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Summary note on the moduli of punctured tori and related Fuchsian differential equations

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1 Setting of the problem, due to Keen, Rauch and Vasquez [KRV]

Let us consider an elliptic curve with Legendre module t

$$E(t) : y^2 = x(x-1)(x-t),$$

and suppose $E(t) = \mathbf{C}/L$ for a certain lattice $L = \mathbf{Z} + \mathbf{Z}\tau$ on $\mathbf{C}$. Let $E'(t) = E(t) - \{\infty\}$ be the one point punctured torus. We have

$$E'(t) = \mathbf{H}/G_0,$$

for some discrete group G_0 acting on the upper half plane $\mathbf{H}$. G_0 is generated by two elements B_1 and B_2 those correspond to a homology basis of $E(t)$. We set $p = trace(B_1), q = trace(B_2), r = trace(B_1B_2)$. The triple (p, q, r) is called the Fricke parameter for $E'(t)$. We are requested to get an explicit description of the Fricke parameter in terms of the complex moduli t.

This problem is studied mainly by [KRV]. We are interested in it because it is the simplest case of the study of the moduli space of punctured Riemann surfaces and at the same time is a first step for the study of modukar functions for a quadrangle group. We do't show any essentially new result on this original problem than those of [KRV]. But we show some relations between this problem and the theory of deformations for Fuchsian differential equations with an apparent singularity.

2 Lamé equation

Let us consider the universal covering space $\mathbf{H}$ of $\mathbf{C} - L$, and let pr be the natural projection. By taking the quotient we have the isomorphism $\mathbf{H}/G_0 \sim (\mathbf{C} - L)/L$.

Theorem 2.1 *(Keen, Rauch and Vasquez [KRV])*
Let ν be a local inverse of $pr : \mathbf{H} \to \mathbf{C} - L$. Then $\nu(\zeta)$ is given by a ratio of two independent solutions of

$$\frac{d^2w}{d\zeta^2} + \frac{1}{4}(\wp(\zeta; L) + C(L))w = 0 \tag{2.1}$$

for some nice constant $C(L)$. We have

$$\lambda^2 C(\lambda L) = C(L) \quad \lambda \in \mathbf{C}^*, C(\overline{L}) = \overline{C}(L). \tag{2.2}$$

Note that this is a differential equation on the elliptic curve $\mathbf{C}/L$ with unique regular singularity at $\bar{0}$ (the orbit of 0 under L) and its monodromy group is equal to G_0.

— Relation to the Fuchsian differential equation on $\mathbf{P}^1$ —

Set $z_1 = \wp(\zeta)$. Let us change the differential equation (2.1) into the form

$$\eta'' + p(z_1)\eta' + q(z_1)\eta = 0, \quad \eta(z_1) = w(\zeta)) \tag{2.3}$$

in terms of the variable z_1. So (2.1) is transformed to

$$\frac{d^2\eta}{dz_1{}^2} + \frac{1}{2}\left(\frac{1}{z_1 - e_1} + \frac{1}{z_1 - e_2} + \frac{1}{z_1 - e_3}\right)\frac{d\eta}{dz_1} + \frac{C + z_1}{16(z_1 - e_1)(z - e_2)(z - e_3)}\eta = 0. \tag{2.4}$$

Here we used the convention $e_1 = \wp(1/2), e_2 = \wp(\tau/2), e_3 = \wp((1+\tau)/2)$. By the transformation $z_1 = (e_2 - e_1)x + e_1$ we obtain

$$\begin{aligned} &\frac{d^2\eta}{dx^2} + P(x)\frac{d\eta}{dx} + Q(x,H)\eta = 0, \\ &P(x) = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right), \\ &Q(x,H) = \frac{1}{16x(x-1)} - \frac{t(t-1)H}{x(x-1)(x-t)}. \end{aligned} \tag{2.5}$$

where $t = (e_3 - e_1)/(e_2 - e_1)$ and

$$C = -e_3 + \frac{(e_3 - e_1)(e_2 - e_3)}{e_2 - e_1}H.$$

(2.5) is the general form of the Fuchsian differential equation with the Riemann scheme

$$\begin{pmatrix} 0 & 1 & t & \infty \\ 0 & 0 & 0 & 1/4 \\ 1/2 & 1/2 & 1/2 & 1/4 \end{pmatrix}. \tag{2.6}$$

- Reduction of the problem -

Now our problem can be devided in two problems

[Problem 1]:

Describe the Fricke parameter in terms of the nice constant C in (2.1).

[Problem 2]:

Describe the nice constant C (that is equivalent to describe the nice accessary parameter H) in terms of the moduli variable t in (2.5).

3 Hermitian condition

We investigate the [Problem 2] in a little bit different aspect.

For the moment we suppose $t \in \mathbf{R}$ and $t < 0 < 1 < \infty$. Let x_0 be a reference point in $\mathbf{H}$. Let γ_i be a circuit on $\mathbf{C} - \{t, 0, 1\}$ around i starting from x_0 $(i = t, 0, 1, \infty)$. Let A_i be the circuit matrix of the equation (2.5) with respect to a fixed system of independent solutions at $x = x_0$.

Let G be the projective monodromy group of (2.5), and set $B_1 = A_0A_t, B_2 = A_1A_0, B_3 = B_1B_2 = A_tA_1$. Let G_0 denote the subgroup of G generated by the system $\{B_1, B_2\}$.

Definition 3. 1 *The triple (p,q,r) = (Trace(B_1)), Trace(B_2), Trace(B_3)) is called the Fricke parameter for the Fuchsian differential equation (2.5).*

Proposition 3. 1 *In the above situation we have the following normal form of the system* $\{A_t, A_0, A_1\}$ *in the conjugacy class.*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_1 = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix}, A_t = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix} \tag{3.1}$$

with the condition

$$\begin{aligned} &a_1^2 + a_2^2 = 1, b_1^2 + b_2 b_3 = 1, \\ &a_2(b_2 - b_3) = \pm 2i. \end{aligned} \tag{3.2}$$

We have the exception of the system

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}, A_t = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}. \tag{3.3}$$

Remark 3. 1 *1) We note that the accessary parameter H determines the monodromy group G and the subgroup G_0. The conjugacy class of G determines the Fricke parameter up to signature, conversely the Fricke parameter determines G uniquely up to the change of b_2 and b_3 (but it determines unique G_0).*

2) If $H(t)$ is a nice accessary parameter for the moduli t mentioned in Section 2, the group G_0 coincides with the descrete group G_0 for $E'(t)$.

3) The parameter H is nice for t if and only if the group G_0 is a Fuchsian group. Especially, if the system $\{A_t, A_0, A_1\}$ has an invariant Hermitian form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we obtain the nice parameter.

4) If t is real and H is nice, the Schwarzian image of $\mathbf{H}$ by (2.5) makes a Poincaré quadrangle with index $(2, 2, 2, \infty)$.

Proposition 3. 2 *The system (3.1) with (3.2) preserves the Hermitian form*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

if and only if we have

$$a_1, ia_2, b_1 \in \mathbf{R}, \ b_3 = -\overline{b_2}. \tag{3.4}$$

4 Fricke parameter

Proposition 4. 1 *([KRV])*

For a nice parameter $H(t)$ we have

(0) $p^2 + q^2 + r^2 - pqr = 0$,

(i) $q(\tau) = p(-\frac{1}{\tau}), r(\tau) = p(-\frac{1}{1+\tau})$,

(ii) $\varphi(\rho) = (3, 3, 3), \varphi(i) = (2\sqrt{2}, 2\sqrt{2}, 4)$,

(iii) $\varphi(\Re(\tau) = 0) = \{pq = 2r\}, \varphi(\Re(\tau) = -1/2) = \{q = r\}, \varphi(|\, \tau \,| = 1) = \{p = q\}$.

Remark 4. 1 *The property* (0) *is valid also for the equation with any accessary parameter $H(t)$.*

– Application of Hill's method –

We suppose the lattice L takes the form $L = \mathbf{Z}\pi + \mathbf{Z}\pi ik$ The equation (2.1) can be regarded as an equation for real functions $\frac{d^2u}{dx^2} + Q(x)u = 0$ with real variable x and the periodicity $Q(x+\pi) = Q(x)$. By the Fourier expansion we have

$$\frac{d^2u}{dx^2} + (\theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos(2nx))u = 0. \tag{4.1}$$

We note that the solution may not be periodic.

-Floquet transformation-

Let u_1, u_2 be a system of independent solutions at $x = 0$. Then we have

$$\begin{pmatrix} u_1(x+\pi) \\ u_2(x+\pi) \end{pmatrix} = F \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$
$$\text{with} \quad F \in \mathrm{GL}(2, \mathbf{C}) \tag{4.2}$$

The above transformation F is called the Floquet transformation. This is nothing but the circuit transformation $C = A_0 A_t$, and we note $F \in \mathrm{SL}(2, \mathbf{C})$.

Let us consider the eigen function for F in the form

$$u(z) = e^{\mu z} \sum_{n=-\infty}^{\infty} b_n e^{2niz}. \tag{4.3}$$

So $F \circ u(z) = e^{\mu\pi} u(z)$, and we have the eigen value $e^{\mu\pi}$. Consequently we have another eigen value $e^{-\mu\pi}$, and

$$Trace(F) = Trace(C) = e^{\mu\pi} + e^{-\mu\pi}. \tag{4.4}$$

Let (4.3) substitute in (4.1), then we have the following system of infinite number of linear equations for $\{b_n\}$:

$$(\mu + 2ni)^2 b_n + \sum_{m=-\infty}^{\infty} \theta_m b_{n-m} = 0, \quad n = 0, \pm 1, \pm 2, \cdots \tag{4.5}$$

We define a formal determinant

$$\Delta(i\mu) = \mathrm{Det}\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{(i\mu+2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \cdots & \cdots & \cdots \\ \cdots & \frac{-\theta_1}{0^2-\theta_0} & \frac{(i\mu+0)^2-\theta_0}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} & \cdots & \\ \cdots & \cdots & \frac{-\theta_1}{2^2-\theta_0} & \frac{(i\mu-2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix}. \tag{4.6}$$

Theorem 4. 1 *(Hill, see also Whitacker-Watson 19.4) We suppose that θ_0 does not take the form $4m^2$ and that $\sum_{-\infty}^{\infty} \mid \theta_n \mid$ is convergent. Then $\Delta(i\mu)$ converges, and it holds*

$$\Delta(i\mu) = \Delta(0) - \sin^2(\frac{1}{2}\pi i\mu)/\sin^2(\frac{1}{2}\pi\sqrt{\theta_0}). \tag{4.7}$$

Consequently $e^{\mu\pi}$ is an eigen value if and only if we have

$$\Delta(0) - \sin^2(\frac{1}{2}\pi i\mu)/\sin^2(\frac{1}{2}\pi\sqrt{\theta_0}) = 0 \tag{4.8}$$

As an easy corollary of this theorem we have

Theorem 4. 2 *(see also Keen, Rauch and Vasquez [KRV])*

$$Trace(F) = 2 - \Delta(0)\sin^2(\frac{1}{2}\pi\sqrt{\theta_0}) \tag{4.9}$$

Remark 4. 2 *Suppose τ is pure imaginary. Then t and $C(L)$ take real values. The Lamé equation (2.1) can be considered as a Hill's equation by restricting it on the real axis (but rewriting in terms of σ functions. In that case the $Trace(F)$ becomes to be the first trace p of the Fricke parameter. We can proceed this method to get the traces q and r. Even for a general accessary parameter we get real values for p and q, but in general we obtain r with complex values. We can't see from that formula the criterion that r to be a real number.*

5 R-H problem of moving singularity type

We consider the following problem. Set a Riemann scheme

$$\begin{pmatrix} 0 & 1 & t & \infty \\ 0 & 0 & 0 & 1/4 \\ 1/2 & 1/2 & 1/2 & 1/4 \end{pmatrix} \tag{5.1}$$

for a Fuchsian differential equation

$$\begin{aligned} & y'' + p_0(x,t)y' + q_0(x,t,H_0)y = 0 \\ & p_0(x,t) = \frac{1-1/2}{x} + \frac{1-1/2}{x-1} + \frac{1-1/2}{x-t} \\ & q_0(x,t,H_0) = \frac{1/16}{x(x-1)} - \frac{t(t-1)H_0}{x(x-1)(x-t)} \end{aligned} \tag{5.2}$$

without apparent singularity. Let us regard (t, H_0) to be the deformation parameter of (5.2). Any representation of $\pi_1(\mathbf{P} - \{0,1,t,\infty\}, *)$ on $GL(2,\mathbf{C})$ admitting the Riemann scheme (5.1) appears as a monodromy group of (5.2) ?

5.1 Relation with the equation with an apparent singularity

Set a Riemann scheme with an apparent singularity $x = \lambda$

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ 0 & 0 & 0 & 0 & 1/4 \\ -1/2 & 1/2 & 1/2 & 2 & 1/4 \end{pmatrix} \tag{5.3}$$

for a Fuchsian differential equation

$$y'' + p(x,t,\lambda)y' + q(x,t,\lambda,\mu,H)y = 0. \tag{5.4}$$

For more general Riemann scheme

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ 0 & 0 & 0 & 0 & \rho_\infty \\ k_0 & k_1 & \theta & 2 & \rho_\infty + k_\infty \end{pmatrix} \tag{5.5}$$

we have the corresponding differential equation with

$$p(x,t) = \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{1}{x-\lambda}$$

$$q(x,t,H)=\frac{\kappa}{x(x-1)}-\frac{t(t-1)H}{x(x-1)(x-t)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}$$
$$H=\frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2-\{\kappa_0(\lambda-1)(\lambda-t)+$$
$$\kappa_1\lambda(\lambda-t)+(\theta-1)\lambda(\lambda-1)\}\mu+\kappa(\lambda-t)],$$
$$\kappa=\frac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2-\frac{1}{4}\kappa_\infty^2 \tag{5.6}$$

for the coefficient function. Here μ and H are accessary parameters, and H is described in terms of t,λ,μ by the fact that λ is non-logarithmic singularity. At this moment our Riemann scheme requires $-\kappa_0=\kappa_1=\theta=1/2, \kappa_\infty=0$.

Proposition 5. 1 *Let us fix t, and suppose $|\lambda|<\varepsilon$. The solution $u(x,\lambda)$ of (5.4) at $x=0$ is holomorphic on $\tilde{U-\{0\}}\times|\lambda|<\varepsilon$, where $\tilde{}$ means the universal covering. So $u(x,0)$ is a solution of (5.4).*

Remark 5. 1 *We can not perform this procedure in case $x=0$ is a logarithmic singularity.*

5.2 Isomonodromy deformation

Suppose we have $t\neq 0,1,\infty$ and $\lambda\neq 0,1,t,\infty$. The isomonodromy defermation of (5.4) is given by the Hamiltonian system

$$\frac{d\lambda}{dt}=\frac{\partial H}{\partial\mu},\frac{d\mu}{dt}=-\frac{\partial H}{\partial\lambda}. \tag{5.7}$$

Now we consider (5.7) in the tublar neiborhood $\{|\lambda|<\varepsilon\}\times(t,\mu)-$space. Let us take an initial data $(t,\lambda,\mu)=(t_0,0,\mu_0)$, then we can find the integral curve by following (5.7).

Proposition 5. 2 *For any initial data $(t,\lambda,\mu)=(t_0,0,\mu_0)$ $(t_0\neq 0)$ the integral curve intersects the axis $\lambda=0$ transversally, so any monodromy group of (5.2) is obtained as the one of an isomonodromic deformation of (5.4).*

[Proof]. We have

$$\frac{\partial H}{\partial\mu}(t,0,\mu)=t/2,-\frac{\partial H}{\partial\lambda}(t,0,\mu)=-1/16+(\mu-\mu^2t). \tag{5.8}$$

So it holds $\frac{d\mu}{d\lambda}(t,0,\mu)\neq 0$ for $t\neq 0$.

q.e.d.

Remark 5. 2 *This Proposition offers a mapping from the monodromy space of (5.2) to that of (5.4). According to the Riemann-Hilbert correspondence this map induces the natural injective map to the complex variety of the set of representations of $\pi_1(\mathbf{P}-0,1,t,\infty,*)$. It seems to be an open mapping. So our problem in the beginning of this section will have the positive answer.*

6 Some comments

(1) In the Riemann scheme (5.3) we can let λ move to the infinity. This process is visualized by the deformation $\lambda'\to 0$ using the transformed variable and singularity : $x'=t/x,\lambda'=t/\lambda$. So we have

$$\begin{pmatrix}0 & 1 & t & \lambda' & \infty\\ 0 & 0 & 0 & 0 & -1/4\\ 0 & 1/2 & 1/2 & 2 & 1/4\end{pmatrix}. \tag{6.1}$$

In that limit procedure we don't have the Proposition corresponding to Prop. 5.1. The Hamiltonian equation reduses a system corresponding the logarithmic derivative of a certain hypergeometric differential equation on $\lambda' = 0$. So the integral orbit for the system does not go out from the wall $\lambda' = 0$, and its orbit does not mean the isomonodromy. We don't know the meaning of this degenerate eqation.

(2) Let us consider the isomonidromy orbit of (5.4) for the nice accessary parameter. We always have a fixed monodromy group. Suppose t is real, then the nice parameter H is also real. Its image of the upper half plane by the Schwarz map differs depending on the value of H. We don't know what happens during this deformation for the figures of Schwarz map.

(3) It is quite difficult to proceed the observation of (2). We prefer to consider more basic case. Namely the hypergeometric differential equation with an apparent singularity. There are articles by Klein [K], Schilling [S] and Ritter [R] those study this problem. Also we regard it a simplest case study of the differential equation which induces a quadrangle group.

(4) Let us fix two Riemann schemes they are mutually contiguous, and let $P_1(x)$ and $P_2(x)$ be the solutions. So these two functions have multibaluedness with the same monodromy behavior. If we take the linear combination $aP_1(x)+bP_2(x)$, it satisfies a differential equation with the same monodromy. Klein [K] appointed that this differential eqation will have the apparent singularity and vice versa. So we are interested where we have the apparent singularity depending on a, b.

References

[1] Dubrovin, B. and Mazzocco, M., Monodromy of certain Painlevé - VI transcendents and reflection groups, preprint (1998) arxiv:math. AG/9806056

[2] Hitchin, N.J., The self-duality equations on a Riemann surface, Proc. London Math. Soc.,55(1987), 59-126.

[3] Iwasaki, K., Kimura, H., Shimomura, S. and Yoshida, M., From Gauss to Painlevé, Vieweg (19911).

[4] Kawai, S., Isomonodromic deformation of Fuchsian type projective connections on elliptic curves, RIMS Kokyuroku 1022(1997), 53 - 57.

[5] Klein, Vorlesung über die Hypergeometrische Function, Springer.

[6] Keen, L., Rauch, H.E. and Vasquez, A.T., Moduli of punctured tori and the accessory parameter of Lamé's equation, Trans. Act. A.M.S., 255(1979), 201 -230.

[7] K. Okamoto, On Fuchs' problem on a torus I, Funkcial Ekvac 14(1971), 137 -152.

[8] K. Okamoto, Sur le problème de Fuchs sur un tore II, J. Fac. Sci. Univ. Tokyo, Sect.IA 24(1977), 357 -?.

[9] Ritter, E., Ueber die hypergeometrische Function mit einem Nebenpunkt, Math. Ann.,48(1897), 1-36.

[10] Schilling, F., Ueber die Theorie der symmetriscen S-Functionen mit einem einfachen Nebenpunkte, Math. Ann.,52(1899), 481-522.

[11] Whitacker,E.T. and Watson, G.N., A course in modern analysis, Cambridge U.P.(1950).

[12] Yoshida, M., Fuchsian Differential Equations, Vieweg (1987).