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# Absence of eigenvalues of the Maxwell operators

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## 1 Introduction

F. Rellich (1943) has shown that if $u\in L^{2}(U)$ is a solution to the eigenvalue problem

$$-\Delta u=ku, \quad k>0 \tag{1.1}$$

in an exterior domain $U$ of $\mathbf{R}^{d}$, then $u$ is identically zero. T. Kato (1959) extended this result to the Schrödinger equation

$$-\Delta u+q(x)u=ku, \ x\in U, \ k>0 \tag{1.2}$$

where

$$q(x)=o(|x|^{-1}), \qquad |x|\rightarrow\infty.$$

In addition, there are many works on a class of second order elliptic equations.

On the other hand, an analogue to Rellich's theorem for symmetric elliptic systems is well known (cf. P.D.Lax - R.S.Phillips and N. Iwasaki). Our major concern is whether an analogue to Kato's result holds for such systems or not. As for Dirac operators, many works are devoted to the study of this problem ([4], [13], [12] and [5]).

In this paper, we focus our attention to optical systems in general inhomogeneous media. In order to attack this problem, we shall take the first order approach instead of the usual second order approach. It is an improved version of Vogelsang's strategy, which is to show a series of weighted $L^{2}$ estimates based on the virial theorem.

## 2 Maxwell operators

Let $\varepsilon$ and $\mu$ be $3\times 3$ real symmetric matrices defined in an exterior domain $U$ of $\mathbf{R}^{3}$. They are supposed to be uniformly positive definite in $U$: There exists a positive constant $\delta_{0}$ such that

$$(\varepsilon(x)\zeta,\zeta)\geq\delta_{0}|\zeta|^{2}, \quad (\mu(x)\zeta,\zeta)\geq\delta_{0}|\zeta|^{2}, \quad \forall\zeta\in\mathbf{C}^{3}, \ \forall x\in U. \tag{2.1}$$

Let us define two $6 \times 6$ matrices as follows:

$$A = \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}.$$

The eigenvalue problem we shall discuss is as follows:

$$Au = i\lambda\Gamma u. \tag{2.2}$$

# 3 Isotropic media

First of all, we consider the case that $\varepsilon$ and $\mu$ are scalar matrices, called isotropic media. Let $I_a$ be an interval $[a, \infty)$ for $a \geq 0$. We denote the positive part and the negative part of a real-valued function $f$ defined in $I_a$ by $[f]_+$ and $[f]_-$, respectively:

$$[f]_+ = \max(0, f(r)), \quad [f]_- = \max(0, -f(r)).$$

In what follows, $f'$ denotes the derivative of $f(r)$. For a positive number $\delta$ and $k = 1, 2$, we define the subset $m_\delta^k(I_a)$ of $C^k(I_a)$ as

$$\begin{aligned} m_\delta^k(I_a) = \{q(r) \in C^k(I_a; \mathbf{R});\ \inf_{I_a} q(r) = q_\infty > 0, \\ (\frac{d}{dr})^j q(r) = o(r^{-j/2} q^{1+\delta j}),\ 1 \leq \forall j \leq k,\ [q']_- = o(r^{-1}q)\}. \end{aligned} \tag{3.1}$$

In addition, define $m_0^k(I_a) = m_\delta^k(I_a) \cap L^\infty(I_a)$, which is independent of $\delta$.

For $a > 0$, define $D_a = \{x \in \mathbf{R}^3;\ |x| > a\}$. Henceforth, we always choose $a$ so large that $D_a \subset U$. We shall use the polar coordinates, $r = |x|$, $\omega = x/|x|$. For $q \in m_\delta^2(I_a)$ with $a > 0$, we say that $F(x) \in C^1(U)^{3\times 3}$ belongs to the class $S_\delta(q)$ if

$$\partial_r^j (F(x) - q(r)) = o((q^\delta r^{-1/2})^{j+1}), \qquad j = 0, 1. \tag{3.2}$$

**Theorem 3.1** *Suppose that $\varepsilon(x)$ and $\mu(x)$ are positive scalar functions such that*

$$\varepsilon \in S_{1/2}(q_1),\ \mu \in S_{1/2}(q_2), \quad q_j \in m_{1/2}^2(I_a) \cap L^\infty(I_a),\ j = 1, 2. \tag{3.3}$$

*If $u \in L^2(U)$ be a solution to (2.2), then $u$ is identically zero in $U$.*

We shall consider the case when $q_1$ or $q_2$ diverges at infinity.

**Theorem 3.2** *Let $q_j \in m_{1/4}^2(I_a)$, $j = 1, 2$ and suppose that $q_1^{-1}q_2$ or $q_2^{-1}q_1$ is bounded in $I_a$. If $\varepsilon(x)$ and $\mu(x)$ are respectively positive scalar functions belonging to $S_{1/4}(q_1)$ and $S_{1/4}(q_2)$ such that*

$$q_1 q_2' - q_1' q_2 = o(r^{-1} q_1 q_2),$$

*then the conclusion of Theorem 3.1 is still true.*

**Remark 3.1** *D. Eidus has studied the same problem by the second order approach. He has obtained an analogous result (Theorem 4.4 of [1]) for $U = \mathbf{R}^3$ under the assumption that $\varepsilon$ and $\mu$ belong to $C^2(\mathbf{R}^3)$ and they satisfy a stronger asymptotic property*

$$|\varepsilon - \varepsilon_0| + |\mu - \mu_0| + |\nabla\varepsilon| + |\nabla\mu| = o(|x|^{-1}).$$

**Remark 3.2** *If both $q_1^{-1}q_2$ and $q_2^{-1}q_1$ are bounded, we can replace $m_{1/4}^2(I_a)$ and $S_{1/4}(q_j)$ in Theorem 3.2 by $m_{1/2}^2(I_a)$ and $S_{1/2}(q_j)$, respectively.*

**Remark 3.3** *A similar result for Dirac operators with the potential growing at infinity has been obtained in [5].*

# 4 Nonisotropic media

To describe our conditions in nonisotropic media, we introduce the function space $\mathcal{M}(U)$ as the set of all real positive symmetric matrices of third order whose components are continuously differentiable functions in $U$ satisfying that there exist a symmetric matrix $F_\infty(x) \in C^1(U)^{3\times 3}$ and a positive constant $F_0$ such that as $|x| \to \infty$

$$(4.1)\quad F(x) - F_\infty(x) = o(|x|^{-1}),\ F_\infty(x) - F_0 I = o(|x|^{-1/2}),\ \nabla F(x) = o(|x|^{-1}).$$

**Theorem 4.1** *Suppose that $\varepsilon$ and $\mu$ belong to $\mathcal{M}(U)$ and there exists a positive constant $\kappa$ such that $\varepsilon_\infty(x) = \kappa\mu_\infty(x)$ for all $x$ in a neighborhood of infinity. If $u \in L^2(U)$ is a solution to (2.2), then $u$ has a compact support.*

**Corollary 4.2** *In addition to the assumptions of Theorem 4.1, we assume that there exists a scalar function $\kappa \in C^1(U)$ such that $\varepsilon(x) = \kappa(x)\mu(x)$. If $u \in L^2(U)$ is a solution to (2.2), then $u$ is identically zero in $U$.*

**Remark 4.1** *If $u \in L^2(U)$ is a solution to (2.2), then $u \in H^1_{\mathrm{loc}}(U)$.*

We remark that each hypothesis of Theorems 4.1, 3.1 and 3.2 implies that if $a$ is taken to be so large, there exists a positive number $\kappa$ such that

$$(4.2)\qquad (r\Gamma)' > \kappa, \quad \forall x \in D_a.$$

This can be verified as follows. If

$$\Gamma_0(r) = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix},$$

then it holds that

$$(r\Gamma)' = (r\Gamma_0)' + (r\Gamma - r\Gamma_0)'. \tag{4.3}$$

Since $\min_{I_a} q_j > 0$ and $[q_j']_- = o(r^{-1})$, if $a$ is taken to be large enough, we have

$$\inf_{I_a}(rq_j)' > 0. \tag{4.4}$$

In view of

$$(r\Gamma - r\Gamma_0)' = o(1),$$

(4.2) follows from (4.3) and (4.4).

If $U = \mathbf{R}^3$ and there exists a positive constant $\beta$ such that

$$\partial_r(r\Gamma)(x) > \beta I, \tag{4.5}$$

holds for all $x \in \mathbf{R}^3$, we can easily show the absence of nonzero eigenvalues. Let $\mathcal{B}^1(U)$ be the subset of $C^1(U)$ consisting of all functions $f$ satisfying

$$|f| + |\nabla f| \in L^\infty(U).$$

**Theorem 4.3** *Let $U = \mathbf{R}^3$ and $\varepsilon$, $\mu \in \mathcal{B}^1(\mathbf{R}^3)^{3\times 3}$ satisfy (2.1). Suppose (4.5). If $u \in L^2(\mathbf{R}^3)$ satisfies (2.2), then $u = 0$ in $\mathbf{R}^3$.*

**Remark 4.2** *Theorem 4.3 also improves Theorem 4.4 of [1].*

# 5 The Polar coordinates

Let $r = |x|$ and $\omega = x/|x|$. It holds

$$\partial_{x_j} = \omega_j \partial_r + r^{-1}\Omega_j,$$

where $\Omega$ is a vector field on $\mathbf{S}^2$. Define respectively two important matrices $J_\omega$ and $J_\Omega$ as $J_\omega u = \omega \wedge u$ and $J_\Omega u = \Omega \wedge u$: It is easily seen that

$$J_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad J_\Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

**Lemma 5.1**

$$\mathrm{curl} = J_\omega \partial_r + r^{-1} J_\Omega$$

*and*

$$J_\omega \mathrm{curl} u = -\partial_r u + r^{-1} G u + (\mathrm{div} u)\omega,$$

*where $G$ is a selfadjoint operator in $L^2(\mathbf{S}^{d-1})$.*

**Remark 5.1** *G is given explicitly as*

$$G = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix},$$

*where*

$$L_1 = x_2\partial_3 - x_3\partial_2, \quad L_2 = x_3\partial_1 - x_1\partial_3, \quad L_3 = x_1\partial_2 - x_2\partial_1.$$

Let

$$\alpha = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \ \mathcal{J}_\omega = \begin{pmatrix} J_\omega & 0 \\ 0 & J_\omega \end{pmatrix}.$$

Define

$$\hat{J}_\Omega = J_\Omega - J_\omega, \ \mathcal{J}_\Omega = \begin{pmatrix} \hat{J}_\Omega & 0 \\ 0 & \hat{J}_\Omega \end{pmatrix}, \ \mathcal{G} = \begin{pmatrix} G+1 & 0 \\ 0 & G+1 \end{pmatrix}.$$

**Lemma 5.2** *If $v = ru$, then it satisfies*

$$\{-\mathcal{J}_\omega\partial_r - r^{-1}\mathcal{J}_\Omega\}\alpha v = \lambda\Gamma v. \tag{5.1}$$

**Lemma 5.3** *Suppose that $\varepsilon$ and $\mu$ are scalar functions belonging to $C^1(U)$. Let $v = ru$. It holds that*

$$\{\partial_r - r^{-1}\mathcal{G} - Q\}\alpha v = \lambda\mathcal{J}_\omega\Gamma v, \tag{5.2}$$

*where*

$$Q\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} \omega\varepsilon^{-1}(\nabla\varepsilon, v_+) \\ \omega\mu^{-1}(\nabla\mu, v_-) \end{pmatrix}, \quad v_\pm \in \mathbf{C}^3. \tag{5.3}$$

Split $Q = Q_1 + Q_2$ with

$$Q_1\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} q_1^{-1}(\nabla q_1, v_+)\omega \\ q_2^{-1}(\nabla q_2, v_-)\omega \end{pmatrix},$$

$$Q_2\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} \omega\{\varepsilon^{-1}(\nabla\varepsilon, v_+) - q_1^{-1}(\nabla q_1, v_+)\} \\ \omega\{\mu^{-1}(\nabla\mu, v_-) - q_2^{-1}(\nabla q_2, v_-)\} \end{pmatrix}.$$

If the hypothesis of Theorem 3.2 is fulfilled and $\lim_{r\to\infty} q(r)$ exists, then $Q_1^* = Q_1$, $Q_2 = o(r^{-1/2})$ and $\partial_r Q_1 = o(r^{-1})$.

In what follows, we denote the inner product and the norm of $L^2(\mathbf{S}^2)^6$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Then, we note that

$$\langle\hat{\mathcal{J}}_\Omega v, v\rangle = \langle v, \hat{\mathcal{J}}_\Omega v\rangle$$

and

$$\int\langle\partial_r v, v\rangle r^2 dr = \int\langle(\partial_r + r^{-1})v, v\rangle r^2 dr = \int\langle\partial_r v, v\rangle dr.$$

# 6 The virial theorem

Note that $(\alpha)^* = \alpha$, $\alpha^2 = I$. Define

$$F_v(r) = -\lambda r \mathrm{Re}\langle \mathcal{J}_\omega \partial_r \alpha v, v\rangle.$$

First of all, we need the following property on regularity of solutions.

**Lemma 6.1** *Suppose that $F \in \mathcal{M}(\mathbf{R}^3)$. There exists a positive constant $C_F > 0$ such that*

$$\int |\nabla v|^2 dx \leq C_F \int \{|\mathrm{curl} v|^2 + |\mathrm{div} Fv|^2 + |v|^2\} dx \tag{6.1}$$

*for all $v \in C_0^1(\mathbf{R}^3)^3$.*

The next is a kind of virial theorems.

**Lemma 6.2** *Let $v = ru$. Then,*

$$\lambda^2 \int_s^t \langle \partial_r (r\Gamma) v, v\rangle dr = F_v(t) - F_v(s).$$

# 7 Proof of Theorem 4.3

Theorem 4.3 follows from the virial theorem. Since $u \in H^1(\mathbf{R}^3)$, we see that

$$\int_0^\infty r^{-1} |F_v| dr < \infty.$$

Thus, it holds that

$$\liminf_{r\to 0} |F_v|(r) = 0, \quad \liminf_{r\to\infty} |F_v(r)| = 0.$$

Performing $s = s_j \to 0$ and $t = t_j \to \infty$ in Lemma 6.2, we obtain

$$\lambda^2 \int_0^\infty \langle \partial_r [r\Gamma] v, v\rangle dr = 0,$$

which implies $v = 0$ since $\partial_r [r\Gamma] > 0$. □

**Remark 7.1** *From Lemma 6.2 and the fact that*

$$\liminf_{r\to\infty} |F_v(r)| = 0,$$

*it follows that $F_v(r) \leq 0$ for every sufficient large $r$.*

The essential difficulty arises when the virial condition (4.2) is valid only in a neighborhood of infinity.

# 8 Isotropic cases

In this section we shall consider the isotropic case.

Define

$$q_0(r)=\sqrt{q_1q_2},\ \Lambda_\infty(r)=\begin{pmatrix} q_1I & 0\\ 0 & q_2I\end{pmatrix},\ \Gamma_\infty(r)=\begin{pmatrix} q_2^{-1/2}q_1^{1/2}I & 0\\ 0 & q_1^{-1/2}q_2^{1/2}I\end{pmatrix}$$

and

$$Q_3=\frac{1}{4q_1q_2}\begin{pmatrix} (q_1q_2'-q_1'q_2)I & 0\\ 0 & (q_1'q_2-q_1q_2')I\end{pmatrix}.$$

**Lemma 8.1** *Let $v=\Gamma_\infty^{1/2}ru$. Then,*

$$\{-\mathcal{J}_\omega\partial_r-r^{-1}\mathcal{J}_\Omega-\mathcal{J}_\omega Q_3\}\alpha v=\lambda Vv \tag{8.1}$$

*and*

$$\{\partial_r-r^{-1}\mathcal{G}-Q+Q_3\}\alpha v=\lambda\mathcal{J}_\omega Vv, \tag{8.2}$$

*where $V\in C^1(D_a)$ satisfies that*

$$V^*=V,\ V=q_0(1+V_2),\ \partial_r^jV_2=o(r^{-(j+1)/2}),\ j=0,1. \tag{8.3}$$

**Proof:** Define

$$\check{\Gamma}_\infty=\begin{pmatrix} q_1^{-1/2}q_2^{1/2}I & 0\\ 0 & q_2^{-1/2}q_1^{1/2}I\end{pmatrix}.$$

Using

$$\alpha\begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}=\begin{pmatrix} g & 0\\ 0 & f\end{pmatrix}\alpha,$$

we observe that if $u$ is a solution to (2.2), $\tilde{u}=\Gamma_\infty^{1/2}ru$ satisfies

$$\begin{aligned}-\check{\Gamma}_\infty^{-1/2}\left\{\mathcal{J}_\omega\partial_r+r^{-1}\mathcal{J}_\Omega\right\}\alpha\tilde{u}-\mathcal{J}_\omega[\partial_r,\check{\Gamma}_\infty^{-1/2}]\alpha\tilde{u}&\\ =\lambda\left(q_0\Gamma_\infty^{1/2}+(\Gamma-\Lambda_\infty)\Gamma_\infty^{-1/2}\right)\tilde{u}.&\end{aligned} \tag{8.4}$$

Let

$$V_2=\check{\Gamma}_\infty^{1/2}(\Gamma-\Lambda_\infty)\Gamma_\infty^{-1/2},\ V=q_0I+V_2.$$

Note that $v=r\tilde{u}$ satisfies

$$\partial_rv=r(\partial_r+r^{-1})\tilde{u}.$$

Since $\check{\Gamma}_\infty\Gamma_\infty=I$, we arrive at the first identity (8.1). If we multiply (8.1) by $\mathcal{J}$ we obtain (8.2) in view of Lemma 5.1.

# 9 A weighted virial relation

In the polar coordinates $(r,\omega) \in [0,\infty) \times \mathbf{S}^{d-1}$, we see that $v = \Gamma_\infty^{1/2} r u$ satisfies

$$\left\{-\mathcal{J}_\omega \partial_r - r^{-1}\mathcal{J}_\Omega - \mathcal{J}_\omega Q_3\right\} \alpha v - \lambda V v = 0.$$

For each pair of $(s,t)$, $0 \le s < t < \infty$, we shall consider a cutoff function $\chi(r) \in C^\infty([0,\infty))$ such that

$$0 \le \chi \le 1,\ \mathrm{supp}\chi \subset [s-1, t+1],\ \chi(r) = 1 \text{ on } [s,t].$$

If $\varphi(r) \in C^3([0,\infty))$, then $\zeta = \chi(r) e^{\varphi} v$ satisfies

$$\left\{-\mathcal{J}_\omega \partial_r - r^{-1}\mathcal{J}_\Omega + \mathcal{J}_\omega(\varphi' - Q_3)\right\} \alpha\zeta - \lambda V\zeta = \mathcal{J}_\omega \chi' e^{\varphi} \alpha v (:= \mathcal{J}_\omega f_\chi) \tag{9.1}$$

and

$$\left[\partial_r - r^{-1}\mathcal{G} - \varphi'\right] \alpha\zeta - \tilde{Q}\zeta = -f_\chi, \tag{9.2}$$

where

$$\tilde{Q} = Q\alpha - Q_3\alpha + \lambda \mathcal{J}_\omega V.$$

We recall that

$$Q = Q_1 + Q_2,\ Q_1^* = Q_1,\ Q_2 = o(r^{-1/2}),\ \partial_r Q_1 = o(r^{-1}).$$

From the virial relation, it follows that

**Lemma 9.1**

$$\int_{s-1}^{t+1} \left[\lambda^2 \langle \partial_r(rV)\zeta, \zeta\rangle - 2\lambda \mathrm{Re}\langle r\mathcal{J}_\omega(\varphi' + Q_3)\alpha\zeta, \partial_r\zeta\rangle\right] dr = -\int_{s-1}^{t+1} \langle r\mathcal{J}_\omega f_\chi, \partial_r\zeta\rangle dr.$$

By (9.1) we can show a kind of Carleman estimates as follows.

**Proposition 9.2**

$$\begin{aligned} \int_N^\infty \left[\lambda^2 \langle (rV)' e^{\varphi} v, e^{\varphi} v\rangle + k_\varphi q^{-1} \|e^{\varphi} v\|^2 + r\varphi' \|\partial_r(e^{\varphi} v/\sqrt{q})\|^2\right] dr \\ \le C \int_{s-1}^{s} \{(1 + |\varphi'| q^{-1}) r |\chi'|^2 + r q^{-1} |\varphi''| |\chi'|\} \|e^{\varphi} v\|^2 dr \end{aligned} \tag{9.3}$$

*for any $N \ge s$. Here,*

$$k_\varphi = r\varphi'\{(\varphi'' + (r^{-1} - o(r^{-1}))\varphi'\} - \frac{1}{2}(r\varphi'')' - o(1)\varphi' - o(q^{1/2})\varphi' - o(1)|\varphi' + r\varphi''|^2.$$

We need much space to present the proof of this proposition. So we just mention to the following important inequality.

**Lemma 9.3** *Suppose that (8.3) and (4.2). Then, it holds that*

$$(9.4)\quad |2\lambda \mathrm{Re}\int_{s-1}^{t+1}\langle r\mathcal{J}_\omega Q_3\alpha\zeta, \partial_r\zeta\rangle| \\ \leq \int_{s-1}^{t+1}\{\lambda^2\langle rV'\zeta,\zeta\rangle + |\varphi' + r\varphi''|\|h^{-1}\alpha\zeta\|^2\}dr, \quad t > s \gg 1.$$

Now we are going to show

$$(\log r)^n v, \quad r^n v, \quad \exp\{nr^\rho\}v \in L^2(D_a), \quad \forall n \in \mathbf{N},\ \forall \rho \in (0,1).$$

Choosing respectively $q(r) = \log^{1/2} r$, $r^{b/2}$ and finally $e^{r^b(\log r)^2}$ as the weight function of (??), we obtain three kind of weighted inequalities. The first one is as follows.

$$(9.5)\quad \int_{s-1}^{t+1}(\log r)^n\|\chi u\|^2 dr \leq C\{\int_{s-1}^{t+1} o(1)(1+n^2(\log r)^{-2})(\log r)^n\|\chi u\|^2 dr \\ +\{\int_{s-1}^{s}+\int_{t}^{t+1}\}n(\log r)^{n-1}\|u\|^2 dr.$$

We shall use

$$\liminf_{N\to\infty} N\int_N^{N+1}\|u\|^2 dr = 0.$$

By letting $t \to \infty$ in (9.5), an induction procedure implies that if $v \in L^2(D_a)^6$,

$$(\log r)^{n/2} v \in L^2(I_a)^6, \quad \forall n = 0,1,2,\dots.$$

In view of

$$r^m = \exp\{m\log r\} = \sum_{n=0}^{\infty}(m\log r)^n/n!,$$

we can conclude that $r^m v \in L^2(I_a)^6$. In the same manner, we see that

$$(9.6)\quad \int_s^\infty \sum_{n=2}^{N}\frac{1}{n!}(mr^b)^n\|u\|^2 dr \\ \leq C\int_{s-1}^{\infty} r^{-2(1-b)}m^2\sum_{n=2}^{N}\frac{1}{(n-2)!}(mr^b)^{n-2}\|u\|^2 dr + C_m(u)$$

for all $N = 2,3,\dots$. Finally, we arrive at

$$e^{nr^b}v \in L^2(I_a)^6, \quad \forall n = 1,2,\dots.$$

for any $b \in (0,1)$.

Applying the weighted inequality with $e^{2\varphi} = e^{nr^b(\log r)^2}$, we can conclude that

**Lemma 9.4** *For every $n \in \mathbf{N}$ and every $s \geq a+1$,*

$$\int_s^\infty e^{nr^b(\log r)^2}\|v\|^2 dr \leq C\int_{a-1}^a ne^{nr^b(\log r)^2}\|v\|^2 dr. \tag{9.7}$$

**Proof:** To prove this, we have to show that $k_\chi > 0$. Indeed, if $e^\varphi = \{r^b(\log r)^2\}^n$, it holds that

$$\varphi'/n = (r^b(\log r)^2)' = br^{b-1}(\log r)^2 + 2r^{b-1}\log r,$$

$$\varphi''/n = b(b-1)r^{b-2}(\log r)^2 + 2br^{b-2}(\log r) + 2(b-1)r^{b-2}\log r + 2r^{b-2}.$$

Therefore,

$$r\varphi'(\varphi'' + r^{-1}\varphi') = n^2b^2r^{b-2}(\log r)^2 br^b(\log r)^2(1+o(1)) = n^2b^3r^{2b-2}(\log r)^4(1+o(1))$$

and

$$(r\varphi'')' + \varphi' o(1) = nb(b-1)^2 r^{b-2}(\log r)^2 + no(r^{b-1}(\log r)^2).$$

Let $X = nr^{b-1}(\log r)^2$. Then, there exists a positive number $\sigma_0$ such that

$$\lambda q_0 + b^3X^2 - o(X) - o(X^2) \geq \sigma_0(1+X^2), \quad \forall X \geq 0.$$

□

Now, we are in the final step for proving Theorem 3.1. Let $\phi = r^b(\log r)^2$. From (9.7), it follows that

$$\int_{s+1}^\infty \|v\|^2 dr \leq Cn\exp\{2n(\phi(s) - \phi(s+1))\}\int_{s-1}^s \|v\|^2 dr.$$

Since $\phi(r)$ is monotone increasing, we see

$$0 < e^{\varphi(s)-\varphi(s+1)} < 1.$$

Letting $n \to \infty$, we conclude that $v = 0$ in $D_{s+1}$. On account of unique continuation theorem for the time harmonic Maxwell equations, we see that $v = 0$ in $U$. □

# 10 Potentials diverging at infinity

In this section we shall prove Theorem 3.2.

If $q_1$ and $q_2$ are in $m^2_{1/4}(I_a)$, then it holds that $q_0 = (q_1q_2)^{1/2} \in m^2_{1/2}(I_a)$. Furthermore, if $q_1$ and $q_2$ are in $m^2_{1/2}(I_a)$ and both $q_1q_2^{-1}$ and $q_2q_1^{-1}$ are bounded, then $q_0 = (q_1q_2)^{1/2} \in m^2_{1/2}(I_a)$.

It suffices to consider only the case where $q_2q_1^{-1} \in L^\infty(I_a)$. We can treat the other case in the same manner. Define

$$\tilde{\Gamma}_\infty = \begin{pmatrix} I & 0 \\ 0 & q_1^{-1}q_2 I \end{pmatrix}$$

If $v = q_0^{-1/2}\tilde{\Gamma}_\infty^{1/2} ru$, then $\zeta = \chi e^{\varphi} v$ satisfies

$$(10.1)\quad \{-\mathcal{J}_\omega\partial_r - r^{-1}\mathcal{J}_\Omega + \mathcal{J}_\omega\left(\varphi' - \frac{1}{2}(q_0^{-1}q_0' + Q_4)\right)\}\alpha\zeta - \lambda\tilde{\Gamma}\zeta = -\mathcal{J}_\omega\chi' e^{\varphi} v (:= \mathcal{J}_\omega g_\chi),$$

where

$$Q_4 = \frac{1}{2q_1q_2}\begin{pmatrix} (q_1q_2' - q_1'q_2)I & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{\Gamma} = q_0 + \tilde{\Gamma}_\infty^{1/2}(\Gamma - q_0 I)\Gamma_\infty^{-1/2}.$$

Thus, it holds that

$$(10.2)\quad \lambda^2\int_{s-1}^{t+1}\langle\partial_r[r\tilde{\Gamma}]\zeta,\zeta\rangle dr - 2\lambda\mathrm{Re}\int_{s-1}^{t+1}\langle r\mathcal{J}_\omega(\varphi' - \frac{1}{2}q_0^{-1}q_0' + Q_4)\alpha\zeta, \partial_r\zeta\rangle dr = 2\mathrm{Re}\int_{s-1}^{t+1}\langle r\mathcal{J}_\omega g_\chi, \partial_r\zeta\rangle dr.$$

If

$$h_0(r) = q_0(q_0' + \frac{1}{2}r^{-1}q_0)^{-1/2}$$

then, we have

**Lemma 10.1** *Let $q_1$ and $q_2$ belong to $m_{1/4}^2(I_a)$ and $q_2q_1^{-1}$ be bounded at infinity. Suppose that $\varepsilon$ and $\mu$ are scalar functions belonging to $S_{1/4}(q_1)$ and $S_{1/4}(q_2)$, respectively. Moreover, we assume that*

$$q_1q_2' - q_1'q_2 = o(r^{-1}q_1q_2).$$

*Then, it holds that*

$$(10.3)\quad |2\lambda\mathrm{Re}\int_{s-1}^{t+1}\langle r\mathcal{J}_\omega(\varphi' - \frac{1}{2}q_0^{-1}q_0' + Q_4)\alpha\zeta, \partial_r\zeta\rangle dr| \le \int_{s-1}^{t+1}\{\lambda^2\langle r\tilde{\Gamma}'\zeta,\zeta\rangle + |\varphi' + r\varphi''|\|h_0^{-1}\alpha\zeta\|^2\}dr, \quad t > s \gg 1.$$

By this lemma, we can show the analogue to Proposition 9.2.

**Proposition 10.2**

$$(10.4) \quad \int_N^\infty \left[\lambda^2\langle (r\tilde{\Gamma})' e^{\varphi} v, e^{\varphi} v\rangle + (-k_\varphi) q_0^{-1} \|e^{\varphi} v\|^2 + r\varphi' \|\partial_r (e^{\varphi} v/\sqrt{q_0})\|^2\right] dr$$
$$\leq C \int_{s-1}^{s} \{(1 + |\varphi'| q_0^{-1}) r |\chi'|^2 + r q_0^{-1} |\varphi''| |\chi'|\} \|e^{\varphi} v\|^2 dr$$

*for any* $N \geq s$.

Our choice of $v$ gives

**Lemma 10.3** *If* $u \in L^2(U)$ *is a solution to (2.2), then* $\tilde{v} = q_0^{-1/2} \tilde{\Gamma}_\infty^{1/2} r u$ *satisfies*

$$\langle \mathcal{J}_\omega \partial_r \alpha \tilde{v}, \tilde{v}\rangle \in L^1(I_a).$$

In view of Lemma 10.3 and Proposition 9.2, we can prove that

$$v = 0 \text{ if } |x| > a \gg 1$$

in the same manner as in the previous section.

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