}(t),$$
$$\mathrm{a}(t) = \frac{\sin 2\nu t}{2\|\underline{\xi}\|}, \quad \mathrm{b}(t) = \frac{1 - \cos 2\nu t}{4\|\underline{\xi}\|^2} \quad \nu = c\hbar^{-1}\|\underline{\xi}\|, \quad \|\underline{\xi}\|^2 = |\underline{\xi}|^2 + m^2c^2 \quad \textit{and}$$
$$\bar{\Theta} = (\underline{\xi}_1 + i\underline{\xi}_2)\bar{\theta}_1 - \underline{\xi}_3\bar{\theta}_2, \quad \underline{\Pi} = (\underline{\xi}_1 - i\underline{\xi}_2)\underline{\pi}_1 - \underline{\xi}_3\underline{\pi}_2.$$

Moreover, $\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathcal{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are solutions of the Hamilton-Jacobi equation

$$(2.17)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) + \mathcal{H}\Big(\dfrac{\partial \mathcal{S}}{\partial \bar{x}},\bar{\theta},\dfrac{\partial \mathcal{S}}{\partial \bar{\theta}}\Big) = 0, \\ \mathcal{S}(0,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \langle \bar{x}|\underline{\xi}\rangle + \langle \bar{\theta}|\underline{\pi}\rangle, \end{cases}$$

and the continuity equation,

$$(2.18)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathcal{D} + \dfrac{\partial}{\partial \bar{x}}\Big(\mathcal{D}\dfrac{\partial \mathcal{H}}{\partial \xi}\Big) + \dfrac{\partial}{\partial \bar{\theta}}\Big(\mathcal{D}\dfrac{\partial \mathcal{H}}{\partial \pi}\Big) = 0, \\ \mathcal{D}(0,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = 1, \end{cases}$$

respectively. In the above, the argument of $\mathcal{D}$ is $(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$, while those of $\mathcal{H}_\xi$ and $\mathcal{H}_\pi$ are $(\mathcal{S}_{\bar{x}},\bar{\theta},\mathcal{S}_{\bar{\theta}})$, respectively. $\mathcal{F}$ is the Fourier transformation for functions on the superspace $\mathfrak{R}^{3|3}$.

Problem. Extend the procedure mentioned above for the free Dirac equation to (2.6) (hint: see [29, 30] which treats the analogous case for the Weyl equation).

2.3.2. *Sung's example for Melin's inequality for system of PDE.* Let $H(q,p) = \sum_{|\alpha+\beta|\le 2} a_{\alpha\beta}q^\alpha p^\beta$ where $a_{\alpha\beta} \in \mathbb{R}$ and $(q,p) \in \mathbb{R}^{2n}$. Let $H_2(q,p) = \sum_{|\alpha+\beta|=2} a_{\alpha\beta}q^\alpha p^\beta$ and $P((q,p),(q',p'))$ be the polarized form of $H_2(q,p)$. Let $\sigma(\cdot,\cdot)$ be the standard symplectic form on $\mathbb{R}^{2n}$. F is the Hamiltonian map of H_2 defined by $\sigma((q,p),F(q',p')) = P((q,p),(q',p'))$ and $\mathrm{tr}^+ p_2$ is defined as the sum of the positive eigenvalues of $-iF$.

Let

$$H^W(q,D_q)u(q) = (2\pi)^{-2n}\iint dq'dp\,H\Big(\frac{q+q'}{2},p\Big)e^{i(q-q')p}u(q') \quad \text{for} \quad u\in\mathcal{S}(\mathbb{R}^n).$$

Theorem 2.6 (Melin). *$\langle H^W(q,D_q)u,u\rangle \ge 0$ for any $u\in\mathcal{S}(\mathbb{R}^n)$ if and only if $\inf H(q,p)+\mathrm{tr}^+ H_2 \ge 0$. In particular, if $H(q,\xi)\ge 0$, then $H^W(q,D)\ge 0$.*

This claim is not generalized straight fowardly to the system of PDE:

Example.(Hörmander [24]). Let

$$\mathbb{P}(q,p) = \begin{pmatrix} q^2 & qp \\ qp & p^2 \end{pmatrix} \quad \text{for} \quad (q,p)\in\mathbb{R}^2,$$

then $\mathbb{P}(q,p)\ge 0$ but for $u_1 = v''$, $u_2 = i(v-qv')$ and $0 \not\equiv v\in\mathcal{S}(\mathbb{R})$,

$$\langle \mathbb{P}^W(q,D_q)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\rangle = -\frac{1}{2}\int dq\,(v')^2 < 0.$$

Problem. Is it posssible to characterize vectors v such that $\langle \mathbb{P}^W(q,D_q)v,v\rangle \le 0$?

Let

$$\mathbb{H}(q,p) = \begin{pmatrix} aq^2+bp^2 & \alpha qp \\ \alpha qp & cq^2+dp^2 \end{pmatrix} \quad \text{for} \quad (q,p)\in\mathbb{R}^2,\ a,\ b,\ c,\ d\ge 0 \quad \text{and} \quad ad+bc\ne 0.$$

Theorem 2.7 (Sung[52]). *Let a, b, c, $d\ge 0$ and $ad+bc\ne 0$. For $\mathbb{H}^W(q,D_q)\ge 0$, it is neccessary and sufficient that $(\lambda_1,\lambda_2)\in\Omega$ or $(\lambda_2,\lambda_1)\in\Omega$ where*

$$\lambda_1 = \frac{\sqrt{ad}-\sqrt{bc}+\alpha}{\sqrt{ad}+\sqrt{bc}},\quad \lambda_2 = \frac{\sqrt{ad}-\sqrt{bc}-\alpha}{\sqrt{ad}+\sqrt{bc}},\quad \Omega = \{(x,y)\,|\,N(x,y)\ge 0\},$$

and

$$N(x,y)=\begin{pmatrix} 1 & \zeta_0 x & 0 & 0 & 0 & 0 & \cdots \\ \zeta_0 x & 1 & \zeta_1 y & 0 & 0 & 0 & \cdots \\ 0 & \zeta_1 y & 1 & \zeta_2 x & 0 & 0 & \cdots \\ 0 & 0 & \zeta_2 x & 1 & \zeta_3 y & 0 & \cdots \\ 0 & 0 & 0 & \zeta_3 y & 1 & \zeta_4 x & \cdots \\ 0 & 0 & 0 & 0 & \zeta_4 x & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \textit{with} \quad \zeta_n=\Big(\frac{(2n+1)(2n+2)}{(4n+1)(4n+5)}\Big)^{1/2}.$$

Problems. (1) Construct a good parametrix for the following operators:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{H}^W(q,-i\hbar\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{H}^W(q,-i\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{H}^W(q,\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

(2) Extend the result of Sung to more general positive definite matrices? Find the condition like Melin's characterization.

2.3.3. *Gelfand's question for the meaning of ellipticity.* Let a matrix be given by

$$\mathbb{B}(p)=\begin{pmatrix}p_1^2-p_2^2 & -2p_1p_2\\ 2p_1p_2 & p_1^2-p_2^2\end{pmatrix}$$

which is weakly but not strongly elliptic system. How about the characteristic behavior of the solution caused by "weakly but not strongly elliptic system" of the following equations?

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{B}^W(-i\hbar\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{B}^W(-i\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\mathbb{B}^W(\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

Problem. Can we characterize the ellipticity of the systems of PDE by checking the behavior of solutions of the heat type for $t\to\infty$?

2.3.4. *Is the Euler equation attackable by superanalysis?* The Euler equation on $\mathbf{R}^3$ is given by

$$\text{(2.19)}\qquad \begin{cases} u_t+(u\cdot\nabla)u+\nabla p=0,\\ \operatorname{div}u=0,\\ u(0,x)=\underline{u}(x), \quad \text{where} \quad u={}^t(u_1(t,x),u_2(t,x),u_3(t,x)).\end{cases}$$

This equation is the one of the most charming one which is not solved for the long time.

Taking the rotation $du=v$, we get

$$\text{(2.20)}\qquad \begin{cases} v_t+(u\cdot\nabla)v=(v\cdot\nabla)u,\\ v(0,x)=\underline{v}(x).\end{cases}$$

Putting $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = du = \begin{pmatrix} u_{2,3} - u_{3,2} \\ u_{3,1} - u_{1,3} \\ u_{1,2} - u_{2,1} \end{pmatrix}$, $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$, we have, for each $i = 1, 2, 3$,

$$\sum_{j=1}^{3} v_j u_{i,j} = \sum_{j=1}^{3} d_{ij} v_j \quad \text{where} \quad d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.21}$$

$D = (d_{ij})$ is called the deformation matrix of the fluid flow with $\sum_{i=1}^{3} d_{ii} = \operatorname{div} u = 0$.

Therefore

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \sum_{j=1}^{3} u_j \mathbb{I}_3 \frac{\partial}{\partial x_j}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{2.22}$$

Problem. The above equation (2.20) in $\mathbb{R}^2$ has no right-hand side and solved nicely which garantees the classical solution for (2.19) in dimension 2. In spite of this fact, whether one can make use of the solution of this vorticity equation nicely to the Euler equation in $\mathbb{R}^3$?

On the other hand, it is well-known that we may apply the method of characteristics to

$$\sum_{j=1}^{n} a_j(q,u)\mathbb{I}_l \frac{\partial u_k}{\partial q_j} = b_k(q,u) \quad \text{for} \quad k = 1, 2, \cdots, l, \tag{2.23}$$

assuming $(a_1(q,u), \cdots, a_n(q,u)) \neq 0$.

Especially, we have the following:

Theorem 2.8. *Let $a_j(t,q)$ be C^1 near $(\underline{t}, \underline{q})$, and let $b_k(t,q,u)$ be C^1 near $(\underline{t}, \underline{q}, \underline{u})$, $\underline{u} = \phi(\underline{q})$, and ϕ is C^1 near $\underline{q}$. If $q = x(t, \underline{t}\,; \underline{q})$ is a solution of*

$$\dot{q}_j = a_j(t,q), \quad q_j(\underline{t}, \underline{t}\,; \underline{q}) = \underline{q}_j,$$

and $U(t,\underline{q}) = (U_1(t,\underline{q}), \cdots, U_l(t,\underline{q}))$ is a solution of

$$\dot{U}_k = b_k(t, x(t,\underline{t}\,;\underline{q}), U), \quad U_k(\underline{t}, \underline{q}) = \phi_k(\underline{q}).$$

Putting $u(t,\bar{q}) = U(t, y(t,\underline{t}\,;\bar{q}))$ where $y = y(t,\underline{t}\,;\bar{q})$ is the inverse function of $\bar{q} = x(t,\underline{t}\,;\underline{q})$, then it satisfies

$$\frac{\partial u_k}{\partial t} + \sum_{j=1}^{n} a_j(t,q)\mathbb{I}_l \frac{\partial u_k}{\partial q_j} = b_k(t,q,u) \quad \textit{with} \quad u(\underline{t},\underline{q}) = \phi(\underline{q}) \tag{2.24}$$

Probelm. Extends the above theorem to the case $a_j(t,q)$ are $l \times l$-matrices.

2.3.5. *The generalized Hopf-Cole transformation of Maslov.* Let $V(t,q) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 : \mathbb{R})$ be given. For a solution $\psi \in C^2(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ satisfying

$$\begin{cases} \nu\psi_t = \dfrac{\nu^2}{2}\Delta\psi + V\psi, \\ \psi(0) = \underline{\psi} = e^{-\nu^{-1}\phi}, \end{cases} \tag{2.25}$$

we put $u(t,q) = -\nu\nabla\log\psi(t,q)$, that is, $u = {}^t(u_1,u_2,u_3) = {}^t(-\nu\frac{\psi_{q_1}}{\psi}, -\nu\frac{\psi_{q_2}}{\psi}, -\nu\frac{\psi_{q_3}}{\psi})$. Then, u satisfies

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla V = \dfrac{\nu}{2}\Delta u, \\ u(0) = \nabla\phi. \end{cases} \tag{2.26}$$

Example. Let $V(t,q) = \sum_{j=1}^{3} \frac{1}{2}\omega_j^2 q_j^2$. We have a solution of (2.25) as

$$\psi(t,\bar{q}) = (E_t\underline{\psi})(\bar{q}) = (2\pi\nu)^{-3/2}\int_{\mathbb{R}^3} d\underline{q}\, D(t,\bar{q},\underline{q})^{1/2} e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})$$
$$= \prod_{j=1}^{3}\left(\frac{\omega_j}{2\pi\nu\sin\omega_j t}\right)^{1/2}\int_{\mathbb{R}^3} d\underline{q}\, e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q}).$$

Here, we put

$$S(t,\bar{q},\underline{q}) = \sum_{j=1}^{3}\left[\frac{\omega_j}{2}(\cot\omega_j t)(\underline{q}_j^2 + \bar{q}_j^2) - \frac{\omega_j}{\sin\omega_j t}\underline{q}_j\bar{q}_j\right] \quad \text{and} \quad D(t,\bar{q},\underline{q}) = \prod_{j=1}^{3}\frac{\omega_j}{\nu\sin\omega_j t}.$$

Therefore, we get

$$u_j(t,\bar{q}) = \frac{\int_{\mathbb{R}^3} d\underline{q}\,(\omega_j\cot\omega_j t\bar{q}_j - \frac{\omega_j}{\sin\omega_j t}\underline{q}_j)e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})}{\int_{\mathbb{R}^3} d\underline{q}\, e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})}$$
$$= \omega_j\cot\omega_j t\bar{q}_j - \frac{\int_{\mathbb{R}^3} d\underline{q}\,\frac{\omega_j}{\sin\omega_j t}\underline{q}_j e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})}{\int_{\mathbb{R}^3} d\underline{q}\, e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})}.$$

Taking especially

$$\phi(q) = \frac{1}{2}\phi_{jk}q_j q_k,$$

we calculate explicitly as

$$u_j(t,\bar{q}) \to \quad \text{I lost the result} \quad \text{when} \quad \nu \to 0.$$

Problem. Does there exists Ehrenfest type theorems for the above (2.25) and what does it imply in (2.26)? (see, Hepp [23]).

3. Witten's approach

3.1. Morse theory from susyQM.

Definition 3.1. *Let $\mathbb{H}$ be a Hilbert space and let H and Q be selfadjoint operators, and P be a bounded self-adjoint operator in $\mathbb{H}$ such that*

$$H = Q^2 \geq 0, \quad P^2 = I, \quad [Q,P]_+ = QP + PQ = 0.$$

Then, we say that the system (H,P,Q) has supersymmetry or it defines a susyQM(=supersymmetric Quantum Mechanics).

Under this circumstance, we may decompose

$$\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f \quad \text{where} \quad \mathbb{H}_f = \{u \in \mathbb{H} \,|\, Pu = -u\}, \quad \mathbb{H}_b = \{u \in \mathbb{H} \,|\, Pu = u\}.$$

Using this decomposition and identifying an element $u = u_b + u_f \in \mathbb{H}$ as a vector $\begin{pmatrix} u_b \\ u_f \end{pmatrix}$, we have a representation

$$P = \begin{pmatrix} I_b & 0 \\ 0 & -I_f \end{pmatrix} = (\text{or simply denoted by}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since P and Q anti-commute and Q is self-adjoint, Q has always the form

$$Q = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}, \tag{3.1}$$

where A, called the annihilation operator, is an operator which maps $\mathbb{H}_b$ into $\mathbb{H}_f$, and its adjoint A^*, called the creation operator, maps $\mathbb{H}_f$ into $\mathbb{H}_b$. Thus, P commutes with H, and $\mathbb{H}_b$ and $\mathbb{H}_f$ are invariant under H, i.e. $H\mathbb{H}_b \subset \mathbb{H}_b$ and $H\mathbb{H}_f \subset \mathbb{H}_f$. That is, there is a one-to-one correspondence between densely defined closd operators A and self-adjoint operators Q (supercharges) of the above form.

Definition 3.2. *We define a supersymmetric index of H if it exists by*

$$\operatorname{ind}_s(H) \equiv \dim(\operatorname{Ker}(H|\mathbb{H}_b)) - \dim(\operatorname{Ker}(H|\mathbb{H}_f)) \in \bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}.$$

Remark. If the operator A is semi-Fredholm, we have the relation

$$\operatorname{ind}_s(H) = \operatorname{ind}_F(A) \equiv \dim(\operatorname{Ker} A) - \dim(\operatorname{Ker} A^*).$$

Corollary 3.1 (Spectral supersymmetry). *The operator A^*A on $(\ker A)^\perp$ is unitarily equivalent to the operator AA^* on $(\ker A^*)^\perp$. In particular, the spectra of A^*A and AA^* are equal away from zero,*

$$\sigma(A^*A) \setminus \{0\} = \sigma(AA^*) \setminus \{0\}.$$

Proposition 3.2. *For any supercharge Q and any bouned continuous function f defined on $D(Q)$, we have*

$$Qf(Q^2) = f(Q^2)Q, \quad f(Q^2) = \begin{pmatrix} f(A^*A) & 0 \\ 0 & f(AA^*) \end{pmatrix},$$
$$f(A^*A)A^* = A^*f(AA^*), \quad f(AA^*)A = Af(A^*A).$$

In order to check whether the supersymmetry is broken or unbroken, E. Witten [59] introduced the so-called Witten index.

Definition 3.3. *Let (H, P, Q) be susyQM with* (3.1).
(I) Putting, for $t > 0$

$$\Delta_t(H) = \operatorname{tr}(e^{-tA^*A} - e^{-tAA^*}) = \operatorname{str} e^{-tH},$$

we define, if the limit exists, the (heat kernel regulated) Witten index W_H of (H, P, Q) by

$$W_H = \lim_{t\to\infty} \Delta_t(H).$$

We define also the (heat kernel regulated) axial anomaly $\mathcal{A}_H$ of (H, P, Q) by

$$\mathcal{A}_H = \lim_{t\to 0} \Delta_t(H).$$

(II) Putting, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$\Delta_z(H) = -z \operatorname{tr}[(A^*A - z)^{-1} - (AA^* - z)^{-1}] = -z \operatorname{str}(H - z)^{-1},$$

we define, if the limit exists, the (resolvent regulated) Witten index W_R of (H, P, Q) by

$$W_R = \lim_{\substack{z\to 0 \\ |\Re z| \le C_0 |\Im z|}} \Delta_z(H) \quad \text{for some } C_0 > 0.$$

Similarly, we define the (resolvent regulated) axial anomaly $\mathcal{A}_R$ by

$$\mathcal{A}_R = -\lim_{\substack{z\to \infty \\ |\Re z| \le C_1 |\Im z|}} \Delta_z(H) \quad \text{for some } C_1 > 0.$$

We have

Theorem 3.3. *Let Q be a supercharge on $\mathcal{H}$. If $\exp(-tQ^2)$ is trace class for some $t > 0$, then Q is Fredholm and*

$$\operatorname{ind}_t(Q)(\text{independent of } t) = \operatorname{ind}_F(Q) = \operatorname{ind}_s(H).$$

If $(Q^2 - z)^{-1}$ is trace class for some $Z \in \mathbb{C} \setminus [0, \infty)$, then Q is Fredholm and

$$\operatorname{ind}_z(Q)(\text{independent of } z) = \operatorname{ind}_F(Q) = \operatorname{ind}_s(H).$$

In the next subsection, we consider the case where A is not semi-Fredholm. To treat this case, Bollé et al [8] introduced the notion of Krein's spectral shift function which is not presented here.

Example 1. Let (M,g), $g = \sum_{i,j=1}^d g_{ij}(q)dq^i dq^j$ be a d-dimensional smooth Riemannian manifold. We put $\Lambda(M) = \cup_{k=0}^d \Lambda^k(M)$ or $\Lambda_0(M) = \cup_{k=0}^d \Lambda_0^k(M)$, where

$$\Lambda^k(M) = \{\omega = \sum_{1\le i_1<\cdots<i_k\le m} \omega_{i_1\cdots i_k}(q)dq^{i_1}\wedge\cdots\wedge dq^{i_k} \,|\, \omega_{i_1\cdots i_k}(q)\in C^\infty(M:\mathbb{C})\},$$

$$\Lambda_0^k(M) = \{\omega\in\Lambda^k(M)\,|\,\omega_{i_1\cdots i_k}(q)\in C_0^\infty(M:\mathbb{C})\},\quad \tilde\Lambda^k(M) = \{\omega\in\Lambda^k(M)\,|\,\|\omega\|<\infty\}.$$

Let d be an exterior differential acting on $\omega_{i_1\cdots i_k}(q)dq^{i_1}\wedge\cdots\wedge dq^{i_k}$ as

$$d\omega = \sum_{j=1}^d \frac{\partial\omega_{i_1\cdots i_k}(q)}{\partial q^j}dq^j\wedge dq^{i_1}\wedge\cdots\wedge dq^{i_k}.$$

P is defined by $P\omega = (-1)^k\omega$ for $\omega\in\Lambda^k(M)$.

Put $\mathbb{H} = \overline{\Lambda(M)}$ where $\overline{\Lambda(M)} = \cup_{k=0}^d \overline{\Lambda^k(M)}$ with $\overline{\Lambda^k(M)}$ is the closure of $\tilde\Lambda^k(M)$ in L^2-norm $\|\cdot\|$. Denoting the adjoint of d in $\overline{\Lambda(M)}$ by d^* and putting

$$Q_1 = d+d^*,\quad Q_2 = i(d-d^*),\quad H = Q_1^2 = Q_2^2 = dd^*+d^*d,$$

we have that (H,Q_α,P) has the supersymmetry on $\mathbb{H}$ for each $\alpha=1,2$.

Example 2 (Witten's deformed Laplacian [59]). For any real-valued function ϕ on M, we put

$$d_\lambda = e^{-\lambda\phi}de^{\lambda\phi},\quad d_\lambda^* = e^{\lambda\phi}d^*e^{-\lambda\phi}$$

where λ is a real parameter. We have $d_\lambda^2 = 0 = d_\lambda^{*2}$.

$$Q_{1\lambda} = d_\lambda + d_\lambda^*,\quad Q_{2\lambda} = i(d_\lambda - d_\lambda^*),\quad H_\lambda = d_\lambda d_\lambda^* + d_\lambda^* d_\lambda.$$

Defining P as before, we have the supersymmetric system (H_λ,Q_α,P) on $\mathbb{H}$ for each $\alpha=1,2$.

Using the above deformed Laplacian, Witten rederived the Morse theory which is outside the scope of our mathematical power to be treated rigorously.

The most important thing of his rederivation is to regard the operator H_λ as the quantized one from the action

$$\mathcal{S}_\lambda = \frac12\int dt\left[g_{ij}\Big(\frac{dq^i}{dt}\frac{dq^j}{dt} + i\bar\psi^i\frac{D\psi^j}{Dt}\Big) + \frac14 R_{ijkl}\bar\psi^i\psi^k\bar\psi^j\psi^l - \lambda^2 g^{ij}\frac{\partial\phi}{\partial q^i}\frac{\partial\phi}{\partial q^j} - \lambda\frac{D^2\phi}{Dq^iDq^j}\bar\psi^i\psi^j\right],$$

where

$$\frac{D\psi^j}{Dt} = \frac{d\psi^j}{dt} + \Gamma^j_{kl}\dot q^k\psi^l,\quad \frac{D^2\phi}{Dq^iDq^j} = \frac{\partial^2\phi}{\partial q^i\partial q^j} - \Gamma^l_{ij}\frac{\partial\phi}{\partial q^l}.$$

That is, to consider the path-integral

$$\int[dq][d\psi][d\bar\psi]\,e^{-\mathcal{S}_\lambda}, \tag{3.2}$$

and its "generator" which is the Hamiltonian H_λ to be obtained. Here, we used the summation convention and ψ^i and $\bar\psi^i$ are anti-commuting fields tangent to M, which becomes the creation and annihilation operators after quantization.

Instanton or tunneling paths satisfying the classical mechanics defined by

$$\begin{aligned}\bar{\mathcal{S}}_\lambda &= \frac12\int dt\left(g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} + \lambda^2 g^{ij}\frac{\partial\phi}{\partial q^i}\frac{\partial\phi}{\partial q^j}\right)\\ &= \frac12\int dt\left|\frac{dq^i}{dt}\pm\lambda g^{ij}\frac{\partial\phi}{\partial q^j}\right|^2 \mp\lambda\int dt\frac{d\phi}{dt}\quad\text{where}\quad |b^i|^2 = g_{ij}b^ib^j,\end{aligned}$$

give the main contribution to the behavior of (3.2) when $\lambda\to\infty$. This is a typical example of physicist's usage of the stationary or steepest descent method to path-integral, which is beyond the mathmatical power existing. But, in this case at hand, we are in the way of giving the mathmatical proof of Witten's procedure by constructing "good parametrix" for a system of heat type equations.

3.2. Atiyah-Singer index theorem by path-integral. Alvarez-Gaumé [2] gave a formal expression below which gives Gauss-Bonnet-Chern theorem:

Let (M,g) be a smooth Riemannian manifold of dimension d whose Ricci curvature is denoted by $R_{ijk\ell}$. We may extend the Riemannian metric $g = \frac{1}{2}g_{ij}(q)dq^i dq^j$ to the supersymmetric one on the supermanifold $\tilde{M}$. More precisely, for a local patch $U \subset M \to \iota(U) \sim U \subset \mathbb{R}^d$, we take $\tilde{U} = \{(x,\theta) \in \mathfrak{R}^{d|d} \,|\, \pi_{\mathrm{B}} x \in U\}$. Glueing these patches suitably, we get $\tilde{M}$.

For a given Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j \in C^\infty(T\mathbb{R}^d : \mathbb{R}),$$

we get as a supersymmetric extension, following physicist's prescription,

$$\mathcal{L}(x,\dot{x},\psi,\bar{\psi}) = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k + \frac{i}{2}g_{jk}(\psi^j\frac{D\bar{\psi}^k}{dt} + \bar{\psi}^j\frac{D\psi^k}{Dt}) - \frac{1}{4}R_{ijkl}\psi^i\bar{\psi}^j\psi^k\bar{\psi}^l \tag{3.3}$$

In other word, we may define a supersymmetric Hamiltonian $\mathcal{H}(x,\xi,\theta,\pi)$ of $H(q,p)$ by

$$\mathcal{H}(x,\xi,\theta,\pi) = \frac{1}{2}g^{ij}(\xi_i - \frac{i}{2}(g_{ik,l}-g_{il,k})\theta^k\pi^l)(\xi_j - \frac{i}{2}(g_{jm,n}-g_{jn,m})\theta^m\pi^n) + \frac{1}{2}R_{ikjl}\theta^j\theta^l\pi^i\pi^k$$

which belongs to $\mathcal{C}_{\mathrm{SS}}(\mathfrak{R}^{2d|2d} : \mathfrak{R}_{\mathrm{ev}})$. Here, the functions $g^{ij} = g^{ij}(x)$ of $x \in \mathfrak{R}^{d|0}$ etc. appeared above are Grassmann extensions of the corresponding ones $g^{ij} = g^{ij}(q)$ of $q \in \mathbb{R}^d$ etc.

Then, this $(\tilde{M},\mathcal{L})$ gives a susyQM whose susy-index is formally expressed by

$$\mathrm{ind}_s(\mathcal{H}) = \mathrm{tr}\,(-1)^F e^{-\beta\mathcal{H}} = \int_{PBC}[d\gamma][d\psi][d\bar{\psi}]e^{-\int_0^\beta dt\,\mathcal{L}(\gamma(t),\dot{\gamma}(t),\psi(t),\bar{\psi}(t))},$$

where PBC stands for the periodic boundary condition with period β, that is, $\gamma(t+\beta) = \gamma(t)$, $\psi(t+\beta) = \psi(t)$ and $\bar{\psi}(t+\beta) = \bar{\psi}(t)$. By its very definition of susy index, this gives us the Euler number $\chi(M)$. On the other hand, independence of the above quantity $\mathrm{tr}\,(-1)^F e^{-\beta H}$ w.r.t. β and the good parametrix of $e^{-\beta\mathcal{H}}$ gives the density of Gauss-Bonnet-Chern.

3.3. Aharonov-Casher's theorem and related topics. Let $A = (A_1, A_2) \in C^\infty(\mathbb{R}^2 : \mathbb{R}^2)$. Put

$$\mathcal{D}_{A,m} = \sum_{j=1}^{2} c\sigma_j(\hat{p}_j - A_j(q)) + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix}, \tag{3.4}$$

$$D = \hat{p}_1 - A_1(q) + i(\hat{p}_2 - A_2(q)), \quad D^* = \hat{p}_1 - A_1(q) - i(\hat{p}_2 - A_2(q)) \quad \text{with} \quad \hat{p}_j = \frac{1}{i}\frac{\partial}{\partial q_j}.$$

We put also

$$B = \nabla \times A = \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2} \quad \text{and} \quad F = \frac{1}{2\pi}\int_{\mathbb{R}^2} dq\, B(q). \tag{3.5}$$

Theorem 3.4 (Aharonov-Casher [1]). *Under above condition, we have*
(I) the spectrum $\sigma(\mathcal{D}_{A,m})$ is symmetric with respect to 0 except possibly at $\pm mc^2$ and

$$(-mc^2, mc^2) \cap \sigma(\mathcal{D}_{A,m}) = \emptyset,$$

$$\mathcal{D}_{A,m}\psi = mc^2\psi \Longleftrightarrow \psi = \begin{pmatrix}\psi_1 \\ 0\end{pmatrix},\; D^*D\psi_1 = 0 \;(i.e.\, D\psi_1 = 0),$$

$$\mathcal{D}_{A,m}\psi = -mc^2\psi \Longleftrightarrow \psi = \begin{pmatrix}0 \\ \psi_2\end{pmatrix},\; DD^*\psi_2 = 0 \;(i.e.\, D^*\psi_2 = 0).$$

(II) Moreover, assuming that $B \in C_0^\infty(\mathbb{R}^2 : \mathbb{R})$, we have the following;
(a) If $F > 0$, then $mc^2 \in \sigma_p(\mathcal{D}_{A,m})$ and $-mc^2 \notin \sigma_p(\mathcal{D}_{A,m})$ and the multiplicity of the eigenvalue mc^2

equals $\{F\}$.
(b) If $F < 0$, *then* $-mc^2 \in \sigma_p(\not{D}_{A,m})$ *and* $mc^2 \notin \sigma_p(\not{D}_{A,m})$ *and the multiplicity of the eigenvalue* $-mc^2$ *equals* $\{|F|\}$. *(Here,* $\{a\}$ *stands for the largest integer strictly less than a.)*

Theorem 3.5 (Aharonov-Casher [1]). *Put*

$$Q = \sum_{j=1}^{2} \sigma_j(\hat{p}_j - A_j) = \not{D}_{A,0}, \quad P = \sigma_3, \quad H = Q^2 = \begin{pmatrix} (\hat{p} - A)^2 + B & 0 \\ 0 & (\hat{p} - A)^2 - B \end{pmatrix}.$$

Then, (H, Q, P) *has supersymmetry in* $\mathbb{H} = L^2(\mathbb{R}^2 : \mathbb{C}^2)$. *Moreover, if* $0 \neq B \in C_0^\infty(\mathbb{R}^2 : \mathbb{R})$, *we have*

$$\mathrm{ind}_s(H) = (\mathrm{sign}\, F)\{|F|\}.$$

Theorem 3.6 (Bollé et al. [8]). *Under the same assumption as above, we have*

$$\Delta_t(H) = W_H = F = W_R = \Delta_z(H).$$

Remark. This theorem was first recognized by Kihlberg et al [37] by the calculation using path-integral: That is,

$$(3.6) \quad \begin{aligned} \Delta_t(H) &= \int dq d\psi d\bar{\psi} \int_{PBC} [dq][d\psi][d\bar{\psi}]\, e^{-\int_0^t ds\, \mathcal{L}(q(s), \dot{q}(s), \psi(s), \bar{\psi}(s))}, \\ \text{with} \quad \mathcal{L}(q, \dot{q}, \psi, \bar{\psi}) &= \frac{1}{2}\dot{q}_i^2 - i\dot{q}_j A_j(q) + \bar{\psi}(\partial_s - B(q))\psi, \quad \dot{q}(s) = \frac{d}{ds} q(s). \end{aligned}$$

Their criterion of evaluation of the right-hand side of above is (i) in the limit $t \to 0$, to use constant configuration or (ii) to evaluate the functional integration they use the change of variables according to the Nicolai mapping and construct a lattice approximation.

Theorem 3.7 (Anghel [4]). *Under the same assumption as above, we have*

$$(3.7) \qquad \mathrm{ind}_s(H) = F - \frac{\mathrm{sign}(F)}{2} - \frac{1}{2}[\eta_F(0) + \mathrm{sign}(F)h],$$

where $\eta_F(0)$ *is the eta invariant associated to* $T = -i\frac{\partial}{\partial\theta} - F$ *on* $C^\infty(S^1)$ *and* $h = \dim\ker T$.

Remark. In this paper, Anghel used the Atiyah-Singer Index Theorem for a manifold with boundary.

Problem: May we derive the formula (3.7) without the Index Theorem? In other word, may we derive the Index Theorem with boundary by using susyQM?

On the other hand, we notice the following physicists dscription:

Claim 3.8 ('t Hooft [54]). *The massless fermion functional integral vanishes when the Fermi field is coupled to a gauge field with nontrivial topology.*

Claim 3.9 (Callan, Dashen and Gross [10], Jackiw and Rebbi [36]). *The functional integral over the fermi fields in the presence of the pseudoparticle vanishes because it represents a transition in which a conservation law is violated.*

Claim 3.10 (Kiskis [38]). *If the gauge field to which the fermions are coupled has nontrivial topology, then the spectrum of* $\not{D}_{A,0}$ *includes either a zero-eigenvalue bound state or a zero-eigenvalue unbound resonance.*

In other word, let $A = (A_1, A_2) \in C^\infty(\mathbb{R}^2 : \mathbb{R}^2)$ *satisfying* $0 \neq B = dA \in C_0^\infty(\mathbb{R}^2 : \mathbb{R})$ *with*

$$\not{D}_{A,0} = \sigma_1(-i\partial_{q_1} - A_1(x)) + \sigma_2(-i\partial_{q_2} - A_2(x)).$$

Then, the spectrum of $\not{D}_{A,0}$ *must include either a bound state or an unbound resonance at zero eigenvalus. Either one of these is sufficient to give*

$$\frac{\int [d\psi][d\bar{\psi}]\, e^{-[\int_{\mathbb{R}^2} dq\, \bar{\psi}(q)\not{D}_{A,0}\psi(q)]}}{\int [d\psi][d\bar{\psi}]\, e^{-[\int_{\mathbb{R}^2} dq\, \bar{\psi}(q)\not{D}_{0,0}\psi(q)]}} = \frac{\det \not{D}_{A,0}}{\det \not{D}_{0,0}} = 0.$$

4. Wigner's semi-circle law in R.M.T.

In the random matrix theory (=R.M.T.), the following problem is considered as the first one to be solved.

Let $\mathfrak{U}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ by

$$
(4.1)\qquad \begin{aligned} d\mu_N(H) &= \prod_{k=1}^{N} d(\Re H_{kk}) \prod_{j<k}^{N} d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H), \\ P_{N,J}(H) &= Z_{N,J}^{-1} \exp\Big[-\frac{N}{2J^2} \operatorname{tr} H^* H\Big] \end{aligned}
$$

where $H = (H_{jk})$, $H^* = (H^*_{jk}) = (\overline{H}_{kj}) = {}^t\overline{H}$, $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N/2}(J^2\pi/N)^{3N/2}$.

Let $E_\alpha = E_\alpha(H)$ $(\alpha = 1, \cdots, N)$ be real eigenvalues of $H \in \mathfrak{U}_N$.

We put

$$
(4.2)\qquad \rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^{N} \delta(\lambda - E_\alpha(H)),
$$

where δ is the Dirac's delta. Denoting

$$
\langle f \rangle_N = \langle f(\cdot) \rangle_N = \int_{\mathfrak{U}_N} d\mu_N(H)\, f(H),
$$

for a "function f" on $\mathfrak{U}_N$, we get

Theorem 4.1 (Wigner's semi-circle law).

$$
(4.3)\qquad \lim_{N\to\infty} \langle \rho_N(\lambda) \rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1}\sqrt{4J^2 - \lambda^2} & \text{for } |\lambda| < 2J, \\ 0 & \text{for } |\lambda| > 2J. \end{cases}
$$

Seemingly, there exist several methods to prove this fact. Here, we want to explain a new derivation of this fact using odd variables(Efetov [15], Fyodorov [20], Brézin [9], Zirnbauer [62]).

Following facts are essential: (1) Let $A = A_1 + iA_2 = (A_{jk})$, where A_1, A_2 are real symmetric $N \times N$-matrices with $A_1 > 0$. Putting x_j, $y_j \in \mathbb{R}$, we have

$$
\int_{(\mathbb{R}\times\mathbb{R})^N} \prod_{j=1}^{N} \frac{dx_j dy_j}{\pi}\, e^{-\sum_{j,k=1}^N (x_j - iy_j) A_{jk} (x_k + iy_k)} = \frac{1}{\det A},
$$

$$
\int_{(\mathbb{R}\times\mathbb{R})^N} \prod_{j=1}^{N} \frac{dx_j dy_j}{\pi} (x_a - iy_a)(x_b + iy_b) e^{-\sum_{j,k=1}^N (x_j - iy_j) A_{jk} (x_k + iy_k)} = \frac{(A^{-1})_{a,b}}{\det A}.
$$

(2) Let θ_k, $\bar\theta_l \in \mathfrak{R}_{\mathrm{od}}$.

$$
\int_{\mathfrak{R}^{0|2N}} \prod_{k=1}^{N} d\bar\theta_k d\theta_k e^{-\sum_{j,k=1}^N \bar\theta_j A_{jk} \theta_k} = \det A,
$$

$$
\int_{\mathfrak{R}^{0|2N}} \prod_{k=1}^{N} d\bar\theta_k d\theta_k \theta_a \bar\theta_b e^{-\sum_{j,k=1}^N \bar\theta_j A_{jk} \theta_k} = (A^{-1})_{a,b} \det A.
$$

(A) Based on the above facts, physicists derived the following formula:

$$
(4.4)\qquad \langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im \int_{\Omega} dQ\, \big(\{(\lambda - i0)I_2 - Q\}^{-1}\big)_{bb} \exp[-N\mathcal{L}(Q)]
$$

where I_n stands for $n \times n$-identity matrix and

$$\begin{aligned} &\mathcal{L}(Q) = \mathrm{str}\,[(2J^2)^{-1}Q^2 + \log((\lambda - i0)I_2 - Q)], \\ (4.5) \qquad &\mathfrak{Q} = \{Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \rho_1, \rho_2 \in \mathfrak{R}_{\mathrm{od}}\} \cong \mathfrak{R}^{2|2},\ dQ = \frac{dx_1dx_2}{2\pi} d\rho_1 d\rho_2, \\ &(((\lambda - i0)I_2 - Q)^{-1})_{bb} = \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}. \end{aligned}$$

Here in (4.4), **the parameter N appears only in one place.** This formula is formidably charming but **not yet directly justified**, like Feynman's expression of the kernel of the Schrödinger equation using his measure (2.1).

(B) In physics literatures, for example in [20],[62], they claim without proof that they may apply the method of steepest descent to (4.4) when $N \to \infty$. More precisely, as

$$\delta\mathcal{L}(Q)\tilde{Q} = \frac{d}{d\epsilon}\mathcal{L}(Q + \epsilon\tilde{Q})\bigg|_{\epsilon=0},$$

they seek solutions of

$$\delta\mathcal{L}(Q) = \mathrm{str}\,(\frac{Q}{J^2} - \frac{1}{\lambda - Q}) = 0.$$

As a candidate of effective saddle points, they take

$$Q_c = (\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 4J^2})I_2,$$

and they have

$$\lim_{N\to\infty} \langle \rho_N(\lambda)\rangle_N = \pi^{-1}\Im(\lambda - Q_c)^{-1}_{bb} = w_{sc}(\lambda). \quad \square$$

Remark. Not only the expression (4.4) nor the applicability of the saddle point method to it are not so clear. To get the mathematical rigour, we **dare to loose such a beautiful expression** like (4.4), but we have the two formulae (4.6) and (4.7).

$$(4.6) \qquad \begin{aligned} \langle \mathrm{tr}\, \frac{1}{(\lambda - i0)I_N - H}\rangle_N = i\frac{1}{(N-1)!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1} \iint_{\mathbf{R}_+\times\mathbf{R}} dsd\tau\,(1 + (\tau + i\lambda)^{-1}s) \\ \times \exp[-N(\frac{1}{2J^2}(\tau^2 + 2i\lambda s + s^2) - \log s(\tau + i\lambda))]. \end{aligned}$$

$$(4.7) \qquad \langle \rho_N(\lambda)\rangle_N = \left(\frac{N}{2\pi J^2}\right)^{1/2} \frac{1}{2\pi(N-1)!}\left(\frac{N}{J^2}\right)^N \iint_{\mathbf{R}^2} dt\,ds\, \exp[-N\phi_\pm(t,s,\lambda)]a_\pm(t,s,\lambda;N),$$

where

$$\begin{aligned} \phi_\pm(t,s,\lambda) &= \frac{1}{2J^2}(t^2 + s^2 + \lambda^2) - \log(\lambda \mp it)(\lambda \mp is), \\ a_\pm(t,s,\lambda : N) &= \frac{1}{(\lambda \mp it)(\lambda \mp is)} - \frac{1}{2}(1 - N^{-1})\left[\frac{1}{(\lambda \mp it)^2} + \frac{1}{(\lambda \mp is)^2}\right]. \end{aligned}$$

We get, in Inoue & Nomura [35],

Theorem 4.2 (A refined version of Wigner's semi-circle law). *For each λ with $|\lambda| < 2J$, when $N \to \infty$, we have*

$$(4.8) \quad \langle \rho_N(\lambda)\rangle_N = \frac{\sqrt{4J^2 - \lambda^2}}{2\pi J^2} - \frac{(-1)^N J}{\pi(4J^2 - \lambda^2)}\cos(N[\frac{\lambda\sqrt{4J^2 - \lambda^2}}{2J^2} + 2\arcsin(\frac{\lambda}{2J})])N^{-1} + O(N^{-2}).$$

When λ satisfies $|\lambda| > 2J$, there exist constants $C_\pm(\lambda) > 0$ and $k_\pm(\lambda) > 0$ such that

$$(4.9) \qquad \left|\langle \rho_N(\lambda)\rangle_N\right| \le C_\pm(\lambda)\exp[-k_\pm(\lambda)N]$$

with $k_\pm(\lambda) \to 0$ *and* $C_\pm(\lambda) \to \infty$ *for* $\lambda \searrow 2J$ *or* $\lambda \nearrow -2J$, *respectively.*

Theorem 4.3 (The spectrum edge problem). *Let* $z \in [-1,1]$. *We have*

$$\begin{aligned} &\langle \rho_N(2J - zN^{-2/3})\rangle_N = N^{-1/3} f(z/J) + O(N^{-2/3}) \quad \text{as } N \to \infty, \\ &\langle \rho_N(-2J + zN^{-2/3})\rangle_N = -N^{-1/3} f(z/J) + O(N^{-2/3}) \quad \text{as } N \to \infty, \end{aligned} \tag{4.10}$$

where

$$f(w) = \frac{1}{4\pi^2 J}(\mathrm{Ai}'(w)^2 - \mathrm{Ai}''(w)\,\mathrm{Ai}(w)), \quad \mathrm{Ai}(w) = \int_{\mathbb{R}} dx \exp[-\frac{i}{3}x^3 + iwx].$$

Problem. Do we calculate analogously if we replace GUE with GOE(=Gaussian Orthogonal Ensemble) or GSE(=Gaussian Symplectic Ensemble)?

Problem. Not only Airy functions above, but also the relation of R.M.T. to other Painleve transcendent is pointed out recently. Interprete these relations using superanalysis.

On the other hand, using eigenvalues denoted by $\lambda_j (j = 1, \cdots, N)$, we may consider the following integral:

$$\rho_N^\beta(\lambda) = N \int \underset{\mathbb{R}^{N-1}}{\cdots} \int d\lambda_2 \cdots d\lambda_N P_N^\beta(\lambda, \lambda_2, \cdots, \lambda_N)$$

where

$$P_N^\beta(\lambda_1, \lambda_2, \cdots, \lambda_N) = C_N^\beta e^{-\beta \sum_{j=1}^N \lambda_j^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta,$$

and

$$\begin{aligned} C_N^\beta &= \int \underset{\mathbb{R}^N}{\cdots} \int d\lambda_1 \cdots d\lambda_N \, e^{-\beta W} \\ &= (2\pi)^{N/2} \beta^{-N/2 - \beta N(N-1)/4} [\Gamma(1+\beta/2)]^{-N} \prod_{j=1}^N \Gamma(1+\beta j/2), \end{aligned}$$

$$\text{with} \quad W = \frac{1}{2}\sum_{j=1}^N \lambda_j^2 - \sum_{1 \le j < k \le N} \log|\lambda_j - \lambda_k|.$$

It is well-known that $\beta = 2$ is equivalent to the above GUE.

Problem. Can we apply our method to have a suitable limit when $N \to \infty$ for all β?

5. Gelfand's problem for dynamical systems

5.1. Out line of the problem. The study of dynamical systems geverned by

$$\frac{d}{dt} q_j(t) = F_j(q_1(t), \cdots, q_n(t)) \quad (j = 1, 2, \cdots, n) \tag{5.1}$$

is related to that of a partial differential equation(PDE) of the first order

$$\frac{\partial}{\partial t} u(t,q) = \sum_{j=1}^n F_j(q_1, \cdots, q_n) \frac{\partial}{\partial q_j} u(t,q). \tag{5.2}$$

By the so-called spectral method of the theory of dynamical systems due to Koopman, the theory of dynamical systems may to a significant degree be interpreted as a theory relative to a linear partial differential equation of first order.

For example, if Ω is an invariant set of the flow defined by (5.1) (i.e. if T_t is defined by $q(t) = T_t q(0)$, Ω should satisfy $T_t\Omega = \Omega$), there exists an invariant measure μ of the flow T_t (i.e. for any Borel set $\omega \subset \Omega$, $\mu(T_{-t}\omega) = \omega$) such that $i \sum_{j=1}^n F_j(q)\partial/\partial q_j$ is self adjoint on $L^2(\Omega, d\mu)$.

Gelfand [21] asked whether in the above story, we may replace (5.2) by

$$\frac{\partial}{\partial t}u_j(t,q) = \sum_{k=1}^{d} A_{j,\ell}^{(k)}(q)\frac{\partial}{\partial q_k}u_\ell(t,q) \quad \text{for} \quad j,\ell = 1,2,\cdots,n, \tag{5.3}$$

where $A^{(k)}$ are $n\times n$-matrices whose elemants are denoted by $A_{j,\ell}^{(k)}$. Gelfand's first question in this direction is, whether there exists an invariant measure $\tilde{\mu}$ on an invariant set $\tilde{\Omega}$ such that $i\sum_{k=1}^{d} A_{j,\ell}^{(k)}(q)\frac{\partial}{\partial q_k}$ becomes self adjoint on $L^2(\tilde{\Omega}; d\tilde{\mu})$?

5.2. Our formulation by an example. Here, we may take 2×2-systems of PDE and explain our formulation for Gelfand's problem.

We consider the initial value problem

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1(t,q)\\ \psi_2(t,q)\end{pmatrix} = \sum_{j=1}^{3}\begin{pmatrix}a^j(q) & c^j(q)\\ d^j(q) & b^j(q)\end{pmatrix}\frac{\partial}{\partial q_j}\begin{pmatrix}\psi_1(t,q)\\ \psi_2(t,q)\end{pmatrix} \quad \text{with} \quad \begin{pmatrix}\psi_1(0,q)\\ \psi_2(0,q)\end{pmatrix} = \begin{pmatrix}\underline{\psi}_1(q)\\ \underline{\psi}_2(q)\end{pmatrix}. \tag{5.4}$$

For the hyperbolicity, we assume

$$(a(q)p - b(q)p)^2 + 4(c(q)p)(d(q)p) \geq 0 \quad \text{for} \quad |p| = 1. \tag{5.5}$$

Here, we abbribiate $\sum_{j=1}^{3} a^j(q)p_j = a(q)p$, etc.

For the matrix

$$\begin{aligned}\mathbb{H}(q,p) &= -\begin{pmatrix}a(q)p & c(q)p\\ d(q)p & b(q)p\end{pmatrix}\\ &= -\frac{a(q)p + b(q)p}{2} - \frac{a(q)p - b(q)p}{2}\sigma_3 - \frac{c(q)p + d(q)p}{2}\sigma_1 - \frac{c(q)p - d(q)p}{2}\sigma_2,\end{aligned}$$

we may associate a Hamiltonian $\mathcal{H}(x,\xi,\theta,\pi)$ on $\mathcal{T}^*(\mathfrak{R}^{3|2}) = \mathfrak{R}^{6|4}$ given by

$$\mathcal{H}(x,\xi,\theta,\pi) = -\mathfrak{a}(x)\xi + i\mathfrak{b}(x)\xi\langle\theta|\pi\rangle - c(x)\xi\theta_1\theta_2 - d(x)\xi\pi_1\pi_2, \tag{5.6}$$

with

$$\mathfrak{a}^j(x) = \frac{a^j(x) + b^j(x)}{2},\ \mathfrak{b}^j(x) = \frac{a^j(x) - b^j(x)}{2},\ \mathfrak{a}(x)\xi = \sum_{j=1}^{3}\mathfrak{a}^j(x)\xi_j,\ \mathfrak{b}(x)\xi = \sum_{j=1}^{3}\mathfrak{b}^j(x)\xi_j.$$

It yields the superspace version of the equation (5.4) represented by

$$i\frac{\partial}{\partial t}u(t,x,\theta) = \mathcal{H}\Big(x,-i\frac{\partial}{\partial x},\theta,-i\frac{\partial}{\partial \theta}\Big)u(t,x,\theta) \quad \text{with} \quad u(0,x,\theta) = \underline{u}(x,\theta). \tag{5.7}$$

As $\mathcal{H}$ is even, we may consider the classical mechanics corresponding to $\mathcal{H}(x,\xi,\theta,\pi)$:

$$(5.8)_{ev}\quad \begin{cases}\dfrac{d}{dt}x_j = \dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\xi_j} = -\mathfrak{a}^j(x) + i\mathfrak{b}^j(x)\langle\theta|\pi\rangle - c^j(x)\theta_1\theta_2 - d^j(x)\pi_1\pi_2,\\[2mm] \dfrac{d}{dt}\xi_j = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial x_j} = \mathfrak{a}_{x_j}(x)\xi - i\mathfrak{b}_{x_j}(x)\xi\langle\theta|\pi\rangle + c_{x_j}(x)\xi\theta_1\theta_2 + d_{x_j}(x)\xi\pi_1\pi_2\end{cases}$$

$$(5.8)_{od}\quad \begin{cases}\dfrac{d}{dt}\theta_1 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\pi_1} = i\mathfrak{b}(x)\xi\theta_1 + d(x)\xi\pi_2,\\[2mm] \dfrac{d}{dt}\theta_2 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\pi_2} = i\mathfrak{b}(x)\xi\theta_2 - d(x)\xi\pi_1,\\[2mm] \dfrac{d}{dt}\pi_1 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\theta_1} = -i\mathfrak{b}(x)\xi\pi_1 + c(x)\xi\theta_2,\\[2mm] \dfrac{d}{dt}\pi_2 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\theta_2} = -i\mathfrak{b}(x)\xi\pi_2 - c(x)\xi\theta_1\end{cases}$$

and at time $t = 0$, the initial data are given by

$$(x(0), \xi(0), \theta(0), \pi(0)) = (\underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}). \tag{5.9}$$

If there exists a unique solution of (5.8) with (5.9), we denote it by

$$\mathcal{T}_t(\underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) = (x(t), \xi(t), \theta(t), \pi(t)) = (x(t; \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \xi(t; \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \theta(t; \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \pi(t; \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})).$$

Therefore, it is natural to ask whether there exists a set $\tilde{\Omega} \subset \mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3|2}$ such that

$$\mathcal{T}_t\tilde{\Omega} = \tilde{\Omega}?$$

whether $-i\mathcal{H}(x, -i\partial_x, \theta, -i\partial_\theta)$ is self-adjoint on $L^2(\tilde{\Omega}, \tilde{\mu})$? Here, $\tilde{\mu}$ is an invariant measure on $\tilde{\Omega}$ related to the symplectic measure $dx \wedge d\xi + d\theta \vee d\pi$ on $\mathcal{T}^*\mathfrak{R}^{3|2}$.

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