数理解析研究所講究録 1225 巻 2001 年 34-45

Rapid decay of solutions to the non-stationary Stokes equations in exterior domains

Hideo Kozono
Mathematical Institute, Tohoku University
小薗 英雄 (東北大・理)
e-mail:kozono@math.tohoku.ac.jp

Introduction

Let Ω be an *exterior* domain in $\mathbb{R}^n (n \geq 3)$, i.e., a domain having a compact complement $\mathbb{R}^n \setminus \Omega$ with the smooth boundary $\partial\Omega$. Consider the initial-boundary value problem of the Stokes equations in $\Omega \times (0, \infty)$:

$$\text{(S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = 0 & \text{in } x \in \Omega,\ 0 < t < \infty, \\ \operatorname{div} u = 0 & \text{in } x \in \Omega,\ 0 < t < \infty, \\ u = 0 \quad \text{on } \partial\Omega, & u(x,t) \to 0 \quad \text{as } |x| \to \infty, \\ u|_{t=0} = a, & \end{cases}$$

where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and pressure of the fluid at the point $(x,t) \in \Omega \times (0,\infty)$, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ is the given initial velocity vector field.

It was shown by Solonnikov [22], [23] that for every $a \in L^q_\sigma(\Omega)$ with $1 < q < \infty$, there exists a unique solution u of (S) in $C([0,\infty); L^q_\sigma(\Omega))$ with $\partial_t u, \partial_x^2 u \in C((0,\infty); L^r(\Omega))$ for all $q \leq r < \infty$. As for asymptotic behaviour of $u(t)$ as $t \to \infty$, Iwashita [8] proved the following $L^q - L^r$-estimates

$$\|u(t)\|_{L^r} \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|a\|_{L^q} \quad \text{for } 1 < q \leq r \leq \infty, \tag{0.1}$$

$$\|\nabla u(t)\|_{L^r} \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|a\|_{L^q} \quad \text{for } 1 < q \leq r \leq n, \tag{0.2}$$

where $C = C(n,q,r)$ is a constant independent of $t > 0$ and $a \in L^q_\sigma(\Omega)$. See also Chen [3].

The first purpose of this note is to investigate the above $L^q - L^r$-estimates for $q = 1$. It is an open question whether (S) has a solution when a belongs to $L^1(\Omega)$. For every $a \in L^1(\Omega)$ with certain regularity, we shall establish (0.1) and (0.2) with some additional term on the right hand side. In this decade, many authors discussed on the L^2 decay of weak solutions to the Navier-Stokes equations in exterior domains([16], [9], [17], [1], [2], [12]). In particular, they made an effort to get the optimal decay rate in L^2 as $t \to \infty$. In exterior domains, the best decay rate up to the present was given by Borchers-Miyakawa

[2]; if the solution u of (S) satisfies $\|u(t)\|_{L^2} = O(t^{-\alpha})$ as $t \to \infty$, then weak solutions v of the Navier-Stokes equations with the same initial data a are subordinate to the estimate

$$\|v(t)\|_{L^2} = \begin{cases} O(t^{-\alpha}) & \text{provided } 0 \le \alpha \le n/4, \\ O(t^{-n/4}) & \text{provided } n/4 < \alpha. \end{cases} \tag{0.3}$$

This indicates that the decay order in L^2 of the Navier-Stokes flows seems to be dominated by the *linear* Stokes flow, and their best rate might be $t^{-n/4}$ which is formally obtained by taking $q = 1$ and $r = 2$ in (0.1). Our result may be regarded as concrete characterization of the initial data a with which the Stokes flow $u(t)$ exhibits this marginal behaviour as $t \to \infty$. Simultaneously, it is an interesting question whether or not $\|u(t)\|_{L^2} = o(t^{-n/4})$. Our second purpose is to show that more rapid decay of $\|u(t)\|_{L^r}$ than (0.1) occurs only in a special situation. Indeed, we shall prove that $\|u(t)\|_{L^r} = o(t^{-\frac{n}{2}(1-\frac{1}{r})})$ for some $1 < r \le \infty$ if and only if there holds

$$\int_0^\infty dt \int_{\partial\Omega} T[u,p](y,t) \cdot \nu dS_y = 0,$$

where $T[u,p] = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p\}_{i,j=1,\cdots,n}$ denotes the stress tensor, and $\nu = (\nu_1, \cdots, \nu_n)$ and dS denote the unit outward normal and the surface element of $\partial\Omega$, respectively.

1 Results

Before stating our results, we first introduce some function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ denote the set of all C^∞ vector functions $\phi = (\phi_1, \cdots, \phi_n)$ with compact support in Ω, such that div $\phi = 0$. $L_\sigma^r(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r \equiv \|\cdot\|_{L^r(\Omega)}$; $(\cdot,\cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space over Ω, $1 \le r \le \infty$. It is known that for $1 < r < \infty$, $L_\sigma^r(\Omega)$ is characterized as

$$\begin{aligned} &L_\sigma^r(\Omega) \\ &= \{u \in L^r(\Omega); \text{div } u = 0 \quad \text{in } \Omega,\ u \cdot \nu = 0 \text{ on } \partial\Omega \text{ in the sense } W^{1-1/r',r'}(\partial\Omega)^* \} \end{aligned} \tag{1.1}$$

and that there holds the Helmholtz decomposition

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega) \quad (\text{direct sum}), 1 < r < \infty,$$

where $G^r(\Omega) = \{\nabla p \in L^r(\Omega); p \in L_{loc}^r(\bar{\Omega})\}$. We denote by P_r the projection operator from $L^r(\Omega)$ onto $L_\sigma^r(\Omega)$ along $G^r(\Omega)$. Then the Stokes operator A_r is defined by $A_r = -P_r\Delta$ with the domain $D(A_r) = \{u \in W^{2,r}(\Omega) \cap L_\sigma^r(\Omega); u|_{\partial\Omega} = 0\}$. It is proved by Giga-Sohr [6] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}\}_{t\ge 0}$ of class C_0 in $L_\sigma^r(\Omega)$ for $1 < r < \infty$. Hence one can define the fractional power A_r^α for $0 \le \alpha \le 1$. There holds an embedding $D(A_r^\alpha) \subset W^{2\alpha,r}(\Omega)$ with

$$\|u\|_{W^{2\alpha,r}(\Omega)} \le C(\|u\|_r + \|A_r^\alpha u\|_r), \quad u \in D(A_r^\alpha), \tag{1.2}$$

where $C = C(n, r, \alpha)$ is independent of u.

For $a \in L^r_\sigma(\Omega)$, $u(t) = e^{-tA}a$ gives a unique solution of (S) together with a scalar function p such that

$$\nabla p \in C((0,\infty); L^r(\Omega)). \tag{1.3}$$

We call such p the pressure *associated* with u. In particular, if $1 < r < n$, by (S) and the Sobolev embedding([6, Corollary 2.2]), we may take p so that $p \in C((0,\infty); L^{nr/(n-r)}(\Omega))$.

Throughout this paper, we impose the following assumption on the initial data.

Assumption. For some $n/(n-2) < q_* < \infty$ and $\varepsilon > 0$ the initial data a belongs to $L^1(\Omega) \cap D(A^\varepsilon_{q_*})$.

Our first result now reads:

Theorem 1. *Let the Assumption hold. Then we have*

$$\|e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \qquad 1 < r \le \infty, \tag{1.4}$$

$$\|\nabla e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \qquad 1 \le r \le n, \tag{1.5}$$

for all $t > 2$ with $C = C(n, q_*, \varepsilon, r)$ independent of a.

Remarks. 1. In (1.4), we do not know whether $r = 1$ is possible. It is shown by the author [15] that $u \in C([0,\infty); L^1(\Omega))$ with its associated pressure $p \in C((0,\infty); L^{n/(n-1)}(\Omega))$ if and only if the net force exerted to $\partial\Omega$ by the fluid is equal to zero:

$$\int_{\partial\Omega} T[u,p](y,t)\cdot \nu dS_y = 0 \quad \text{for all } 0 < t < \infty, \tag{1.6}$$

where $T[u,p] = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p\}_{i,j=1,\cdots,n}$ denotes the stress tensor, and $\nu = (\nu_1, \cdots, \nu_n)$ and dS denote the unit outward normal and the surface element of $\partial\Omega$, respectively. Hence, it seems to be difficult to take $r = 1$ in (1.4) for all a satisfying the Assumption.

2. On the other hand, in (1.5), we may include $r = 1$. This is closely related to the fact that $\nabla\Gamma_t$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, where $\Gamma_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ denotes the Gauss kernel. In the half-space $\mathbb{R}^n_+$, Giga-Matsui-Shimizu [7] obtained a shaper estimate than (1.5) like $\mathcal{H}^1 - L^r$-type.

We next investigate the more rapid decay than (1.4):

Theorem 2. *Let a be as in the Assumption. If*

$$\|e^{-tA}a\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \textit{for some } 1 < r \le \infty \tag{1.7}$$

as $t \to \infty$, then there holds

$$\int_0^\infty dt \int_{\partial\Omega} T[u,p](y,t)\cdot \nu dS_y = 0. \tag{1.8}$$

Conversely, if (1.8) holds, then we have

$$\|e^{-tA}a\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}), \qquad \text{for all } 1 < r \le \infty \tag{1.9}$$

$$\|\nabla e^{-tA}a\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \qquad \text{for all } 1 < r \le n \tag{1.10}$$

as $t \to \infty$.

Remarks. 1. Theorem 2 shows a significant difference of asymptotic behaviour of solutions between the whole space $\mathbb{R}^n$ and *exterior* domains. In $\mathbb{R}^n$, the Stokes semi-group e^{-tA} is essentially identical with the heat operator so that we have $\lim_{t\to\infty} \|e^{-tA}a\|_{L^1(\mathbb{R}^n)} = 0$ for all $a \in L^1(\mathbb{R}^n)$ with div $a = 0$. Hence both (1.9) and (1.10) are always true in $\mathbb{R}^n$. Furthermore, if we impose some momentum condition on a, then the better decay than (1.9) and (1.10) can be obtained. See Miyakawa [18, Lemma 3.3].

2. (1.7) is a condition on the solution $u(t)$ to (S) at $t = \infty$. On the other hand, (1.8) is restriction on the solution on the whole interval $t \in (0, \infty)$. So, it turns out that the more rapid decay in L^r than $t^{-\frac{n}{2}(1-\frac{1}{r})}$ as $t \to \infty$ has a influence even on the global behaviour on $(0, \infty)$ of the solution $u(t)$.

3. As we have seen in (0.1), for $a \in L^q_\sigma(\Omega)$ with the *lower* integral exponent q, the *better* decay of $\|e^{-tA}a\|_r$ for $q \le r \le \infty$ as $t \to \infty$ is expected. Theorem 2 states that, in general situations in exterior domains, we cannot realize the better decay than Theorem 1, and that the condition on the net force exerted $\partial\Omega$ such as (1.6) and (1.8) controls the asymptotic behaviour of the solutions $u(x,t)$ as $|x| \to \infty$, $t \to \infty$. As for the influence of the net force on the solutions of the *stationary* problems, see e.g., Finn [4], [5] and Kozono-Sohr[13]. See also [14].

2 Representation formula

In this section, we shall establish a representation formula of the solution to (S) for the initial data a satisfying the Assumption. To this end, we need to investigate behaviour of the boundary integral $\int_{\partial\Omega} T[u,p](y,t)\cdot\nu(y)dS_y$ as $t \to 0$. We observe also its behaviour as $t \to \infty$. In what follows we shall denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes constants depending only on the quantities appearing in the parenthesis.

Lemma 2.1 *Let* $n/(n-2) < q_* < \infty$ *and let* q *be as* $1/q - 1/n = 1/q_*$, *i.e.,* $q = nq_*/(n+q_*)$. *For every* $1 < l < q$, *there is a constant* $C = C(n, q_*, l)$ *such that*

$$\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_y \le Ct^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{q_*})}(\|a\|_1 + \|a\|_{q_*}) \tag{2.1}$$

for all $1 \le t < \infty$ *and all* $a \in L^1(\Omega) \cap L^{q_*}_\sigma(\Omega)$, *where* $u(t) = e^{-tA}a$ *with its associated pressure* p. *If, in addition,* a *satisfies the Assumption, then there holds*

$$\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_y \le Ct^{\alpha-1}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q*}) \tag{2.2}$$

with $\alpha \equiv (\frac{1-1/q}{1-/q_*})\varepsilon$ *for all* $0 < t \le 1$, *where* $C' = C(n, q_*, \varepsilon)$.

Let us recall the fundamental tensor $\{E_{ij}(x,t)\}$, $i,j = 1,\cdots,n$ to (S) defined by

$$E_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}(\Gamma(\cdot,t) * G)(x), \quad i,j = 1,\cdots,n,$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad G(x) = \frac{1}{(n-2)\omega_n}|x|^{2-n} \quad (\omega_n\text{: area of the unit sphere in } \mathbb{R}^n).$$

Our representation formula now reads:

Theorem 2.1 (Representation formula) *Let* a *be as in the Assumption. The solution* $u(t) = e^{-tA}a$ *to (S) can be represented as*

$$\begin{aligned} u_i(x,t) &= \int_\Omega \Gamma(x-y,t)a_i(y)dy \\ &\quad + \int_0^t d\tau \int_{\partial\Omega} \sum_{j,k=1}^n E_{ij}(x-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1,\cdots,n, \end{aligned}$$

for all $(x,t) \in \Omega \times (0,\infty)$, *where* $T_{jk}[u,p](y,\tau) = \frac{\partial u_j}{\partial y_k}(y,\tau) + \frac{\partial u_k}{\partial y_j}(y,\tau) - \delta_{jk}p(y,\tau)$, $j,k = 1,\cdots,n$ *and* $\nu(y) = (\nu_1(y),\cdots,\nu_n(y))$ *is the unit outward normal to* $y \in \partial\Omega$.

Knightly [10, (6)] gave the representation formula to solutions of the Navier-Stokes equations by assuming $u(x,t) = o(1)$, $\nabla u(x,t), p(x,t) = o(|x|)$ as $|x| \to \infty$ locally uniformly in t. Mizumachi [19, Proposition 1] also showed under the stronger hypothesis than ours on the boundary integral in Lemma 2.1 which is due to Solonnikov [22].

3 $L^1 - L^r$ estimates; Proof of Theorem 1

To prove (1.4), we shall first restrict ourself to the case

$$(3.1) \qquad 1 < r < n/(n-2).$$

By Theorem 2.1, $u(t) = e^{-tA}a$ can be expressed as

$$(3.2) \qquad u(x,t) = v(x,t) + w(x,t) \quad \text{for all } (x,t) \in \Omega \times (0,\infty),$$

where $v(x,t) = (v_1(x,t),\cdots,v_n(x,t))$ and $w(x,t) = (w_1(x,t),\cdots,w_n(x,t))$ with

$$\begin{aligned} v_i(x,t) &\equiv \int_\Omega \Gamma(x-y,t)a_i(y)dy, \quad i = 1,\cdots,n, \\ w_i(x,t) &\equiv \int_0^t d\tau \int_{\partial\Omega} \sum_{j,k=1}^n E_{ij}(x-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1,\cdots,n. \end{aligned}$$

By the Hausdorff-Young inequality, we have

$$\|v(t)\|_r \le \|\Gamma(\cdot,t)\|_r\|a\|_1 \le Ct^{-\frac{n}{2}(1-\frac{1}{r})}\|a\|_1 \quad \text{for all } t>0 \tag{3.3}$$

with $C = C(n,r)$ independent of a. As for the estimate of $\|w(t)\|_r$, we notice that the fundamental tensor $\{E_{ij}\}_{i,j=1,\cdots,n}$ can be expressed as

$$E_{ij}(\cdot,t) = (\delta_{ij} + R_iR_j)\Gamma(\cdot,t), \quad i,j=1,\cdots,n, \tag{3.4}$$

where $R_i = \dfrac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}}$, $i=1,\cdots,n$ denote the Riesz transforms. Since R_i is bounded from $L^r(\mathbb{R}^n)$ into itself, we have

$$\|\partial_x^m\partial_t^k E_{ij}(\cdot,t)\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{m}{2}-k}, \quad m,k=0,1,\cdots \quad \text{for all } t>0, \tag{3.5}$$

which yields

$$\begin{aligned}
\|w(t)\|_r &\le \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \|E_{ij}(\cdot-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)\|_r dS_y \\
&\le \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} |T_{jk}[u,p](y,\tau)\nu_k(y)|\|E_{ij}(\cdot-y,t-\tau)\|_r dS_y \\
&\le C\int_0^t (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\left(\int_{\partial\Omega}(|\nabla u(y,\tau)|+|p(y,\tau)|)dS_y\right)d\tau
\end{aligned} \tag{3.6}$$

By (2.2) there holds

$$\begin{aligned}
&\int_0^1 (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\left(\int_{\partial\Omega}(|\nabla u(y,\tau)|+|p(y,\tau)|)dS_y\right)d\tau \\
&\le C(t-1)^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1+\|a\|_{q_*}+\|A^\varepsilon a\|_{q_*})\int_0^1 \tau^{\alpha-1}d\tau \\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1+\|a\|_{q_*}+\|A^\varepsilon a\|_{q_*})
\end{aligned} \tag{3.7}$$

for all $t>2$. Next, we take $1<l<q\equiv nq_*/(n+q_*)$ so that

$$1/q_* < 1/l - 2/n.$$

For such l, there holds

$$-\frac{n}{2}\left(\frac{1}{l}-\frac{1}{q_*}\right)+1<0. \tag{3.8}$$

By (2.1) and (3.8) we have

$$\begin{aligned}
&\int_1^{t/2} (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\left(\int_{\partial\Omega}(|\nabla u(y,\tau)|+|p(y,\tau)|)dS_y\right)d\tau \\
&\le C(\|a\|_1+\|a\|_{q_*})\int_1^{t/2}(t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\tau^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{q_*})}d\tau \\
&\le C(\|a\|_1+\|a\|_{q_*})t^{-\frac{n}{2}(1-\frac{1}{r})}\left(1-(t/2)^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{q_*})+1}\right) \\
&\le C(\|a\|_1+\|a\|_{q_*})t^{-\frac{n}{2}(1-\frac{1}{r})}
\end{aligned} \tag{3.9}$$

for all $t>2$. It follows from (3.1) that $-\frac{n}{2}(1-\frac{1}{r})>-1$, and hence again by (2.1) and (3.8) we have

$$\begin{aligned}
&\int_{t/2}^{t}(t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\left(\int_{\partial\Omega}(|\nabla u(y,\tau)|+|p(y,\tau)|)dS_y\right)d\tau\\
&\le C(\|a\|_1+\|a\|_{q_*})t^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{q_*})}\int_{t/2}^{t}(t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}d\tau\\
&\le C(\|a\|_1+\|a\|_{q_*}))t^{-\frac{n}{2}(\frac{1}{l}-\frac{1}{q_*})+1-\frac{n}{2}(1-\frac{1}{r})}\\
&\le C(\|a\|_1+\|a\|_{q_*})t^{-\frac{n}{2}(1-\frac{1}{r})} \qquad (3.10)
\end{aligned}$$

for all $t>2$. Gathering (3.7), (3.9) and (3.10), we obtain from (3.6)

$$\|w(t)\|_r\le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*}),\quad t>2 \tag{3.11}$$

provided $1<r<n/(n-2)$. Now it follows from (3.2), (3.3) and (3.11) that

$$\|e^{-tA}a\|_r\le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*}),\quad t>2 \tag{3.12}$$

provided $1<r<n/(n-2)$. In case $n/(n-2)\le r\le\infty$, we may take $\tilde r$ so that $1<\tilde r<n/(n-2)$. Then by (0.1) and (3.12)

$$\begin{aligned}
\|e^{-tA}a\|_r &= \|e^{-\frac{t}{2}A}e^{-\frac{t}{2}A}a\|_r\\
&\le Ct^{-\frac{n}{2}(\frac{1}{\tilde r}-\frac{1}{r})}\|e^{-\frac{t}{2}A}a\|_{\tilde r}\\
&\le Ct^{-\frac{n}{2}(\frac{1}{\tilde r}-\frac{1}{r})}t^{-\frac{n}{2}(1-\frac{1}{\tilde r})}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*})\\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*}) \qquad (3.13)
\end{aligned}$$

for all $t>4$. From (3.12) and (3.13) we obtain (1.4).

Next, we shall prove (1.5). In case $1<r\le n$, we have by (0.2) and (1.4) just proved that

$$\begin{aligned}
\|\nabla e^{-tA}a\|_r &= \|\nabla e^{-\frac{t}{2}A}e^{-\frac{t}{2}A}a\|_r\\
&\le Ct^{-\frac{1}{2}}\|e^{-\frac{t}{2}A}a\|_r\\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*}), \qquad (3.14)
\end{aligned}$$

for all $t>4$, which yields (1.5) except for $r=1$.

Finally, it remains to prove (1.5) for $r=1$. Similarly to (3.3) , we can show easily that $\|\nabla v(t)\|_1\le Ct^{-1/2}\|a\|_1$ for all $t>0$ with $C=C(n)$ independent of a. Hence it suffices to prove

$$\|\nabla w(t)\|_1\le Ct^{-\frac{1}{2}}(\|a\|_1+\|a\|_{q_*}+\|A^{\varepsilon}a\|_{q_*})\quad\text{for all } t>2. \tag{3.15}$$

It is well-known that $\nabla\Gamma(\cdot,t)\in\mathcal{H}^1$ with $\|\nabla\Gamma(\cdot,t)\|_{\mathcal{H}^1}\le Ct^{-\frac{1}{2}}$, where $\mathcal{H}^1$ denotes the Hardy space on $\mathbb{R}^n$. Since the Riesz transform R_i, $i=1,\cdots,n$ is bounded from $\mathcal{H}^1$ into itself, we have by (3.4)

$$\|\nabla E_{ij}(\cdot,t)\|_1\le C\|\nabla\Gamma(\cdot,t)\|_{\mathcal{H}^1}\le Ct^{-\frac{1}{2}},\quad i,j=1,\cdots,n\quad\text{for all } t>0.$$

See e.g., Stein [24, Chapter III, 1.2.4]. Hence, as we have derived (3.6) from (3.5), we obtain

$$\|\nabla w(t)\|_1 \le C\int_0^t (t-\tau)^{-\frac{1}{2}} \left(\int_{\partial\Omega} (|\nabla u(y,\tau)| + |p(y,\tau)|)dS_y\right) d\tau. \tag{3.16}$$

Now it is easy to see that the same procedure as in (3.7), (3.9) and (3.10) works to the estimate of the right hand side of (3.16), and we get (3.15). This completes the proof of Theorem 1.

4 More rapid decay; Outline of the proof of Theorem 2

Without loss of generality, we may assume

$$\|e^{-tA}a\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } 1 < r < n/(n-1) \text{ as } t \to \infty. \tag{4.1}$$

Indeed, if (1.7) holds for some $n/(n-1) \le r \le \infty$, then by choosing $1 < r_0 < r_1 < n/(n-1)$ and $0 < \theta < 1$ with $1/r_1 = (1-\theta)/r_0 + \theta/r$, we have

$$\|e^{-tA}a\|_{r_1} \le \|e^{-tA}a\|_{r_0}^{1-\theta}\|e^{-tA}a\|_r^{\theta} = O(t^{-\frac{n}{2}(1-\frac{1}{r_0})(1-\theta)})\cdot o(t^{-\frac{n}{2}(1-\frac{1}{r})\theta}) = o(t^{-\frac{n}{2}(1-\frac{1}{r_1})})$$

as $t \to \infty$, which yields (4.1). By Theorem 2.1, we have similarly to (3.2) that

$$u_i(x,t) = v_i(x,t) + \tilde{w}_i(x,t) + \sum_{j,k=1}^{n} E_{ij}(x,t)\int_0^t d\tau \int_{\partial\Omega} T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1,\cdots,n, \tag{4.2}$$

for all $(x,t) \in \Omega \times (0,\infty)$, where $v = (v_1,\cdots,v_n)$ is the same as in (3.2) and $\tilde{w} = (\tilde{w}_1,\cdots,\tilde{w}_n)$ is defined by

$$\tilde{w}_i(x,t) \equiv \sum_{j,k=1}^{n}\int_0^t d\tau \int_{\partial\Omega} \{E_{ij}(x-y,t-\tau) - E_{ij}(x,t)\}\, T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1,\cdots,n. \tag{4.3}$$

Let us first show that

$$\|v(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for } 1 \le r \le \infty \text{ as } t \to \infty. \tag{4.4}$$

Indeed, defining $\tilde{a}(x) = a(x)$ for $x \in \Omega$ and $= 0$ for $x \in \mathbb{R}^n \setminus \Omega$, we have

$$v(x,t) = \int_{y\in\mathbb{R}^n} \Gamma(x-y,t)\tilde{a}(y)dy = e^{t\Delta}\tilde{a}(x) = e^{\frac{t}{2}\Delta}(e^{\frac{t}{2}\Delta}\tilde{a})(x)$$

for $(x,t) \in \Omega \times (0,\infty)$, where $e^{t\Delta}$ denotes the heat semi-group in $\mathbb{R}^n$. Hence there holds

$$\|v(t)\|_r \le \|e^{\frac{t}{2}\Delta}(e^{\frac{t}{2}\Delta}\tilde{a})\|_{L^r(\mathbb{R}^n)} \le Ct^{-\frac{n}{2}(1-\frac{1}{r})}\|e^{\frac{t}{2}\Delta}\tilde{a}\|_{L^1(\mathbb{R}^n)}, \quad 1 \le r \le \infty, t > 0. \tag{4.5}$$

Since $a \in L^1(\Omega) \cap L^{q_*}_\sigma(\Omega)$ for $q_* > n/(n-2)$, it follows from [15, Lemma 2.2] that

$$\int_{\mathbb{R}^n} \tilde{a}_i(y)dy = \int_\Omega a_i(y)dy = 0, \quad i = 1, \cdots, n.$$

By an elementary argument, we can show that this mean value property yields

$$\|e^{\frac{t}{2}\Delta}\tilde{a}\|_{L^1(\mathbb{R}^n)} \to 0 \quad \text{as } t \to \infty.$$

From this and (4.5) we obtain (4.4).

By a slightly technical calculation, we can also show

$$\|\tilde{w}(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}), \quad 1 < r < n/(n-1) \quad \text{as } t \to \infty. \tag{4.6}$$

On the other hand, there holds

$$\begin{aligned} &\liminf_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{r})} \left\| \sum_{j=1}^n E_{ij}(\cdot, t) \int_0^t f_j(\tau)d\tau \right\|_r \\ &\geq \left(\int_{y\in\mathbb{R}^n} \left| \sum_{j=1}^n E_{ij}(y, 1) \int_0^\infty f_j(\tau)d\tau \right|^r dy \right)^{\frac{1}{r}}, \quad i = 1, \cdots, n \end{aligned} \tag{4.7}$$

for all $1 < r < \infty$, where

$$f_j(\tau) = \int_{\partial\Omega} \sum_{k=1}^n T_{jk}[u, p](y, \tau)\nu_k(y)dS_y, \quad j = 1, \cdots, n.$$

First, if we take l as in (3.8), then Lemma 2.1 yields $\int_0^\infty f_j(\tau)d\tau < \infty$, $j = 1, \cdots, n$. Since $E_{ij}(x, t) = t^{-n/2}E_{ij}(x/\sqrt{t}, 1)$, we have

$$\begin{aligned} & t^{\frac{n}{2}(1-\frac{1}{r})} \left\| \sum_{j=1}^n E_{ij}(\cdot, t) \int_0^t f_j(\tau)d\tau \right\|_r \\ = \ & t^{-\frac{n}{2r}} \left\| \sum_{j=1}^n E_{ij}(\cdot/\sqrt{t}, 1) \int_0^t f_j(\tau)d\tau \right\|_r \\ \geq \ & t^{-\frac{n}{2r}} \left(\int_{|x|\geq R} \left| \sum_{j=1}^n E_{ij}(x/\sqrt{t}, 1) \int_0^t f_j(\tau)d\tau \right|^r dx \right)^{\frac{1}{r}} \\ & \text{(by changing variable } x \to y = x/\sqrt{t}) \\ = \ & \left(\int_{|y|\geq R/\sqrt{t}} \left| \sum_{j=1}^n E_{ij}(y, 1) \int_0^t f_j(\tau)d\tau \right|^r dy \right)^{\frac{1}{r}} \\ \to \ & \left(\int_{y\in\mathbb{R}^n} \left| \sum_{j=1}^n E_{ij}(y, 1) \int_0^\infty f_j(\tau)d\tau \right|^r dy \right)^{\frac{1}{r}} \quad \text{as } t \to \infty, \end{aligned}$$

which implies (4.7).

Now, assume that (1.7) holds. Then it follows from (4.2), (4.4) and (4.6) and (4.7) that

$$\sum_{j=1}^{n} E_{ij}(y,1)\int_0^\infty f_j(\tau)d\tau = 0, \quad i=1,\cdots,n \quad \text{for all } y\in\mathbb{R}^n. \tag{4.8}$$

Since $\widehat{E}_{ij}(\xi,1) = \left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)e^{-|\xi|^2}$, $i,j=1,\cdots,n$, we have by (4.8) that

$$\sum_{j=1}^{n}(\delta_{ij}-\omega_i\omega_j)\int_0^\infty f_j(\tau)d\tau = 0, \quad i=1,\cdots,n$$

for all $\omega=(\omega_1,\cdots,\omega_n)\in\mathbb{R}^n$ with $|\omega|=1$. Obviously, we conclude that

$$\int_0^\infty f_1(\tau)d\tau = \cdots = \int_0^\infty f_n(\tau)d\tau = 0,$$

which implies (1.8).

Conversely, if (1.8) holds, then we have by (4.2), (4.4) and (4.6) that

$$\begin{aligned}\|u(t)\|_r &\le \|v(t)\|_r + \|\tilde{w}(t)\|_r + \sum_{i,j=1}^{n}\|E_{ij}(\cdot,t)\|_r\left|\int_0^t f_j(\tau)d\tau\right| \\ &= o(t^{-\frac{n}{2}(1-\frac{1}{r})})\end{aligned}$$

for all $1<r<n/(n-1)$ as $t\to\infty$. By the same technique as in (3.13) and (3.14), we get (1.8) and (1.9). This proves Theorem 2.

References

[1] Borchers, W., Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exterior domains.* Acta Math. **165**, 189–227 (1990).

[2] Borchers, W., Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exterior domains. II.* Hiroshima Math. J. **21**, 621–640 (1991).

[3] Chen, Z., *Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains.* Pacific J. Math. **159**, 227–240 (1993).

[4] Finn, R., *Estimates at infinity for stationary solutions of the Navier-Stokes equations.* Bull. Math. Soc. Sci. Math. Phys. Répub. Populaire Roumaine **3**, 387–418 (1959).

[5] Finn, R., *On exterior stationary problem for the Navier-Stokes equations and associated perturbation problems.* Arch. Rational Mech Anal. **19**, 363–404 (1964).

[6] Giga, Y., Sohr, H., *On the Stokes operator in exterior domains.* J. Fac. Sci. Univ. Tokyo, Sec IA **36**, 103–130 (1989).

[7] Giga, Y., Matsui, S., Shimizu, Y., *On estimates in Hardy spaces for the Stokes flows in a half space.* Math. Z. **231**, 383–396 (1999).

[8] Iwashita, H., *$L^q - L^r$ estimates for solutions of non-stationary Stokes equations in exterior domains and the Navier-Stokes initial value problems in L^q spaces.* Math. Ann. **285**, 265–288 (1989).

[9] Kato, T., *Strong L^p-solution of the Navier-Stokes equation in $\boldsymbol{R}^m$, with applications to weak solutions.* Math. Z. **187**, 471–480 (1984).

[10] Knightly, G., *Some asymptotic properties of solutions of the Navier-Stokes equations.* Bednarek, A.R., Cesari, L., eds., Academic Press, New York, 139–155, 1977.

[11] Kozono, H., Ogawa, T., *Some L^p estimate for the exterior Stokes flow and an application to the nonstationary Navier-Stokes equations.* Indiana Univ. Math. J. **41**, 789–808 (1992).

[12] Kozono, H., Ogawa, T., *Decay properties of strong solutions for the Navier-Stokes equations in two-dimensional unbounded domains.* Arch. Rational Mech. Anal. **122**, 1–17 (1993).

[13] Kozono, H., Sohr, H., *On stationary Navier-Stokes equations in unbounded domains.* Ricerche Mat. **42**, 69–86 (1993).

[14] Kozono. H., Sohr, H., Yamazaki, M., *Representation formula, net force and energy relation to the stationary Navier-Stokes equations in 3-dimensional exterior domains.* Kyushu J. Math. **51**, 239–260 (1997).

[15] Kozono, H., *L^1-solutions of the Navier-Stokes equations in exterior domains.* Math. Ann. **312** (1998), 319–340.

[16] Masuda, K., *Weak solutions of the Navier-Stokes equations.* Tohoku Math. J. **36**, 623–646 (1984).

[17] Miyakawa, T., Sohr, H., *On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains.* Math. Z. **199**, 455–478 (1988).

[18] Miyakawa, T., *Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations.* Kyushu J. Math. **50**, 1-64 (1996).

[19] Mizumachi, R., *On the asymptotic behavior of incompressible viscous fluid motions past bodies.* J. Math. Soc. Japan **36**, 497–522(1984).

[20] Schonbek, M.E., *Lower bounds of rates of decay for solutions to the Navier-Stokes equations.* J. Amer. Math. Soc. **4**, 423–449 (1991).

[21] Schonbek, M.E., *Asymptotic behavior of solutions to the three-dimensional Navier-Stokes equations.* Indiana Univ. Math. J. **41**, 809–823 (1992).

[22] Solonnikov, V.A., *Estimates of the solutions of a nonstationary linearized system of the Navier-Stokes equations.* Amer. Math. Soc. Transl. **75**, 1-116 (1968).

[23] Solonnikov, V.A., *Estimates for the solutions of a nonstationary Navier-Stokes equations.* J. Soviet Math. **8**, 1-116 (1977).

[24] Stein, E. M., *Harmonic Analysis.* Princeton University Press 1993.

[25] Wahl, W. von, *The equations of Navier-Stokes and abstract parabolic equations.* Braunschweig-Wiessbaden: Vieweg 1985

[26] Wiegner, M., *Decay results for weak solutions of the Navier-Stokes equations in* $\mathbb{R}^n$. J. London Math. Soc.(2) **35**, 303–313 (1987).