数理解析研究所講究録 1227 巻 2001 年 14-60

Density matrices and LR transforms (Genesis of Orthogonal Functions)

Kazuhiko AOMOTO
(Graduate School of Mathematics, Nagoya University)

April 3, 2001

1 Introduction

The two kinds of Chebyschev polynomials $T_n(\cos\theta) = \cos n\theta$ and $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ are linearly related to each other by the formulae

$$\cos n\theta = \frac{1}{2}\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta}\}$$

Both polynomials satisfy the difference equations

$$xu_n = \frac{1}{2}(u_{n+1} + u_{n-1})$$

This is a simplest case of LR-transforms associated with difference operators for orthogonal functions.

A system of orthogonal functions are intimately related with eigenfunctions for a self-adjoint operator through density matrices. Once a family of self-adjoint operators are given, we can discuss the interplay among LR-transforms of self-adjoint operators, linear transforms of density matrices and connection relations between two system of eigen-functions for the operators. This mechanism enables us to give a new orthogonal system from the previous one, and so on.

Let A be an infinite real tri-diagonal matrix $(a_{n,m})_{n,m=-\infty}^{\infty}$ which defines a bounded self-adjoint operator on $l^2(\mathbf{Z})$. There exist the spectral kernels $d\Theta(n, m|\lambda)$ which are the Stieltjes measures on $\mathbf{R}$ such that

$$\delta_{n,m} = \int_{-\infty}^{\infty} d\Theta(n,m;\lambda) \tag{1.1}$$

$$a_{n,m} = \int_{-\infty}^{\infty} \lambda d\Theta(n,m;\lambda) \tag{1.2}$$

The eigenfunction expansion for A is an expression of $d\Theta(n,m;\lambda)$, by using generalized eigenfunctions $\psi^{(\epsilon)}(n;\lambda)$ $(\epsilon = \pm)$ of A satisfying

$$A\psi^{(\epsilon)}(n;\lambda) = \lambda\psi^{(\epsilon)}(n;\lambda) \tag{1.3}$$

and Stieltjes measures called density matrices $d\rho_{\epsilon,\epsilon'}(\lambda)$, as

$$d\Theta(n,m;\lambda) = \sum_{\epsilon=\pm,\epsilon'=\pm} \psi^{(\epsilon)}(n;\lambda)\overline{\psi^{(\epsilon')}(m;\lambda)}d\rho_{\epsilon,\epsilon'}(\lambda) \tag{1.4}$$

Let $f(\lambda)$ be a positive continuous function such that $f(A)$ defines a positive definite operator on $l^2(\mathbf{Z})$.

Assume that there exists a Gauss decomposition of $f(A)$ of the following type

$$f(A) = B_- \cdot B_+ \tag{1.5}$$

where B_+ (or $B_- = {}^tB_+$ the transpose of B_+) denotes an upper triangular (or lower triangular) matrix such that the inverses $B_\pm^{-1}$ are also well-defined.

Then the LR transform of A can be defined as follows.

$$A \to A' = B_-^{-1} \cdot A \cdot B_- = B_+ \cdot A \cdot B_+^{-1} \tag{1.6}$$

In this note we show that this transform is equivalent to a certain linear or projective transform of the density matrices $d\rho_{\epsilon,\epsilon'}(\lambda)$ and evaluate it explicitly in the following four cases

(1) Orthogonal polynomials in a single variable
(2) Inverse scattering case
(3) Periodic case
(4) Orthogonal polynomials in multi-variables
respectively.

This note has been written in collaboration with Dr.Masahiko Ito. Especially the computations for proving Proposition 8 are mostly due to him.

2 Orthogonal polynomials in a single variable

We consider a Stieltjes measure $d\rho(\lambda)$ with infinite increments and whose support is contained in the finite interval $[a, b]$ $(a < b)$ in $\mathbf{R}$. There exist the unique orthonormal polynomials in λ

$$p_0(\lambda),\ p_1(\lambda),\ p_2(\lambda), \ldots$$

(we put $p_{-1}(\lambda) = 0$) such that they satisfy

$$p_n(\lambda) = k_n\lambda^n + (\text{lower degree terms}) \quad k_n > 0 \tag{2.1}$$

$$\int_a^b p_n(\lambda)p_m(\lambda)d\rho(\lambda) = \delta_{n,m} \tag{2.2}$$

The three term recurrence equations hold

$$\lambda p_n(\lambda) = b_{n-1}p_{n-1}(\lambda) + a_n p_n(\lambda) + b_n p_{n+1}(\lambda) \ \ (n \geq 0) \tag{2.3}$$

Let A denote the corresponding tri-diagonal matrix $(a_{n,m})_{n,m=-\infty}^{\infty}$ such that

$$a_{n,n} = a_n, a_{n,n+1} = a_{n+1,n} = b_n \quad n \geq 0 \tag{2.4}$$

The matrix A defines a self-adjoint operator on $l^2(\mathbf{Z}_{\geq 0})$. A has the spectral decomposition (1.1), (1.2) where $d\Theta(n, m; \lambda)$ is represented simply by

$$d\Theta(n, m; \lambda) = p_n(\lambda)p_m(\lambda)d\rho(\lambda) \tag{2.5}$$

Let $f(x)$ be a positive continuous function on $[a, b]$. Then $f(A)$ and $f(A)^{-1}$ define bounded self-adjoint operators. There exist the unique upper triangular and lower triangular matrices B_+ and B_- with positive diagonal elements satisfying (1.5). All $B_\pm$ and $B_\pm^{-1}$ are bounded operators.

The LR transform of A associated with the function $f(\lambda)$ is defined by the correspondence (1.6). A' is again a tri-diagonal self-adjoint operator on $l^2(\mathbf{Z}_{\geq 0})$

Y.Nakamura and Y.Kodama, and also V.Spiridonov and A.Zhedanov have investigated LR-transforms associated with finite matrices and orthogonal polynomials (see [23],[24],[30]). Here we want to relate them to linear (or projective) transforms of density matrices $d\rho(\lambda)$.

In section 7-8 we extend LR-transforms to the case of orthogonal polynomials in multi-variables. In the final section we shall obtain explicit formulae for LR-transforms associated with Koornwinder polynomials.

Proposition 1 *Let $d\rho'(\lambda)$ be the density corresponding to the operator A'. The LR transform (1.6) is equivalent to the linear correpondence*

$$d\rho'(\lambda) = f(\lambda)d\rho(\lambda) \tag{2.6}$$

If $d\rho(\lambda)$ and $d\rho'(\lambda)$ are normalized such that

$$\int_a^b d\rho(\lambda) = \int_a^b d\rho'(\lambda) = 1 \tag{2.7}$$

then (2.6) shoud be modified as

$$d\rho(\lambda) \to d\rho'(\lambda) = \frac{f(\lambda)d\rho(\lambda)}{\int_a^b f(\lambda)d\rho(\lambda)} \tag{2.8}$$

In fact, (2.6) implies the formulae

$$(f(A))_{n,m} = \int_{-\infty}^{\infty} p_n(\lambda)p_m(\lambda)d\rho'(\lambda) \tag{2.9}$$

Let $\{p'_n(\lambda)\}$ be the orthonormal polynomials with respect to the density $d\rho'(\lambda)$. $p_n(\lambda)$ can be expressed uniquely as a linear combination of $p'_m(\lambda)$

$$p_n(\lambda) = \sum_{m=0}^{n} b_{m,n}p'_m(\lambda) \tag{2.10}$$

Let B_+ be the upper triangular matrix $(b_{n,m})_{n,m=0}^{\infty}$. Then (1.5) holds from (2.9). On the other hand

$$(A')_{n,m} = \int_{-\infty}^{\infty} \lambda p'_n(\lambda) p'_m(\lambda) d\rho'(\lambda) \tag{2.11}$$

From (2.9)-(2.11), we deduce (1.6).

In particular, if A is itself positive definite and $f(\lambda) = \lambda$, (1.6) reduces to the original Rutishauser's LR algorithm.

Examples 1. Jacobi polynomials.

Let $d\rho(\lambda) = (1-\lambda)^\alpha (1+\lambda)^\beta d\lambda$ on $[-1,1]$, for $\alpha, \beta > -1$. The Jacobi polynomials $P_n^{(\alpha,\beta)}(\lambda)$ are defined by the equations

$$(1-\lambda)^\alpha (1+\lambda)^\beta P_n^{(\alpha,\beta)}(\lambda) = \frac{(-1)^n}{2^n n!} (\frac{d}{d\lambda})^n \{(1-\lambda)^{\alpha+n} (1+\lambda)^{\beta+n}\} \tag{2.12}$$

Then

$$P_n^{(\alpha,\beta)}(\lambda) = l_n^{(\alpha,\beta)} \lambda^n + \cdots$$

$$l_n^{(\alpha,\beta)} = 2^{-n} \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

and

$$\int_{-1}^{1} (1-\lambda)^\alpha (1+\lambda)^\beta P_n^{(\alpha,\beta)}(\lambda) P_m^{(\alpha,\beta)}(\lambda) d\lambda = 0, \quad n \neq m$$

$$\int_{-1}^{1} (1-\lambda)^\alpha (1+\lambda)^\beta \{P_n^{(\alpha,\beta)}(\lambda)\}^2 d\lambda = h_n^{(\alpha,\beta)}$$

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

The reccurence equations for $P_n^{\alpha,\beta}(x)$ are as follows.

$$\begin{aligned} & 2(n+1)(n+1+\alpha+\beta)(2n+\alpha+\beta) P_{n+1}^{(\alpha,\beta)}(\lambda) \\ = \; & (2n+\alpha+\beta+1)\{(2n+\alpha+\beta+2)(2n+\alpha+\beta)\lambda + \alpha^2 - \beta^2\} P_n^{(\alpha,\beta)}(\lambda) \\ & -2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha,\beta)}(\lambda) \end{aligned} \tag{2.13}$$

$$p_n(\lambda) = \{h_n^{(\alpha,\beta)}\}^{-\frac{1}{2}} P_n^{(\alpha,\beta)}(\lambda)$$

then $p_n(\lambda)$ is the orthonormal polynomials with respect to $d\rho(\lambda)$. We denote by A the tri-diagonal operator on $l^2(\mathbf{Z})_{\geq 0}$ derived from (2.13).

The shift $\alpha \to \alpha + 1$ induces the transform of the densities

$$d\rho(\lambda) \to d\rho'(\lambda) = (1 - \lambda) d\rho(\lambda) \tag{2.14}$$

Since $1 - A$ is positive definite, the Gauss decomposition

$$1 - A = B_- \cdot B_+ \tag{2.15}$$

is uniquely determined. Likewise we have

$$1 + A = B_- \cdot B_+ \tag{2.16}$$

These are Christoffel-Darboux tranforms of contiguity relation.

In fact, if we put

$$\psi_n(\alpha, \beta) = \frac{1}{l_n^{(\alpha,\beta)}} P_n^{(\alpha,\beta)}(\lambda) = \lambda^n + \frac{n(\alpha - \beta)}{2n + \alpha + \beta} \lambda^{n-1} + \cdots$$

then

$$\begin{aligned} \psi_n(\alpha, \beta) &= \psi_n(\alpha + 1, \beta) + v_n \psi_{n-1}(\alpha + 1, \beta) \\ v_n &= -\frac{2n(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)} \end{aligned} \tag{2.17}$$

More exactly saying, $B_\pm^{-1}$ are not bounded on $l^2(\mathbf{Z})$, although $B_\pm$ are bounded. We must modify the operators $B_\pm$ as follows.

We denote by $\mathcal{H}$ the Hilbert space $l^2(\mathbf{Z}_{\geq 0})$ consisting of sequences $u = (u_n)_{n=0}^\infty$, $v = (v_n)_{n=0}^\infty$ etc with the inner product $(u, v) = \sum_{n=0}^\infty u_n \overline{v_n}$. We define another Hilbert space $\mathcal{H}_0$, the closed linear subspace spanned by $B_+ u'$ $(u' \in l^2(\mathbf{Z}_{\geq 0})$. $\mathcal{H}_0$ is isomorphic to the Hilbert space consisting of the sequences $u = (u_n)_{n \geq 0}$ such that $((1 - A)^{-1} u, u) < \infty$. B_+^{-1} is a bounded operator from

$\mathcal{H}_0$ to $\mathcal{H}$, so that $B_+ A B_+^{-1}$ is bounded as a linear mapping from $\mathcal{H}_0$ to $\mathcal{H}$ which is extendable to a bounded operator on $\mathcal{H}$.

Example 2. Askey-Wilson polynomials (see [5]).

Let q be the real modulus such that $0 < q < 1$, and c_1, c_2, c_3, c_4 be real numbers. Askey-Wilson polynomials are defined by using the basic hypergeometric series of order m

$$ {}_m\varphi_{m-1}\left(\begin{matrix} a_1, \cdots, a_m \\ b_1, \cdots, b_{m-1} \end{matrix} ; \lambda \right) = \sum_{\nu=0}^{\infty} \frac{(a_1;q)_\nu \cdots (a_m;q)_\nu}{(b_1;q)_\nu \cdots (b_{m-1};q)_\nu (q;q)_\nu} \lambda^\nu \tag{2.18} $$

as

$$ \begin{aligned} & p_n(\lambda; c_1, c_2, c_3, c_4) \\ = \; & c_1^{-n} (c_1c_2;q)_n \cdot (c_1c_3;q)_n \cdot (c_1c_4;q)_n \cdot {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1}c_1c_2c_3c_4, c_1e^{i\theta}, c_1e^{-i\theta} \\ c_1c_2, c_3c_4, c_1c_4 \end{matrix} ; q \right) \\ = \; & l_n \lambda^n + \cdots \qquad (l_n = 2^n (c_1c_2c_3c_4q^n;q)_n) \end{aligned} \tag{2.19} $$

where $\lambda = cos\theta$. The weight function $w(\lambda)$ $(d\rho(\lambda) = \frac{w(\lambda)}{\sqrt{1-\lambda^2}} d\lambda)$ is given by

$$ w(\lambda) = \frac{\prod_{k=0}^{\infty} (1 - 2(2\lambda^2 - 1)q^k + q^{2k})}{h(\lambda, c_1)h(\lambda, c_2)h(\lambda, c_3)h(\lambda, c_4)} \tag{2.20} $$

where

$$ h(\lambda, a) = \prod_{k=0}^{\infty} (1 - 2a\lambda q^k + q^{2k}a^2) = (ae^{i\theta};q)_\infty (ae^{-i\theta};q)_\infty \tag{2.21} $$

Then the orthogonality relations are

$$ \frac{1}{2\pi} \int_{-1}^{1} p_n(\lambda; c_1, c_2, c_3, c_4) p_m(\lambda; c_1, c_2, c_3, c_4) \frac{w(\lambda)}{\sqrt{1-\lambda^2}} d\lambda = \delta_{n,m} h_n \tag{2.22} $$

$$h_n = \frac{(c_1c_2c_3c_4q^{2n};q)_\infty(c_1c_2c_3c_4q^{n-1};q)_\infty(q^{n+1};q)_\infty^{-1}(c_1c_2q^n;q)_\infty^{-1}}{(c_1c_3q^n;q)_\infty(c_1c_4q^n;q)_\infty(c_2c_3q^n;q)_\infty(c_2c_4q^n;q)_\infty(c_3c_4q^n;q)_\infty} \tag{2.23}$$

The three term recurrence relations for $p_n(\lambda; c_1, c_2, c_3, c_4)$ are expressed as

$$2\lambda p_n(\lambda) = b_{n-1}p_{n-1}(\lambda) + a_n p_n(\lambda) + b'_n p_{n+1}(\lambda) \tag{2.24}$$

$$b_{n-1} = (1-q^n)(1-c_1c_2q^{n-1})(1-c_1c_3q^{n-1})(1-c_1c_4q^{n-1})$$
$$\times\frac{(1-c_2c_3q^{n-1})(1-c_2c_4q^{n-1})(1-c_3c_4q^{n-1})}{(1-cq^{2n-2})(1-cq^{2n-1})},$$
$$b'_n = \frac{1-cq^{n-1}}{(1-cq^{2n-1})(1-cq^{2n})},$$
$$a_n = \frac{q^{n-1}[(1+cq^{2n-1})(sq+s'c) - q^{n-1}(1+q)(s+s'q)c]}{(1-cq^{2n-2})(1-cq^{2n})}$$

($s = c_1 + c_2 + c_3 + c_4,\ \ s' = c_1^{-1} + c_2^{-1} + c_3^{-1} + c_4^{-1},\ \ c = c_1c_2c_3c_4$).

$d\rho(\lambda)$ depends on c_1, c_2, c_3, c_4. In fact each shift

$$T_1 : c_1 \to c_1q;\ \ T_2 : c_2 \to c_2q;\ \ T_3 : c_3 \to c_3q;\ \ T_4 : c_4 \to c_4q \tag{2.25}$$

multiplies $w(\lambda)$ by

$$1+c_1^2-2c_1\lambda\ ,1+c_2^2-2c_2\lambda\ ,1+c_3^2-2c_3\lambda\ ,1+c_4^2-2c_4\lambda \tag{2.26}$$

times respectively.

The corresponding LR transforms of A are defined as the Gauss decompositions of each positive operator

$1+A^2-2c_1A > 0,\ \ 1+A^2-2c_2A > 0,\ \ 1+A^2-2c_3A > 0,\ \ 1+A^2-2c_4A > 0$

Put

$$\psi_n(\lambda; c_1, c_2, c_3, c_4) = \frac{1}{l_n}p_n(\lambda; c_1, c_2, c_3, c_4) \tag{2.27}$$

then, as for T_1 for example, the transform B_+ is equivalent to the following contiguity relation

$$\psi_n(x; c_1, c_2, c_3, c_4) = \psi_n(x; c_1 q, c_2, c_3, c_4) + v_n \psi_{n-1}(x; c_1 q, c_2, c_3, c_4)$$

$$v_n = -\frac{2(1-q)c_1}{(1-aq^{2n-2})(1-aq^{2n-1})(1-c_2c_3q^{n-1})(1-c_2c_4q^{n-1})(1-c_3c_4q^{n-1}}$$

likewise for $T_2,\ T_3,\ T_4$.

3 Inverse scattering, Application of H.Flaschka theory

A be a tri-diagonal matrix which defines a bounded self-adjoint operator on $\mathcal{H} = l^2(\mathbf{Z})$.

We put $a_{n,n} = a_n$ and $a_{n,n+1} = a_{n+1,n} = b_n$ for $-\infty < n < \infty$ and assume the following condition

$$(C) \sum_{n=-\infty}^{\infty} |a_n||n| < \infty, \sum_{n=-\infty}^{\infty} |b_n - \frac{1}{2}||n| < \infty \tag{3.1}$$

The inverse scattering theory for the difference operator A was developed by H.Flaschka (see [10],[32]). We put the spectral parameter $z = \frac{1}{2}(\zeta + \zeta^{-1})$.

If $|\zeta| \leq 1$, then the Jost solutions $\psi^{\pm}(n; z)$(minimal solutions in the sense of S.Elaydi [9]) are uniquely determined as the eigenfunctions (1.3) with the asymptotic behaviours

$$\psi^{\pm}(n; z) \asymp \zeta^{\pm n} \quad n \to \pm\infty \tag{3.2}$$

The connection relations between $\psi^{\pm}(n; z)$ are

$$\psi^{-}(n; z) = \alpha(z)\tilde{\psi}^{\pm}(n; z) + \beta(z)\psi^{\pm}(n; z) \tag{3.3}$$

where $\tilde{\psi}^{\pm}(n; z)$ are defined to be the conjugates of $\psi^{\pm}(n; z)$ when ζ ($|\zeta| = 1$) is replaced by ζ^{-1}. $\alpha(z)$, $\beta(z)$ can be holomorphically extended to the domain $|\zeta| \leq 1$.

The Wronskian and the reflection coefficients are expressed as

$$R(z)=\frac{\beta(z)}{\alpha(z)} \tag{3.4}$$

$$W(\psi_+,\psi_-)=\frac{1}{2}(\zeta^{-1}-\zeta)\alpha(z) \tag{3.5}$$

respectively.

For $\lambda\in[-1,1]$, $\psi^{\pm}(n;\lambda+i0)$, $\alpha(\lambda+i0)$, $\beta(\lambda+i0)$ do exist. Moreover, $\alpha(z)$ has a finite number of simple poles λ_k, $k=1,2,3,...,s$ such that $|\lambda_k|>1$.

Under this circumstance, it holds the following two expansion formulae which are equivalent to each other.

Proposition 2 (1)

$$\begin{aligned} d\Theta(n,m;\lambda) &= \frac{\chi_{[-1,1]}(\lambda)d\lambda}{2\pi\sqrt{1-\lambda^2}|\alpha(\lambda+i0)|^2}\{\psi^+(n;\lambda+i0)\overline{\psi^+(m;\lambda+i0)} \\ &+ \psi^-(n;\lambda+i0)\overline{\psi^-(m;\lambda+i0)}\} \\ &+ \sum_{k=1}^{s}\psi^+(n;\lambda_k)\psi^+(m;\lambda_k)c_k^2\delta(\lambda-\lambda_k)d\lambda \end{aligned} \tag{3.6}$$

where $c_k^2=\frac{\beta(\lambda_k)}{\alpha'(\lambda_k)\sqrt{\lambda_k^2-1}}$ *and* $\chi_{[-1,1]}(\lambda)$ *denotes the indicator function of* $[-1,1]$.

(2)

$$\begin{aligned} d\Theta(n,m;\lambda) &= \frac{\chi_{[-1,1]}(\lambda)d\lambda}{2\pi\sqrt{1-\lambda^2}}\{\psi^+(n;\lambda+i0)\overline{\psi^+(m;\lambda+i0)} \\ &+ \psi^+(n;\lambda-i0)\overline{\psi^+(m;\lambda-i0)} \\ &+ R(\lambda+i0)\psi^+(n;\lambda+i0)\overline{\psi^+(m;\lambda-i0)} \\ &+ R(\lambda-i0)\psi^+(n;\lambda-i0)\overline{\psi^+(m;\lambda+i0)}\} \\ &+ \sum_{k=1}^{s}\psi^+(n;\lambda_k)\psi^+(m;\lambda_k)c_k^2\delta(\lambda-\lambda_k)d\lambda \end{aligned} \tag{3.7}$$

For the proof see [3],[6].
We can rewrite (3.6),(3.7) by using the Fourier expnasions of $\psi^{\pm}(n;z)$

$$\psi^{+}(n;z)=\sum_{m\geq n}K(n,m)\zeta^{m}\quad K(n,n)>0 \tag{3.8}$$

$$F(m)=F_c(m)+F_p(m), \tag{3.9}$$
$$F_c(m)=\frac{1}{2\pi i}\int_{|\zeta|=1}R(z)\zeta^{m-1}d\zeta \tag{3.10}$$
$$F_p(m)=\sum_{k=1}^{s}c_k^2\zeta_k^m \tag{3.11}$$

We denote by $\hat{F}$, $\hat{K}$ the operators defined by the kernel functions $\{F(n+m)\}_{n,m=-\infty}^{\infty}$ and $\{K(n,m)\}_{n,m=-\infty}^{\infty}$. $\hat{F}$ is of Fredholm type and of Hankel type. $\hat{K}$ has a bounded inverse.

Then (3.7) imply the following Gelfand-Levitan-Marchenko decomposition (abreviated by GLM decomposition)

Proposition 3 (1.1), (1.2) *can be expressed in operator form as*

$$1=\hat{K}(1+\hat{F})^{t}\hat{K} \tag{3.12}$$
$$A=\hat{K}A_0(1+\hat{F})^{t}\hat{K}=\hat{K}A_0\hat{K}^{-1} \tag{3.13}$$

where ${}^{t}\hat{K}$ *denotes the transpose of* $\hat{K}$. *We denote by* A_0 *the symmetric tri-diagonal matrix such that* $b_n=\frac{1}{2}$, $a_n=0$.

$1+\hat{F}$ is positive definite so that $\hat{K}$ is uniquely determined by (3.12).
A_0 has the unique decomposition

$$A_0=A_{0,+}+A_{0,-} \tag{3.14}$$

where $A_{0,+}$ and $A_{0,-}$ are upper triangular and lower triangular matrices respectively. $2A_{0,\pm}$ are unitary operators which shift the indices by ±1 respec-

Now let us discuss how the LR transform of A can be expressed in terms of $\hat{F}$.

Since $A,\ A_0$ are bounded, there exists a positive number c such that all 4 operators $A(c) = A+c,\ A_0(c) = A_0+c$ and $A(c)^{-1}, A_0(c)^{-1} > 0$ are positive definite.

We want to find the upper triangular bi-diagonal matrix $A_+(c)$, with (n,n)th entries ξ_n and $(n,n+1)$th entries η_n such that $\xi_n > 0$, and its transpose $A_-(c) = {}^tA_+(c)$, such that the following Gauss decompositon holds.

$$A(c) = A_-(c) \cdot A_+(c) \tag{3.15}$$

i.e.,

$$\xi_n^2 + \eta_{n-1}^2 = a_n + c,\ \ \xi_n\eta_n = b_n \tag{3.16}$$

The equations (3.16) have the unique solution such that ξ_0^2 equals the convergent continued fraction

$$\xi_0^2 = \frac{b_0^2\ |}{|\,a_1+c} - \frac{b_1^2\ |}{|\,a_2+c} - \cdots = -b_0\frac{\psi^+(1;-c)}{\psi^+(0;-c)} \tag{3.17}$$

because, if $z \notin \sigma(A)$, we have

$$b_0\frac{\psi(1;z)}{\psi(0;z)} = \frac{b_0^2\ |}{|\,z-a_1} - \frac{b_1^2\ |}{|\,z-a_2} - \cdots \tag{3.18}$$

We shall call the Gauss decomposition (3.15) thus obtained canonical.

The LR-transform is then defined as

$$A \to A' = A_+(c) \cdot A_-(c) = A_+(c) \cdot A \cdot A_-(c)^{-1} \tag{3.19}$$

A' is also tri-diagonal.

We can now state

Theorem 1 *Let the GLM decompositon of A' be*

$$1 = K' \cdot (1 + \hat{F}') \cdot {}^t\hat{K}' \tag{3.20}$$
$$A' = K' \cdot A_0 \cdot (1 + \hat{F}') \cdot {}^t\hat{K}' \tag{3.21}$$

then A' is the LR-transform of A if and only if

$$\hat{F}' = \hat{F} \cdot A_{0,-}(c) \cdot A_{0,+}(c)^{-1} = A_{0,+}(c) \cdot \hat{F} \cdot A_{0,+}(c)^{-1} \tag{3.22}$$

(Remark that $\hat{F} \cdot A_{0,\pm}(c) = A_{0,\mp}(c) \cdot \hat{F}$.)

If we put

$$g(\zeta) = \frac{\sqrt{c+1} - \sqrt{c-1}}{2}\zeta + \frac{\sqrt{c+1} + \sqrt{c-1}}{2}$$

i.e.,

$$z + c = g(\zeta)g(\zeta^{-1})$$

then (3.22) can be restated as

$$R'(z) = R(z)g(\zeta)^{-1}g(\zeta^{-1}) \tag{3.23}$$

which is nothing else than dressing transformation in the sense of Zakhalov-Shabat. (This fact has been pointed out to the author by S.Kakei.)

Proof 1 *First we show that (3.22) implies (3.19). From (3.20),(3.22) and because of the uniqueness of Gauss decomposition, we have*

$$\hat{K}' = A_+(c) \cdot \hat{K} \cdot g(2A_{0,+}) \tag{3.24}$$

Hence, from (3.21)

$$\begin{aligned} A' = \hat{K}' \cdot A_0 \cdot (1 + \hat{F}') \cdot {}^tK' &= A_+(c)\hat{K}g(2A_{0,+})A_0 g(2A_{0,-})^{-1}\hat{K}^{-1}A_+(c)^{-1} \\ &= A_+(c)\hat{K}A_0\hat{K}^{-1}A_+(c)^{-1} = A_+(c) \cdot A \cdot A_+(c)^{-1} \end{aligned}$$

(3.19) *has thus been obtained.*

Next we show the converse. We remark first that any bounded upper triangular operator which commutes $A_{0,+}$ *is a holomorphic function of* $2A_{0,+}$. *As is seen from (3.12)and (3.19), there exists a holomorphic function* $\tilde{g}(\zeta)$ *of* ζ ($|\zeta| < 1$) *such that*

$$\hat{K}' = A_+(c) \cdot \hat{K} \cdot \tilde{g}(2A_{0,+}) \tag{3.25}$$

Hence from (3.13),(3.20) and (3.21)

$$\tilde{g}(2A_{0,+})\tilde{g}(2A_{0,-}) + \tilde{g}(2A_{0,+})^2\hat{F}' = A_0(c)^{-1}(1+\hat{F}) \tag{3.26}$$

By uniquness of this matrix expression, we have

$$\tilde{g}(2A_{0,+})\tilde{g}(2A_{0,-}) = A_0(c)^{-1} \tag{3.27}$$

$$\tilde{g}(2A_{0,+})^2\hat{F}' = A_0(c)^{-1}\hat{F} \tag{3.28}$$

which imply

$$\tilde{g}(2A_{0,+}) = A_{0,+}(c)^{-1} \tag{3.29}$$

and

$$A_{0,+}(c)^{-2}\hat{F}' = A_0(c)^{-1}\hat{F} \tag{3.30}$$

which are nothing else than (3.22).

4 Periodic Toda lattice

Let A be a periodic tri-diagonal matrix with period N,

$$a_{n+N} = a_n, b_{n+N} = b_n \tag{4.1}$$

We assume that it is positive definite on $l^2(\mathbf{Z})$. Let h be the Floquet multiplier and $A_h = (\tilde{a}_{n,m})_{n,m=0}^{N-1}$ be the $N \times N$ matrix defined by

$$\begin{aligned}\tilde{a}_{n,m} &= hb_{N-1} \ (n,m) = (N-1,0),\\ &= h^{-1}b_{N-1} \ (n,m) = (0,N-1),\\ &= a_{n,m} \text{ otherwise}\end{aligned}$$

The determinant of $z - A_h$ can be written as

$$\det[z - A_h] = -b_0 b_1 \cdots b_{N-1}(h + h^{-1} - \Delta) \tag{4.2}$$

where Δ denotes the polynomial of degree N such that

$$b_0 b_1 \cdots b_{N-1}\Delta = z^N - (a_0 + a_1 + \cdots + a_{N-1})z^{N-1} + \cdots$$

The function h annhilating (4.2) is obtained by the equation

$$h = \frac{\Delta - \sqrt{\Delta^2 - 4}}{2} \tag{4.3}$$

which defines the hyperelliptic curve X of genus $N-1$.

Let $\lambda_1, ..., \lambda_{2N}$ be the roots of the equation $\Delta^2 - 4 = 0$, such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{2N-1} < \lambda_{2N}$$

$|h| = 1$ i.e., $|\Delta| < 4$ holds if and only if

$$\lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup \cdots \cup [\lambda_{2N-1}, \lambda_{2N}] \tag{4.4}$$

In other words, the spectra $\sigma(A)$ are continuous and given by the bands (4.4). When $\lambda \notin \sigma(A)$, we have $|h| < 1$.

Let $\psi^{\pm}(n; z)$ be the Bloch solutions to (1.3) satisfying

$$\psi^{\pm}(n + N; z) = h^{\pm 1}\psi^{\pm}(n; z) \tag{4.5}$$

which are obtained by solving the finite equations

$$(z - A_h)\tilde{\psi} = 0 \tag{4.6}$$

Let $K^{\pm}(n;z)$ be the normalized Bloch solutions such that $K^{\pm}(0;z) = 1$.

We denote by $D(i,j)$ the subdeterminant corresponding to the (n,m)th entries $(i \leq n, m \leq j)$ of $z - A$.

Then $K^{\pm}(n;z)$ can be expressed in terms of $D(i,j)$, in particular

$$K^{+}(1;z) = -\frac{(-1)^N h b_1 \cdots b_{N-1} + b_0 D(2, N-1)}{D(1, N-1)} \tag{4.7}$$

$$K^{-}(1;z) = -\frac{(-1)^N h^{-1} b_1 \cdots b_{N-1} + b_0 D(2, N-1)}{D(1, N-1)} \tag{4.8}$$

Proposition 4 *We have*

$$K^{-}(n;\lambda + i0) = K^{+}(n;\lambda - i0) = \overline{K^{+}(n;\lambda + i0)} \quad \text{for } \lambda \in \sigma(A) \tag{4.9}$$

Put

$$d\rho_{+}(\lambda) = d\rho_{-}(\lambda) = \frac{1}{2\pi} \frac{|D(1, N-1)|}{|b_0 b_1 \cdots b_{N-1}| \sqrt{4 - \Delta^2}}, \quad \lambda \in \sigma(A) \tag{4.10}$$

Then the spectral kernels of A can be expressed as

$$d\Theta(n,m;\lambda) = 2\Re\{K^{+}(n;\lambda + i0)\overline{K^{+}(m;\lambda + i0)}\} d\rho_{+}(\lambda) \tag{4.11}$$

We may put $D(1, N-1) = \prod_{k=1}^{N-1}(z - \mu_k)$ where $\mu_1, \mu_2, \ldots, \mu_{N-1}$ denote the auxiliary spectra such that

$$\lambda_2 < \mu_1 < \lambda_3 < \lambda_4 < \cdots < \mu_{N-1} < \lambda_{2N-1} < \lambda_{2N} \tag{4.12}$$

We want to find the Gauss decompositon of A as in (3.15) (we put $c = 0$).

$$A = A_{-} \cdot A_{+}, \quad A_{-} = {}^{t}A_{+} \tag{4.13}$$

such that $\xi_{n+N} = \xi_n, \quad \eta_{n+N} = \eta_n$ hold.

We can find uniquely $\xi_n, \quad \eta_n$ such that (3.16), (3.17) hold with $c = 0$, i.e.,

$$\xi_0^2 = -b_0 K^+(1;0) \tag{4.14}$$

Remark that (3.17) is a periodic continued fraction in this case.

The LR-transform is now defined by

$$A = A_- \cdot A_+ \to A' = A_+ \cdot A_- = A_+ \cdot A \cdot A_+^{-1} \tag{4.15}$$

The following Propsition 5 is most fundamental.

Proposition 5 *Let $\{\mu'_1, \mu'_2, ..., \mu'_{N-1}\}$ be the auxiliary spectra for A'. Then A' is the LR-transform of A if and only if*

$$z\frac{\prod_{k=1}^{N-1}(z-\mu'_k)}{\prod_{k=1}^{N-1}(z-\mu_k)} = (\xi_0 + \eta_0 K^+(1;z))(\xi_0 + \eta_0 K^-(1;z)) \tag{4.16}$$

The matrices A' i.e., $\xi_n, \eta_n, \mu'_1, ..., \mu'_{N-1}$ can be uniquely obtained by solving (4.16).

Proof 2 *The Bloch solutions for A' are given by*

$$K'_+(n;z) = \frac{\xi_n K^+(n;z) + \eta_n K^+(n+1;z)}{\xi_0 + \eta_0 K_+(1;z)} \tag{4.17}$$

We want to show first that (4.16) implies (4.15).

At $z = \infty$, $K^\pm(n;z)$ are meromorphic and satisfy

$$K^+(n;z) = O(z^{-n}), \quad K'^+(n;z) = O(z^{-n})$$

There exists the unique upper triangular real matrix $\Xi = (\xi_{n,m})_{n,m=-\infty}^{\infty}$ such that

$$K'^+(n;z) = \sum_{m=n}^{\infty} \xi_{n,m} K^+(m;z) \tag{4.18}$$

From (4.16),(1.1) and (1.2) we have the relations of operators

$$1 = \Xi \cdot \frac{\prod_{k=1}^{N-1}(A-\mu'_k)}{\prod_{k=1}^{N-1}(A-\mu_k)} \cdot {}^t\Xi \tag{4.19}$$

$$A' = \Xi \cdot A \cdot \frac{\prod_{k=1}^{N-1}(A-\mu'_k)}{\prod_{k=1}^{N-1}(A-\mu_k)} \cdot {}^t\Xi = \Xi \cdot A \cdot \Xi^{-1} \tag{4.20}$$

Moreover,there exists an upper triangular matrix $Y = (\eta_{n,m})_{n,m=-\infty}^{\infty}$ *such that*

$$(\xi_0 + \eta_0 K^+(1;z))K^+(n;z) = \sum_{m=n}^{\infty} \eta_{n,m} K^+(m;z) \tag{4.21}$$

which is equivalent to the relations

$$\eta_{n,m} = 2\int_{-\infty}^{\infty} \Re\{(\xi_0 + \eta_0 K^+(1;\lambda+i0))K^+(n;\lambda+i0)\overline{K^+(m;\lambda+i0)}\}d\rho_+(\lambda) \tag{4.22}$$

in view of (1.1) and (4.11). Therefore by substitution of A *into* $K^+(1;z)$, *we have*

$$\xi_0 + \eta_0 K^+(1;A) = Y \tag{4.23}$$

In the same way,

$$\xi_0 + \eta_0 K^-(1;A) = {}^tY \tag{4.24}$$

From (4.16), these two equalities imply

$$A\frac{\prod_{k=1}^{N-1}(A-\mu'_k)}{\prod_{k=1}^{N-1}(A-\mu_k)} = Y \cdot {}^tY = {}^tY \cdot Y \tag{4.25}$$

Since Y *and* ${}^tY$ *commute each other,*

$$A = {}^tY \cdot \frac{\prod_{k=1}^{N-1}(A-\mu_k)}{\prod_{k=1}^{N-1}(A-\mu'_k)} \cdot Y = {}^tY \cdot {}^t\Xi \cdot \Xi \cdot Y$$

which is nothing else than the Gauss decomposition of A, *i.e.,*

$$A_- = {}^tY \cdot {}^t\Xi, \quad A_+ = \Xi \cdot Y \tag{4.26}$$

From (4.20)

$$A' = A_+ \cdot Y^{-1} \cdot A \cdot Y \cdot A_+^{-1} = A_+ \cdot A \cdot A_+^{-1}$$

which leads to (4.15).

Next we show that (4.15) implies (4.16).

Put

$$\psi' = A_+(K^+) \tag{4.27}$$

and normalize it such that $K'^+(0;z) = 1$ *as follows.*

$$K'^+(n;z) = \frac{\psi'(n;z)}{\psi'(0;z)} \tag{4.28}$$

which gives (4.17) for A'. *Then there exists the unique upper triangular matrix* Ξ *satisfying (4.18). Hence,*

$$\{A_+(K^+)\}(n;z) = \{(\xi_0 + \eta_0 K^+(1;z))\Xi(K^+)\}(n;z) = \{\Xi \cdot Y(K^+)\}(n;z)$$

In other words,

$$A_+ = \Xi \cdot Y \tag{4.29}$$

As a consequence,

$$\{A'\}_{n,m} = \{A_+ \cdot A_-\}_{n,m} = \{\Xi \cdot Y \cdot {}^tY \cdot {}^t\Xi\}_{n,m}$$
$$= 2\int_{-\infty}^{\infty} \Re\{(\xi_0 + \eta_0 K^+(1, \lambda + i0))(\xi_0 + \eta_0 K^+(1, \lambda - i0))$$
$$K'^+(n, \lambda + i0) K'^+(m, \lambda - i0)\} d\rho_+(\lambda) \qquad (4.30)$$

On the other hand, by definition

$$\{A'\}_{n,m} = 2\int_{\infty}^{\infty} \lambda \Re\{\lambda K'^+(n; \lambda + i0) K'^+(m; \lambda - i0)\} d\rho'_+(\lambda)$$

Therefore by uniqueness of expression

$$\lambda d\rho'_+(\lambda) = (\xi_0 + \eta_0 K^+(1, \lambda + i0))(\xi_0 + \eta_0 K^+(1, \lambda - i0)) d\rho_+(\lambda)$$

Seeing that

$$d\rho'_+(\lambda) = \frac{\prod_{k=1}^{N-1}(\lambda - \mu'_k)}{\prod_{k=1}^{N-1}(\lambda - \mu_k)} d\rho_+(\lambda)$$

we have (4.16).

The hyperelliptic curve X defined by (4.3) has two sheets, physical and unphysical, which correspond to $|h| < 1$ (> 1) respectively, for $\lambda \notin \sigma(A)$.

Since $K^\pm(n; z)$ are meromorphic functions on X, we can represent the functions $K^\pm(n; z)$ by using divisors in X. Since $z = 0$, ∞ are not branch points of X, there are two points in X in each case, lying over $z = 0$, and $z = \infty$ $\langle 0 \rangle$,$\langle \infty \rangle$ in the physical sheet, $\langle 0^* \rangle$,$\langle \infty^* \rangle$ in the unphysical sheet respectively. X has the canonical involution

$$\iota \ : h \to h^{-1}. \qquad (4.31)$$

Obviously $\iota(\langle 0 \rangle) = \langle 0^* \rangle$ and $\iota(\langle \infty \rangle) = \langle \infty^* \rangle$. We denote by D^* the conjugate $\iota(D)$ of a divisor D. Then

Lemma 1 *Fix $n \geq 0$. $K^+(n; z)$ has simple poles at the physical points in X, lying over $z = \mu_1, \mu_2, ..., \mu_{N-1}$ which do not depend on n. We denote the corresponding positive divisor of degree $N - 1$ by D_0. It has also a pole of* *order n at $\langle\infty^*\rangle$. Similarly it has simple zeros at the unphysical points lying over $z = \mu_1, \mu_2, ..., \mu_{N-1}$ (its divisor of degree $N-1$ is denoted by D_n) and a zero of order n at $\langle\infty\rangle$.*

In other words, in terms of divisors,

$$(K^+(n;z)) = n\langle\infty\rangle - n\langle\infty^*\rangle - D_0 + D_n \tag{4.32}$$

$$(K^-(n;z)) = n\langle\infty^*\rangle - n\langle\infty\rangle - D_0^* + D_n^* \tag{4.33}$$

Furthermore

$$(\prod_{k=1}^{N-1}(z-\mu_k)) = -(N-1)\{\langle\infty\rangle + \langle\infty^*\rangle\} + D_0 + D_0^* \tag{4.34}$$

$$(h) = N(\langle\infty\rangle - \langle\infty^*\rangle) \tag{4.35}$$

As for the zeros and poles of $\xi_0 + \eta_0 K^+(1;z)$, we have

Theorem 2 *There exist a positive divisor of degree $N-1$, D_0' and its conjugate $D_0'^*$, such that*

$$(\xi_0 + \eta_0 K^+(1;z)) = \langle 0\rangle - \langle\infty^*\rangle - D_0 + D_0' \tag{4.36}$$

$$(\xi_0 + \eta_0 K^-(1;z)) = \langle 0^*\rangle - \langle\infty\rangle - D_0^* + D_0'^* \tag{4.37}$$

Hence, there exists a positive divisor of degree $N-1$, D_1' such that

$$(K'^+(1;z)) = \langle\infty\rangle - \langle\infty^*\rangle - D_0' + D_1' \tag{4.38}$$

The set of divisor classes of degree $N-1$ in X makes the Jacobi variety of X denoted by $Jac(X)$. As is seen from (4.36), we have the equality as a point of $Jac(X)$

$$D_0' - D_0 \equiv -\langle 0\rangle + \langle\infty^*\rangle \tag{4.39}$$

The new tri-diagonal operator A' has the same spectra as A and therefore we can take the LR-transform of A' again. By repeating this procedure, we get a sequence of tri-diagonal operators

$$A \rightarrow A' \rightarrow A'' \rightarrow \cdots \tag{4.40}$$

and a sequence of corresponding divisor classes

$$D_0 \rightarrow D_0' \rightarrow D_0'' \rightarrow \cdots \tag{4.41}$$

such that

$$D_0' - D_0 \equiv D_0'' - D_0' \equiv \cdots \equiv -\langle 0 \rangle + \langle \infty^* \rangle \tag{4.42}$$

As a conclusion,

Theorem 3 *The sequence of LR-tranforms (4.40) is realized in $Jac(X)$, by the discrete paralell displacement of $\mathbf{p}_m$ by the constant divisor class $-\langle 0 \rangle + \langle \infty^* \rangle$, starting from $\mathbf{p}_0 = D_0$ such that*

$$\mathbf{p}_m = \mathbf{p}_0 + m\{-\langle 0 \rangle + \langle \infty^* \rangle\}, \quad m = 0, 1, 2, 3, ... \tag{4.43}$$

Corollary 1 *The sequence of LR-transforms is periodic with period $M > 0$ if and only if*

$$M\{-\langle 0 \rangle + \langle \infty^* \rangle\} \equiv 0 \tag{4.44}$$

Remark 1 *When A is finite or semi-infinite, the sequence (4.40) never become periodic. In fact, in a finite case, A tends to a diagonal matrix, so that the eigenvalues of A are approximated by these procedure*([25],[26],[27]). *I do not know how they behave, when A is semi-finite.*

Remark 2 *$f(z)$ is a polynomial of degree r, it is possible to extend* (4.16) *to a more general transform* (1.6). *In this situation* (4.16) *must be replaced by the equation*

$$f(z)\frac{\prod_{k=1}^{N-1}(z-\mu_k')}{\prod_{k=1}^{N-1}(z-\mu_k)} = (\xi_0 + \sum_{k=1}^{r} \eta_{0,k} K^+(k;z))(\xi_0 + \sum_{k=1}^{r} \eta_{0,k} K^-(k;z))$$

Since $f(A)$ is no more tri-diagonal, we cannot find tri-diagonal matrices $B_\pm$ satisfying (1.5).

Suppose that $f(A)$ is positive definite and multiple- diagonal of width $2m+1$. Then $f(A)$ is a tri-diagonal matrix in block form, consisting of matrices $A_{n,n}$, $(A_{n,n} = {}^tA_{n,n} > 0)$ $A_{n,n+1}, A_{n+1,n} = {}^tA_{n,n+1}$ of size $m+1$. One can find an upper block bi-triangular matrix B_+ consisting of triangular matrices $B_{n,n}$ and $B_{n,n+1}$ of size $m+1$ such that

$$A_{n,n} = {}^tB_{n,n} \cdot B_{n,n} + {}^tB_{n-1,n} \cdot B_{n-1,n}, \; A_{n,n+1} = {}^tB_{n,n} \cdot B_{n,n+1}$$

If we put $Z_n = {}^tB_{n,n} \cdot B_{n,n}$, then we have the reccurence relations

$$Z_n = A_{n,n+1} \cdot (A_{n+1,n+1} - Z_{n+1})^{-1} \cdot {}^tA_{n,n+1}$$

which give the matrix version of the convergent continued fraction (3.17) such that

$$Z_n \leq A_{n,n+1} \cdot A_{n+1,n+1}^{-1} \cdot {}^tA_{n+1,n}$$

$B_{n,n}$ can be solved uniquely from Z such that all the diagonal elements are positive.

In the next section, in case of $N = 2$, we shall give explicit computation in terms of the sigma functions on the elliptic curve X.

5 Case of period $N = 2$

It is sufficient to give $\{a_0, a_1, b_0, b_1\}$ to define the operator A.
We put $W(z) = b_0^2 b_1^2(\Delta^2 - 4)$, then

$$W(z) = (z-\lambda_1)(z-\lambda_2)(z-\lambda_3)(z-\lambda_4), \;\; 0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \quad (5.1)$$

Moreover

$$d\rho_\pm = \frac{1}{4\pi}\frac{|\lambda - a_1|}{\sqrt{|W(\lambda)|}}, \;\; \lambda_2 < a_1 < \lambda_3 \quad (5.2)$$

$$K^+(1;z) = \frac{b_0 + b_1 h}{z - a_1}, \;\; K^-(1;z) = \frac{b_0 + b_1 h^{-1}}{z - a_1}, \quad (5.3)$$

(4.16) reduces to

$$z\frac{z-a_1'}{z-a_1} = (\xi_0 + \eta_0 K^+(1;z))(\xi_0 + \eta_0 K^-(1;z)) \qquad (5.$$

Put

$$u = \int_{\lambda_4}^{z} \frac{dz}{\sqrt{W(z)}}, v = \int_{\lambda_4}^{\infty} \frac{dz}{\sqrt{W(z)}} > 0, w = \int_{\lambda_4}^{0} \frac{dz}{\sqrt{W(z)}} > 0$$

$$v - c = \int_{\lambda_4}^{a_1} \frac{dz}{\sqrt{W(z)}}, \quad v > \Re c > 0, \Im c < 0$$

$$\omega_1 = \int_{\lambda_2}^{\lambda_3} \frac{dz}{\sqrt{W(z)}} > 0, \omega_2 = i\int_{\lambda_3}^{\lambda_4} \frac{dz}{\sqrt{|W(z)|}} \in i\mathbf{R}_{>0}$$

then, $2\omega_1, 2\omega_2$ are double periods, and $\langle 0\rangle$, $\langle 0^*\rangle$, $\langle\infty\rangle$, $\langle\infty^*\rangle$ correspond to

$$u = w, \quad u = -w, \quad u = v, \quad u = -v$$

respectively. Furthermore,

$$4v = 2\omega_1 \equiv 0$$

i.e.,

$$D_2 - D_0 \sim 0$$

$\sigma(u)$ has the zero $u = 0$, and quasi-periodic

$$\sigma(u + 2\omega_1) = -e^{2(\eta_1 u + \omega_1)}\sigma(u)$$
$$\sigma(u + 2\omega_2) = -e^{2(\eta_2 u + \omega_2)}\sigma(u)$$

(where η_1, η_2 denote constants). We have

$$z = -\frac{\sigma(u+w)\sigma(u-w)\sigma(2v)}{\sigma(u-v)\sigma(u+v)\sigma(v+w)\sigma(v-w)}$$

$$h = C_1 \frac{\sigma^2(u-v)}{\sigma^2(u+v)}$$

$$K^+(1;z) = C_2 \frac{\sigma(u-v)\sigma(u+v+c)}{\sigma(u+v)\sigma(u-v+c)}$$

$$K^{+'}(1;z) = C_3 \frac{\sigma(u-v)\sigma(u+v+c')}{\sigma(u+v)\sigma(u-v+c')}$$

If we put

$$c' - c = v + w = \int_{0^*}^{\infty} \frac{dz}{\sqrt{W(z)}}$$

then the LR-transform represents the paralell displacement on the 1 dimensional complex trorus $\mathbf{C}/(\mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2)$

$$c \to c + v + w \to c + 2(v+w) \to \cdots$$

In order that it is periodic, there exists a positive integer M such that

$$M(v+w) \equiv 0 \quad (2\omega_1, 2\omega_2)$$

6 Multi-Index Hankel Matrices and Orthogonal Polynomials in Multi-Variables

In the next three sections we shall make a multi-dimensional extension of $LR-$transforms developed in the previous sections . Multi-dimensional $LR-$transforms are related with eigenfunction expansions for commuting self-adjoint operators.

We restrict ourselves to orthogonal polynomials case. The problem of finding LR-transforms reduces to obtaining the connection formula between two systems of orthogonal polynomials. Our main result in this section is

Theorem 4. In the course of proof, we shall give a formula for the connection matrix which is a lower triangular matrix, in terms of determinants of the associated muti-dimensional Hankel matrix.

Let $d\rho = d\rho(x)$, $x = (x_1, \cdots, x_n)$ be a Radon measure on the n dimensional Euclidean space $\mathbf{R}^n$ whose support is a bounded closed set $\mathcal{D}$. We assume that all multi-index moments

$$c_{i_1,\cdots,i_n} = \int_{\mathbf{R}^n} x_1^{i_1} \cdots x_n^{i_n} d\rho(x) \tag{6.1}$$

are finite.

Let $\mathcal{H}_\rho$ be the Hilbert space completed by the inner product $(f,g)_\rho$

$$(f,g)_\rho = \int_{\mathbf{R}^n} f(x)g(x)d\rho(x) \tag{6.2}$$

for real continuous functions $f(x), g(x)$ on $\mathbf{R}^n$.

For two sequences of indices $I = (i_1, \cdots, i_n)$ and $J = (j_1, \cdots, j_n)$, we define the sum $I + J$ by the sequence of indices $(i_1 + j_1, \cdots, i_n + j_n)$.

We define the lexicographic ordering $\mathcal{O}$ for the set of multi-indices as follows.

$(i_1, \cdots, i_n)$ is greater than $(j_1, \cdots, j_n)$ if and only if there exists a number r $(1 \le r \le n)$ such that $i_1 = j_1, \cdots, i_{r-1} = j_{r-1}, i_r > j_r$. In this case, we also say the monomial $x_1^{i_1} \cdots x_n^{i_n}$ is greater than the monomial $x_1^{j_1} \cdots x_n^{j_n}$.

Thus we have the sequence of monomials in increasing order

$$1 < x_1 < \cdots < x_n < x_1^2 < x_1x_2 < \cdots < x_n^2 < x_1^3 < \cdots$$

Let N be the unique bijective mapping from the set of positive integers onto the set of multi-indices such that $N(l_1) < N(l_2)$ for two positive integers $l_1 < l_2$.

We have $N(1) = (0, \cdots, 0), N(2) = (1, \cdots, 0), N(3) = (0, 1, 0, \cdots, 0)$, $\cdots, N(n+1) = (0, \cdots, 0, 1)$, and $N(n+2) = (2, 0, \cdots, 0)$ etc.

We assume that $d\rho(x)$ is non-degenerate in the sense that

$$\int_{\mathbf{R}^n} f(x)^2 d\rho(x) > 0$$

for any polynomial $f(x)$ which is not identically zero on $\mathcal{D}$.

Let C be the generalized Hankel matrix with the $N(l), N(m)$th entries $c_{N(l)+N(m)}$ for $l, m = 1, 2, 3, \cdots$. It is a positive definite matrix , so that all the determinants

$$D_{N(r)} = \det((c_{N(l)+N(m)})_{l,m=1}^{r}) > 0$$

for $(0 \leq r < \infty)$. Here we put $D_{N(0)} = 1$.

Gram-Schmit orthonormalization with respect to the lexicographic ordering gives the orthonormalized polynomials $\{p_{i_1,\cdots,i_n}\}_{i_1,\cdots,i_n \geq 0}$ such that

$$p_{i_1,\cdots,i_n} = \xi_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n} + (lower\ \ order\ \ terms) \tag{6.3}$$

where $\xi_{i_1,\cdots,i_n}$ denote normalizing positive constants. Hence we have the orthonormality

$$(p_{i_1,\cdots,i_n}, p_{j_1,\cdots,j_n})_\rho = \delta_{i_1,j_1} \cdots \delta_{i_n,j_n} \tag{6.4}$$

Let $N(l)$ denote the multi-index $(i_1, \cdots, i_n)$. We denote the monomial $x^{N(l)} = x_1^{i_1} \cdots x_n^{i_n}$. Then the polynomials $\tilde{p}_{i_1,\cdots,i_n}(x)$ defined by the determinant

$$\tilde{p}_{i_1,\cdots,i_n}(x) = \frac{1}{D_{N(l-1)}} \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(l)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(l)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(l-1)} & c_{N(l-1)+N(2)} & \cdots & c_{N(l-1)+N(l)} \\ x^{N(1)} & x^{N(2)} & \cdots & x^{N(l)} \end{vmatrix} \tag{6.5}$$

are monic orthogonal polynomials such that the following equations hold.

$$(\tilde{p}_{i_1,\cdots,i_n}(x), \tilde{p}_{j_1,\cdots,j_n}(x)) = 0 \quad \text{for} \ (i_1, \cdots, i_n) \neq (j_1, \cdots, j_n) \tag{6.6}$$

$$(\tilde{p}_{i_1,\cdots,i_n}(x), \tilde{p}_{i_1,\cdots,i_n}(x)) = \frac{D_{N(l)}}{D_{N(l-1)}} \quad \text{for} \ N(l) = (i_1, \cdots, i_n) \tag{6.7}$$

so that we have the orthonormalized polynomials

$$p_{N(l)}(x) = \sqrt{\frac{D_{N(l-1)}}{D_{N(l)}}} \tilde{p}_{N(l)}(x) \tag{6.8}$$

(The above computation can be done in the same way as in [31].)

We have

$$p_{N(l)}(x) = \sqrt{\frac{D_{N(l-1)}}{D_{N(l)}}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + (lower\ \ order\ \ terms)$$

Let $A_1, A_2, \cdots, A_n$ be the bounded linear operators on $\mathcal{H}_\rho$ defined by

$$A_j \varphi(x) = x_j \varphi(x) \qquad \varphi(x) \in \mathcal{H}_\rho \tag{6.9}$$

They can be expressible in matrix form $a^{(j)}_{N(l),N(m)}$ in terms of the basis $x^{N(l)}$ $l = 1, 2, 3, \cdots$

$$x_j p_{N(l)}(x) = \sum_{m \geq 1 \text{ (finite sum)}} a^{(j)}_{N(l),N(m)} p_{N(m)}(x) \quad (1 \leq j \leq n) \tag{6.10}$$

We have

$$a^{(j)}_{N(l),N(m)} = (x_j p_{N(l)}(x), p_{N(m)}(x))_\rho \tag{6.11}$$

A_j are self-adjoint bounded operators and commute each other.

Let $L^2(\mathbf{Z}^n_{\geq 0})$ denote the Hilbert space consisiting of real sequences $u = (u_{i_1,\cdots,i_n})_{i_1,\cdots,i_n \geq 0}$ with the inner product

$$(u, v) = \sum_{i_1,\cdots,i_n \geq 0} u_{i_1,\cdots,i_n} v_{i_1,\cdots,i_n} \qquad u,\ v \in L^2(\mathbf{Z}^n_{\geq 0})$$

The correspondence from the set of real sequences $(u_{i_1,i_2,\cdots,i_n})_{i_1,\cdots,i_n \geq 0}$ to continuous functions $\varphi(x)$

$$(u_{i_1,i_2,\cdots,i_n})_{i_1,\cdots,i_n \geq 0} \to \varphi(x) = \sum_{i_1,\cdots,i_n \geq 0} u_{i_1,i_2,\cdots,i_n} p_{i_1,\cdots,i_n}(x) \tag{6.12}$$

give rise to the isomorphism between the space $L^2(\mathbf{Z}^n_{\geq 0})$ and $\mathcal{H}_\rho$.

Consider the shifts τ_ν for the sequences $i_1 \geq, \cdots, i_n \geq 0$ as

$$\tau_\nu^\pm : (i_1, \cdots, i_n) \to (i_1, \cdots, i_\nu \pm 1, \cdots i_n) \tag{6.13}$$

For $N(l) = (i_1, \cdots, i_n)$, we denote by $\tau_\nu^\pm l$ the number $l^\pm$ such that $N(l^\pm) = \tau_\nu^\pm N(l)$ by abuse of notation (Remark that l^- does not exist when $i_\nu = 0$.)

From the relations (6.5),(6.8) and (6.10) the following Proposition holds.

Proposition 6 *Assume* $1 \le m \le l$. *We can represent explicitly the matrix elements* $a^{(j)}_{N(l),N(m)}$ *as*

$$a^{(j)}_{N(l),N(m)} = \sum_{m\le r\le l} (-1)^{l+m+r} \frac{1}{\sqrt{D_{N(l)}D_{N(l-1)}D_{N(m)}D_{N(m-1)}}}$$

$$\cdot \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(m)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(m)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(m-1)} & c_{N(m-1)+N(2)} & \cdots & c_{N(m-1)+N(m)} \\ c_{N(r)} & c_{N(r)+N(2)} & \cdots & c_{N(r)+N(m)} \end{vmatrix}$$

$$\cdot \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(l-1)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(l-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \rangle c_{\tau_j^- N(r)} & c_{\tau_j^- N(r)+N(2)} & \cdots & c_{\tau_j^- N(r)+N(l-1)} \langle \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(l)} & c_{N(l)+N(2)} & \cdots & c_{N(l)+N(l-1)} \end{vmatrix} \tag{6.14}$$

(The symbol $\rangle \cdots \langle$ denotes the deletion of a line)

Let $f(x)$ be a continuous function on $\mathbf{R}^n$ which is non-negative on $\mathcal{D}$. Consider the new density $d\rho'(x)$ on $\mathbf{R}^n$ with the same support as $\mathcal{D}$.

$$d\rho'(x) = f(x)d\rho(x) \tag{6.15}$$

Then we can define the multiplication operators

$$A'_j : \varphi(x) \to x_j\varphi(x) \quad (1 \le j \le n) \tag{6.16}$$

on the new Hilbert space $\mathcal{H}_{\rho'} = L^2(\mathbf{R}^n; d\rho')$ with the inner product $(\cdots,\cdots)_{\rho'}$.

Let $(c'_{i_1,\cdots,i_n})_{i_1,\cdots,i_n\ge 0}$ be the moments of the density $d\rho'$ and C' be the corresponding generalized Hankel matrix with the $N(l), N(m)$th entries $c'_{N(l)+N(m)}$.

Then $f(A_1,\cdots,A_n)$ is a self-adjoint operator on $\mathcal{H}_{\rho'}$, which is positive definite, because

$$(f(A_1,\cdots,A_n)\varphi(x),\varphi(x))_\rho = \int_{\mathcal{D}} \varphi(x)^2 f(x)d\rho(x) > 0$$

for a continuous function $\varphi(x)$ which does not vanish identically in $\mathcal{D}$.

Let $(p'_{i_1,\cdots,i_n}(x))_{i_1,\cdots,i_n\geq 0}$ be the Gram-Schmidt orthonormalization according to the lexicographic ordering $\mathcal{O}$.

$(\tilde{p}'_{i_1,\cdots,i_n}(x))_{i_1,\cdots,i_n\geq 0}$ are defined similarly to (1.5), replacing $c_{i_1,\cdots,i_n}$ by $c'_{i_1,\cdots,i_n}$.

The operator $f(A_1,\cdots,A_n)$ can be represented by the matrix with the $N(l),N(m)$th elements $(f(A_1,\cdots,A_n)p_{N(l)}(x),p_{N(m)}(x))_\rho$.

We are interested in the connection relations between the two set of orthogonal polynomials $(p_{i_1,\cdots,i_n})_{i_1,\cdots,i_n}$ and $(p'_{i_1,\cdots,i_n})_{i_1,\cdots,i_n}$.

$p_{i_1,\cdots,i_n}$ can be represented as a linear combination of $p'_{j_1,\cdots,j_n}$

$$p_{N(l)}(x) = \sum_{1\leq m\leq l} R_{N(l)/N(m)}p'_{N(m)}(x) \tag{6.17}$$

We put further $R_{N(l)/N(m)}$ to be 0 for $l < m$, so that $R = (R_{N(l)/N(m)})_{l,m\geq 0}$ defines an invertible lower triangular matrix with respect to the lexicogrphic ordering. In particular the diagonal elements are expressed as

$$R_{N(l)/N(l)} = \sqrt{\frac{D'_{N(l)}D_{N(l-1)}}{D_{N(l)}D'_{N(l-1)}}} > 0 \tag{6.18}$$

As for the relations between $\tilde{p}_{i_1,\cdots,i_n}$ and $\tilde{p}'_{i_1,\cdots,i_n}$, we have similarly

$$\tilde{p}_{N(l)} = \sum_{1\leq m\leq l} \tilde{R}_{N(l)/N(m)}\tilde{p}'_{N(m)} \tag{6.19}$$

for an invertible lower triangular matrix $\tilde{R} = (\tilde{R}_{N(l)/N(m)})_{1\leq l,m<\infty}$. Remark that $\tilde{R}_{N(l)/N(l)} = 1$. In view of (6.8),(6.17) and (6.19), the following identities hold.

$$R_{N(l),N(m)} = \sqrt{\frac{D_{N(l-1)}D'_{N(l)}}{D'_{N(l-1)}D_{N(l)}}}\tilde{R}_{N(l),N(m)} \tag{6.20}$$

Theorem 4 *As a matrix expression, we have*

$$f(A_1,\cdots,A_n) = R\cdot {}^tR \tag{6.21}$$

The matrix R is uniquely determined by (1.21).

For every j, we have the following LR–transforms

$$A'_j = R^{-1} \cdot A_j \cdot R \tag{6.22}$$

In particular,

$$f(A'_1, \cdots, A'_n) = R^{-1} \cdot f(A_1, \cdots, A_n) \cdot R = {}^tR \cdot R$$

which is just the interchange of R and ${}^tR$*. R is an invertible matrix so that* R^{-1} *is well-defined.*

For $u = (u_{i_1,\cdots,i_n})_{i_1,\cdots,i_n} \in L^2(\mathbf{Z}^n_{\geq 0})$, (6.10) and (6.12) give the matrix expression

$$(A_j u)_{i_1,\cdots,i_n} = \sum_{j_1,\cdots,j_n \geq 0} a^{(j)}_{(i_1,\cdots,i_n),(j_1,\cdots,j_n)} u_{j_1,\cdots,j_n} \tag{6.23}$$

Let $\mathcal{H}_0$ be the Hilbert space spanned by the sequences ${}^tRu$. $\mathcal{H}_0$ is isomorphic to the space of sequences $v = (v_{i_1,\cdots,i_n})_{i_1,\cdots,i_n}$ in $L^2(\mathbf{Z}^n_{\geq 0})$ such that $(f(A_1, \cdots, A_n)^{-1}v, v) < \infty$. Then the inverse R^{-1} is well-defined as a bounded operator from $\mathcal{H}_0$ to $L^2(\mathbf{Z}^n_{\geq 0})$.

The matrix elements $\tilde{R}_{N(l),N(m)}$ can be expressed by using the following system of determinants $\psi_{l_1,\cdots,l_r}$ for different positive integers $l_1, \cdots, l_r, \cdots,$ from each other.

$$\psi_{l_1} = c_{N(l_1)}, \psi_{l_1,l_2} = \begin{vmatrix} c_{N(l_1)} & c_{N(l_2)} \\ c_{N(2)+N(l_1)} & c_{N(2)+N(l_2)} \end{vmatrix}$$

$$\psi_{l_1,l_2,\cdots,l_r} = \begin{vmatrix} c_{N(l_1)} & c_{N(l_2)} & \cdots & c_{N(l_r)} \\ c_{N(2)+N(l_1)} & c_{N(2)+N(l_2)} & \cdots & c_{N(2)+N(l_r)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(r)+N(l_1)} & c_{N(r)+N(l_2)} & \cdots & c_{N(r)+N(l_r)} \end{vmatrix}$$

(Remark that $N(1) = (0, 0, \cdots, 0)$.)

In the same way we define the determinants $\psi'_{l_1,\cdots,l_r}$ associated with the moments $c'_{N(l)}$

$$\psi'_{l_1} = c'_{N(l_1)}, \psi'_{l_1,l_2} = \begin{vmatrix} c'_{N(l_1)} & c'_{N(l_2)} \\ c'_{N(2)+N(l_1)} & c'_{N(2)+N(l_2)} \end{vmatrix}$$

,

$$\psi'_{l_1,l_2,\cdots,l_r} = \begin{vmatrix} c'_{N(l_1)} & c'_{N(l_2)} & \cdots & c'_{N(l_r)} \\ c'_{N(2)+N(l_1)} & c'_{N(2)+N(l_2)} & \cdots & c'_{N(2)+N(l_r)} \\ \cdots & \cdots & \cdots & \cdots \\ c'_{N(r)+N(l_1)} & c'_{N(r)+N(l_2)} & \cdots & c'_{N(r)+N(l_r)} \end{vmatrix}$$

Then we have

Proposition 7

$$\begin{aligned} \tilde{R}_{N(l),N(m)} &= \frac{1}{D'_{N(m)} \cdots D'_{N(l-1)} D_{N(l-1)}} \\ &\cdot \sum \epsilon \psi'_{1,2,\cdots,m-1,\alpha_{m,m}} \psi'_{1,2,\cdots,m-1,\alpha_{m+1,m},\alpha_{m+1,m+1}} \\ &\cdots \psi'_{1,2,\cdots,m-1,\alpha_{l-1,m},\cdots,\alpha_{l-1,l-1}} \psi_{1,2,\cdots,m-1,\alpha_{l,m},\cdots,\alpha_{l,l-1}} \end{aligned} \tag{6.24}$$

where $\alpha_{m,m}, \alpha_{m+1,m}, \cdots$ move over the set of finite sequences of integers such that the following identities hold as sets

$$\begin{aligned} \{\alpha_{m,m}, \alpha_{m+1,m}\} &= \{m, m+1\}, \\ \{\alpha_{m+1,m+1}, \alpha_{m+2,m}, \alpha_{m+2,m+1}\} &= \{m, m+1, m+2\}, \\ \cdots &= \cdots \\ \{\alpha_{l-2,l-2}, \alpha_{l-1,m}, \cdots, \alpha_{l-1,l-2}\} &= \{m, m+1, \cdots, l-1\}, \\ \{\alpha_{l-1,l-1}, \alpha_{l,m}, \cdots, \alpha_{l,l-1}\} &= \{m, m+1, \cdots, l\} \end{aligned}$$

and that

$$\begin{aligned} \alpha_{m+2,m} &< \alpha_{m+2,m+1}, \\ \cdots &= \cdots \\ \alpha_{l-1,m} &< \alpha_{l-1,m+1} < \cdots < \alpha_{l-1,l-2}, \\ \alpha_{l,m} &< \alpha_{l,m+1} < \cdots < \alpha_{l,l-1} \end{aligned}$$

ϵ denotes the suitably chosen sign $\pm$ depending on the choices of α's.

This Proposition follows by solving (6.19) term by term in view of(6.5). For example,

$$
\begin{aligned}
\tilde{R}_{N(l),N(l)} &= 1, \\
\tilde{R}_{N(l),N(l-1)} &= \frac{1}{D'_{N(l-1)}D_{N(l-1)}} \\
&\cdot \quad (\psi'_{1,2,\cdots,l-2,l}\psi_{1,2,\cdots,l-2,l-1} - \psi'_{1,2,\cdots,l-2,l-1}\psi_{1,2,\cdots,l-2,l}) \\
\tilde{R}_{N(l),N(l-2)} &= \frac{1}{D'_{N(l-1)}D'_{N(l-2)}D_{N(l-1)}} \\
&\cdot \quad (\psi'_{1,2,\cdots,l-2}\psi'_{1,2,\cdots,l-1}\psi_{1,2,\cdots,l-3,l-1,l} \\
&- \quad \psi'_{1,2,\cdots,l-3,l-1}\psi'_{1,2,\cdots,l-2,l}\psi_{1,2,\cdots,l-1} \\
&+ \quad \psi'_{1,2,\cdots,l-2}\psi'_{1,2,\cdots,l-3,l-1,l}\psi_{1,2,\cdots,l-1} \\
&- \quad \psi'_{1,2,\cdots,l-3,l-1}\psi'_{1,2,\cdots,l-1}\psi_{1,2,\cdots,l-2,l})
\end{aligned}
$$

and so on.

Example. (Appell's Polynomials) Suppose the density

$$d\rho(x) = x_1^{\alpha_1}\cdots x_n^{\alpha_n}(1 - x_1 - \cdots - x_n)^{\alpha_{n+1}}dx_1 \wedge \cdots \wedge dx_n \tag{6.25}$$

be defined on the simplex $\mathcal{D} : x_1 \geq 0, \cdots, x_n \geq 0,\ x_1 + \cdots + x_n \leq 1$. We have

$$c_{i_1,\cdots,i_n} = \frac{\Gamma(\alpha_1 + i_1 + 1)\cdots\Gamma(\alpha_n + i_n + 1)\Gamma(\alpha_{n+1} + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_{n+1} + i_1 + \cdots + i_n + n + 1)}$$

The ratio $D(N(l))/D(N(1))$ and the monic polynomial $\tilde{p}_{N(l)}$ for every l are rational functions of $\alpha_1, \cdots, \alpha_{n+1}$. Whence every element $\tilde{R}_{N(l),N(m)}$ is a rational function of $\alpha_1, \cdots, \alpha_{n+1}$.

7 Matrix Form of LR-Transforms and Proof of Theorem 4

Assume that the orthonormal polynomials $p_{N(l)}$ and $p'_{N(l)}$ $l = 1, 2, 3, \cdots$ are expressed as linear combinations of monomials $x^{N(m)}$ $m = 1, 2, 3, \cdots$ as

$$p_{N(l)} = \sum_{m=1}^{l} \xi_{N(l),N(m)} x^{N(m)}$$

$$p'_{N(l)} = \sum_{m=1}^{\cdot} \xi'_{N(l),N(m)} x^{N(m)}$$

We put $\xi_{N(l),N(m)}$ and $\xi'_{N(l),N(m)}$ to be 0 for $l < m$. Let Ξ, Ξ' be the lower triangular matrices with the $N(l), N(m)$th elements $\xi_{N(l),N(m)}$, $\xi'_{N(l),N(m)}$ respectively. Then the orthonormality and the spectral representations for A_j and A'_j imply the matrix relations

$$\Xi \cdot \mathcal{C} \cdot {}^t\Xi = 1 \tag{7.1}$$

$$\Xi' \cdot \mathcal{C}' \cdot {}^t\Xi' = 1 \tag{7.2}$$

and

$$A_j = \Xi \cdot \tau_j^+ \mathcal{C} \cdot {}^t\Xi \tag{7.3}$$

$$A'_j = \Xi' \cdot \tau_j^+ \mathcal{C}' \cdot {}^t\Xi' \tag{7.4}$$

respectively.

Lemma 2 *Let M_ν, $1 \leq \nu \leq n$ be the operator defined by the matrix whose $(i_1, \cdots, i_n; j_1, \cdots, j_n)$th elements are equal to* 1 *if $(j_1, \cdots, j_n) = (i_1, \cdots, i_{\nu-1}, i_\nu + 1, i_{\nu+1}, \cdots, i_n)$ and equal to* 0 *otherwise. Then we have*

$$\tau_\nu^+(\mathcal{C}) = M_\nu \cdot \mathcal{C} \tag{7.5}$$

This lemma shows that (7.3) and (7.4) are equivalent to the followings

$$A_j = \Xi \cdot M_j \cdot \Xi^{-1} \tag{7.6}$$

$$A'_j = \Xi' \cdot M_j \cdot \Xi'^{-1} \tag{7.7}$$

respectively. We have further

$$R = \Xi \cdot \Xi'^{-1} \tag{7.8}$$

(7.6)-(7.8) imply that

$$A'_j = R^{-1} \cdot A_j \cdot R \tag{7.9}$$

This proves the Theorem.

This is a discrete analog of the argument done in [34].

8 Symmetric Polynomials Case

LR-transforms can also be applied to symmetric orthogonal polynomials with respect to a non-degenerate symmetric Radon measure $d\rho(x)$ on $\mathbf{R}^n$ with support $\hat{\mathcal{D}}$ which is a bounded closed set.

Let $\lambda_1, \cdots, \lambda_n$ be a partition, namely a sequence of non-increasing integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

Assume that $\lambda_1 = \cdots = \lambda_{r_1} > \lambda_{r_1+1} = \cdots = \lambda_{r_2} > \cdots > \lambda_{r_{m-1}+1} = \lambda_{r_m}$ for an incresing sequence $0 < r_1 < r_2 < \cdots < r_m$. Let $m_\lambda(x)$ be the symmetric polynomials defined by the symmetrization

$$m_\lambda = \frac{1}{r_1!(r_2-r_1)!\cdots(r_m-r_{m-1})!} \sum_{\sigma\in\mathcal{S}_n} \sigma(x_1^{\lambda_1}\cdots x_n^{\lambda_n})$$

under the permutation group $\mathcal{S}_n$ of degree n.

The symmetric lexicographic ordering $\hat{\mathcal{O}}$ can be introduced for the partitions as follows. The partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ is greater than the parttion $\mu = (\mu_1, \cdots, \mu_n)$ if there exists a positive integer r such that $\lambda_1 = \mu_1, \cdots, \lambda_{r-1} = \mu_{r-1}$ and $\lambda_r > \mu_r$.

The symmetric moments are defined as

$$\hat{c}_\lambda = \frac{1}{n!}\int_{\mathbf{R}^n} m_\lambda(x)d\rho(x). \tag{8.1}$$

Let $\hat{\mathcal{H}}_\rho$ be the Hilbert space consisting of symmetric functions on $\mathbf{R}^n$ with the inner product $(f,g)_\rho$ and the norm $||f||_\rho = \sqrt{(f,f)_\rho}$,

$$(f,g)_\rho = \frac{1}{n!}\int_{\mathbf{R}^n} f(x)g(x)d\rho(x) \tag{8.2}$$

for functions $f(x)$, $g(x)$ on $\hat{\mathcal{D}}$.

Let $\hat{N}$ be the bijective mapping from the set of positive integers onto the set of all partitions such that $\hat{N}(l) > \hat{N}(m)$ for $l > m$. Hence $\hat{N}(1) = (0,\cdots,0)$, $\hat{N}(2) = (1,0,\cdots,0)$, $\hat{N}(3) = (1,1,0,\cdots,0)\cdots$, $\hat{N}(n+1) = (1,1,\cdots,1)$, $\hat{N}(n+2) = (2,0,\cdots,0)$, $\hat{N}(n+3) = (2,1,0,\cdots,0)$, $\hat{N}(n+4) = (2,1,1,0\cdots,0)$, $\hat{N}(2n+1) = (2,1,1,\cdots,1), \cdots$, so that we have

$m_{\hat{N}(1)}(x) = 1$, $m_{\hat{N}(2)}(x) = x_1+\cdots+x_n$, $m_{\hat{N}(3)}(x) = \sum_{1\leq i<j\leq n} x_i x_j$, $m_{\hat{N}(n+1}$
$x_1\cdots x_n$, $m_{\hat{N}(n+2)}(x) = \sum_{j=1}^n x_j^2$, etc.

The generalized Hankel matrix $\hat{C}$ are defined with the $\hat{N}(l), \hat{N}(m)$th elements $\hat{c}_{\hat{N}(l)+\hat{N}(m)}$.

We denote the determinants for each $\hat{N}(l) = \lambda$,

$$\hat{D}_{\hat{N}(l)} = \det((\hat{c}_{\hat{N}(r)+\hat{N}(s)})_{r,s=1}^{l}) \tag{8.3}$$

The symmetric orthogonal polynomials $\tilde{\hat{p}}_\lambda(x)$ parametrized by the partitions $\hat{N}(l) = \lambda$ are given by the formulae

$$\tilde{\hat{p}}_\lambda(x) = \frac{1}{\hat{D}_{N(l-1)}} \begin{vmatrix} \hat{c}_{\hat{N}(1)} & \hat{c}_{\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(l)} \\ \hat{c}_{\hat{N}(2)} & \hat{c}_{\hat{N}(1)+\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(2)+\hat{N}(l)} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{c}_{\hat{N}(l-1)} & \hat{c}_{\hat{N}(l-1)+\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(l-1)+\hat{N}(l)} \\ m_{\hat{N}(1)}(x) & m_{\hat{N}(2)}(x) & \cdots & m_{\hat{N}(l)}(x) \end{vmatrix} \tag{8.4}$$

$$= m_{\hat{N}(l)}(x) + (\textit{lower order symmetric polynomials}) \tag{8.5}$$

The orthogonality and the norms are given by

$$(\tilde{\hat{p}}_\lambda(x), \tilde{\hat{p}}_\mu(x))_\rho = 0 \quad \lambda \neq \mu \tag{8.6}$$

$$= \frac{\hat{D}_{\hat{N}(l)}}{\hat{D}_{\hat{N}(l-1)}} \quad \lambda = \mu \tag{8.7}$$

so that

$$\hat{p}_{\hat{N}(l)}(x) = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)}}} \tilde{\hat{p}}_{\hat{N}(l)}(x) \tag{8.8}$$

are the orthonormal polynomials having the properties

$$(\hat{p}_\lambda(x), \hat{p}_\mu(x))_\rho = 0 \quad \lambda \neq \mu$$

$$= 1 \quad \lambda = \mu \tag{8.9}$$

and

$$p_{\hat{N}(l)}(x) = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)}}} m_{\hat{N}(l)}(x) + \text{ lower order symmetric polynomials} \tag{8.10}$$

Let $e_r \quad (1 \leq r \leq n)$ be the elementary symmetric polynomials $e_r = \sum_{1\leq i_1<\cdots<i_r\leq n} x_{i_1}x_{i_2}\cdots x_{i_r}$. We define the bounded linear operators $\hat{A}_r$ (Pieri operators) on $\hat{\mathcal{H}}_\rho$

$$\hat{A}_r : f(x) \in \hat{\mathcal{H}}_\rho \to e_r(x)f(x) \in \hat{\mathcal{H}}_\rho \tag{8.11}$$

They can be expressed in matrix form as

$$e_r(x)\hat{p}_\lambda(x) \to \sum_\mu \hat{a}^{(r)}_{\lambda,\mu}\hat{p}_\mu(x) \tag{8.12}$$

Let $f(x)$ be a symmetric polynomial in x, such that $f(x)$ can be expressed as a polynomial F in $e_1, \cdots, e_n$ $f(x) = F(e_1, \cdots, e_n)$

The multiplication operator by $f(x)$ on $\hat{\mathcal{H}}_\rho$ can be expressed as $F(\hat{A}_1, \cdots, \hat{A}_n)$.

We assume that $f(x)$ is positive in $\hat{\mathcal{D}}$ so that $F(A_1, \cdots, A_n)$ is a positive definite operator on $\hat{\mathcal{H}}_\rho$.

Let $d\rho'(x) = f(x)d\rho(x)$ be another positive Radon measure on $\mathbf{R}^n$ with the same support $\mathcal{D}$ as $d\rho(x)$.

We denote by $\hat{D}'_{\hat{N}(l)}$ the determinant $\det((\hat{c}'_{\hat{N}(r)+\hat{N}(s)})^l_{r,s=1})$. We can define the orthogonal polynomials $\tilde{\hat{p}}'_\lambda(x)$ and $\hat{p}'_\lambda(x)$ in the same way as (3.4), (3.7) respectively.

$$\hat{p}'_{N(l)}(x) = \sqrt{\frac{\hat{D}'_{N(l-1)}}{\hat{D}'_{N(l)}}}\,\tilde{\hat{p}}'_{N(l)}(x) \tag{8.13}$$

and

$$\begin{aligned} (\hat{p}'_\lambda(x), \hat{p}'_\mu(x))_{\rho'} &= 0 \quad \lambda \neq \mu \\ &= 1 \quad \lambda = \mu \end{aligned} \tag{8.14}$$

We have the connection relations between $\{\hat{p}_{\hat{N}(l)}(x)\}_{l\geq 1}$ and $\{\hat{p}'_{\hat{N}(l)}(x)\}_{l\geq 1}$ in the following form

$$\hat{p}_{\hat{N}(l)}(x) = \sum_{m=1}^{l} \hat{R}_{\hat{N}(l)/\hat{N}(m)}\hat{p}'_{\hat{N}(m)}(x) \tag{8.15}$$

$$\tilde{\hat{p}}_{\hat{N}(l)}(x) = \sum_{m=1}^{l} \tilde{\hat{R}}_{\hat{N}(l)/\hat{N}(m)} \tilde{\hat{p}}'_{\hat{N}(m)}(x) \tag{8.16}$$

where

$$\hat{R}_{\hat{N}(l),\hat{N}(m)} = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)}\hat{D}'_{\hat{N}(m)}}{\hat{D}_{\hat{N}(l)}\hat{D}'_{\hat{N}(m-1)}}}\tilde{\hat{R}}_{\hat{N}(l)/\hat{N}(m)} \tag{8.17}$$

In particular,

$$\hat{R}_{\hat{N}(l),\hat{N}(l)} = \sqrt{\frac{\hat{D}'_{\hat{N}(l)}\hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)}\hat{D}'_{\hat{N}(l-1)}}} > 0 \tag{8.18}$$

because $\tilde{\hat{R}}_{\hat{N}(l),\hat{N}(l)} = 1$. Let $\hat{R}$ and $\tilde{\hat{R}}$ be the corressponding lower triangular operators which are both invertible.

Theorem 5 *We have the LR transforms*

$$F(\hat{A}_1, \cdots, \hat{A}_n) = \hat{R} \cdot {}^t\hat{R} \tag{8.19}$$

$$\hat{A}_r{}' = \hat{R}^{-1} \cdot \hat{A}_r \cdot \hat{R} \tag{8.20}$$

Example 2. (Koornwinder polynomials , Heckman-Opdam BC_2-type Polynomials) (see [12],[18].)

Let $d\rho(x) = (1-x_1)^{\alpha}(1-x_2)^{\alpha}(1+x_1)^{\beta}(1+x_2)^{\beta}(x_1-x_2)^{2\gamma+1}$ defined on $\mathcal{D}: \ -1 \le x_2 \le x_2 \le 1$ in the 2-dimensional Euclidean space.

For a partition $\lambda_1 \ge \lambda_2 \ge 0$

$$\hat{c}_{\lambda_1,\lambda_2} = \int_{\mathcal{D}} (x_1^{\lambda_1}x_2^{\lambda_2} + x_2^{\lambda_1}x_1^{\lambda_2})d\rho(x) \qquad \lambda_1 > \lambda_2 \tag{8.21}$$

$$= \int_{\mathcal{D}} (x_1x_2)^{\lambda_1} d\rho(x) \qquad \lambda_1 = \lambda_2 \tag{8.22}$$

One can consider the symmetric orthogonal polynomials $\tilde{p}_{\hat{N}(l)}(x)$ as functions of $u = x_1 + x_2, \ v = x_1x_2$ (We also denote them by $\tilde{p}^{\alpha,\beta,\gamma}_{\hat{N}(l)}(u,v)$ or simply by $\tilde{p}^{\alpha,}_{N(l)}(u,v)$, $\tilde{p}^{\gamma}_{N(l)}(u,v)$, $\tilde{p}_{N(l)}(u,v)$ etc according as we are interested in dependence on α,β or γ.)

$\mathcal{D}$ is defined by the inequalities

$$1-u+v\geq 0,\ \ 1+u+v\geq 0,\ \ u^2-4v\geq 0$$

The symmetric lexicographic ordering $\hat{\mathcal{O}}$ with respect to x_1, x_2 coincides with the lexicographic ordering $\mathcal{O}$ with respect to u, v.

Let

$$1,u,v,u^2,uv,v^2,u^3,u^2v,uv^2,u^3,\cdots \tag{8.23}$$

be the sequence of monomials in the lexicographic order.

$$\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}(u,v)=u^{\lambda_1-\lambda_2}v^{\lambda_2}+(lower\ \ order\ \ terms)$$

are the monic orthogonal polynomials in u,v, obtained by Gram-Schmidt orthogonalization with respect to the inner product

$$(f(u,v),g(u,v))_\gamma=\int_{\mathcal{D}}f(u,v)g(u,v)\mu^{\alpha,\beta,\gamma}(u,v)dudv \tag{8.24}$$

where $\mu^{\alpha,\beta,\gamma}(u,v)$ denotes the density

$$\mu^{\alpha,\beta,\gamma}(u,v)=2^{2\alpha+2\beta+2\gamma+3}(1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^\gamma \tag{8.25}$$

The first moment can be evaluated by using the 2 dimensional Selberg integral formula :

$$\begin{aligned} c_{0,0} &= \hat{c}_{0,0}=2^{2\alpha+2\beta+2\gamma+2} \\ &\cdot \frac{\Gamma(\alpha+\gamma+\frac{3}{2})\Gamma(\beta+\gamma+\frac{3}{2})\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\gamma+2)}{\Gamma(\alpha+\beta+2\gamma+3)\Gamma(\alpha+\beta+\gamma+\frac{5}{2})\Gamma(\gamma+\frac{3}{2})} \end{aligned} \tag{8.26}$$

The moments $c_{i,j}=\int_{\mathcal{D}}u^iv^j\mu^{\alpha,\beta,\gamma}dudv$ can be expressed explicitly as follows (see [1].)

We put $U=1-u+v$ and $V=1+u+v$. Remark first that

$$\int_{\mathcal{D}}U^rV^s\mu^{\alpha,\beta,\gamma}dudv=c_{0,0}\frac{(\alpha+\gamma+\frac{3}{2})_r(\beta+\gamma+\frac{3}{2})_s(\alpha+1)_r(\beta+1)_s}{(\alpha+\beta+2\gamma+3)_{r+s}(\alpha+\beta+\gamma+\frac{5}{2})_{r+s}} \tag{8.27}$$

where $(\alpha)_r$ denotes the product $\alpha(\alpha+1)\cdots(\alpha+r-1)$. Hence

$$\begin{aligned} c_{i,j} &= \int_{\mathcal{D}} 2^{-i-j}(U-V)^i(U+V-2)^j\mu^{\alpha+\beta+\gamma}dudv \\ &= 2^{-i-j}c_{0,0} \sum_{i\geq\nu_1\geq 0, j\geq\nu_2+\nu_3} (-1)^{j+\nu_1-\nu_2-\nu_3} \\ &\cdot \frac{(\alpha+\gamma+\frac{3}{2})_{\nu_1+\nu_2}(\beta+\gamma+\frac{3}{2})_{i-\nu_1+\nu_3}(\alpha+1)_{\nu_1+\nu_2}(\beta+1)_{i-\nu_1+\nu_3}}{(\alpha+\beta+2\gamma+3)_{i+\nu_2+\nu_3}(\alpha+\beta+\gamma+\frac{5}{2})_{i+\nu_2+\nu_3}} \end{aligned}$$

The norm $(\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}, \tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2})_\gamma$ has been evaluated by Heckman-Opdam in the form of a product of Gamma functions. The following expression is given by van Diejen (see [8]).

$$(\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}, \tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2})_\gamma = 2^{2\alpha+2\beta+2\gamma+2+2(\lambda_1+\lambda_2)+3}\Delta_+(\alpha,\beta,\gamma)\Delta_-(\alpha,\beta,\gamma) \quad (8.28)$$

where

$$\begin{aligned} &\Delta_+(\alpha,\beta,\gamma) \\ = &\frac{\Gamma(\alpha+\beta+\gamma+\lambda_1+\frac{3}{2})\Gamma(\alpha+\gamma+\lambda_1+\frac{3}{2})\Gamma(\alpha+\beta+\lambda_2+1)\Gamma(\alpha+\lambda_2+1)}{\Gamma(\alpha+\beta+2\gamma+2\lambda_1+2)\Gamma(\alpha+\beta+2\lambda_2+1)} \\ \cdot &\frac{\Gamma(\alpha+\beta+2\gamma+\lambda_1+\lambda_2+2)\Gamma(2\gamma+\lambda_1-\lambda_2+1)}{\Gamma(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{3}{2})\Gamma(\gamma+\lambda_1-\lambda_2+\frac{1}{2})} \end{aligned}$$

$$\begin{aligned} &\Delta_-(\alpha,\beta,\gamma) \\ = &\frac{\Gamma(\gamma+\lambda_1+\frac{3}{2})\Gamma(\beta+\gamma+\lambda_1+\frac{3}{2})\Gamma(\lambda_2+1)\Gamma(\beta+\lambda_2+1)}{\Gamma(\alpha+\beta+2\gamma+2\lambda_1+3)\Gamma(\alpha+\beta+2\lambda_2+2)} \\ \cdot &\frac{\Gamma(\alpha+\beta+\lambda_1+\lambda_2+2)\Gamma(\lambda_1-\lambda_2+1)}{\Gamma(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{5}{2})\Gamma(\gamma+\lambda_1-\lambda_2+\frac{3}{2})} \end{aligned}$$

From (6.7), $D_{N(l)}$ is evaluated by the identities

$$D_{N(l)} = \prod_{m=1}^{l} (\tilde{p}^{\alpha,\beta,\gamma}_{N(m)}, \tilde{p}^{\alpha,\beta,\gamma}_{N(m)})_\gamma$$

The operators A_1 ,A_2 and $B = A_1^2 - 4A_2$ are defined by

$$f(u,v) \to uf(u,v) \tag{8.29}$$
$$f(u,v) \to vf(u,v) \tag{8.30}$$
$$f(u,v) \to (u^2 - 4v)f(u,v) \tag{8.31}$$

respectively.

They can be expressed in recurrence form by the use of orthogonal polynomials as follows (see [9] ,[29] for explicit forms .)

$$\begin{aligned} & A_1 : u\tilde{p}_{\lambda_1,\lambda_2}(x) \\ = \; & a^{(1)}_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}\tilde{p}_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2}(x) + a^{(1)}_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}\tilde{p}_{\lambda_1+2,\lambda_2}(x) \\ + \; & a^{(1)}_{\lambda_1,\lambda_2,\lambda_1+1,\lambda_2-1}\tilde{p}_{\lambda_1+1,\lambda_2-1}(x) + a^{(1)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2-1}\tilde{p}_{\lambda_1-1,\lambda_2-1}(x) \\ + \; & a^{(1)}_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2+}\tilde{p}_{\lambda_1+1,\lambda_2+1}(x) + a^{(1)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2}\tilde{p}_{\lambda_1-1,\lambda_2}(x) \\ + \; & a^{(1)}_{\lambda_1,\lambda_2;\lambda_1-2,\lambda_2}\tilde{p}_{\lambda_1-2,\lambda_2}(x) + a^{(1)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2+1}\tilde{p}_{\lambda_1-1,\lambda_2+1}(x) \end{aligned} \tag{8.32}$$

$$\begin{aligned} & A_2 : v\tilde{p}_{\lambda_1,\lambda_2}(x) \\ = \; & a^{(2)}_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}\tilde{p}_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2}(x) + a^{(2)}_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}\tilde{p}_{\lambda_1+2,\lambda_2}(x) \\ + \; & a^{(2)}_{\lambda_1,\lambda_2,\lambda_1+1,\lambda_2-1}\tilde{p}_{\lambda_1+1,\lambda_2-1}(x) + a^{(2)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2-1}\tilde{p}_{\lambda_1-1,\lambda_2-1}(x) \\ + \; & a^{(2)}_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2+}\tilde{p}_{\lambda_1+1,\lambda_2+1}(x) + a^{(2)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2}\tilde{p}_{\lambda_1-1,\lambda_2}(x) \\ + \; & a^{(2)}_{\lambda_1,\lambda_2;\lambda_1-2,\lambda_2}\tilde{p}_{\lambda_1-2,\lambda_2}(x) + a^{(2)}_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2+1}\tilde{p}_{\lambda_1-1,\lambda_2+1}(x) \end{aligned} \tag{8.33}$$

$$\begin{aligned} & B : (u^2 - 4v)\tilde{p}_{\lambda_1,\lambda_2}(x) \\ = \; & b_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}\tilde{p}_{\lambda_1,\lambda_2;\lambda_1+2,\lambda_2}(x) + b_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2}\tilde{p}_{\lambda_1+1,\lambda_2}(x) \\ + \; & b_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2-1}\tilde{p}_{\lambda_1+1,\lambda_2-1}(x) + b_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2-1}\tilde{p}_{\lambda_1-1,\lambda_2-1}(x) \\ + \; & b_{\lambda_1,\lambda_2;\lambda_1+1,\lambda_2+1}\tilde{p}_{\lambda_1+1,\lambda_2+1}(x) + b_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2}\tilde{p}_{\lambda_1-1,\lambda_2}(x) \\ + \; & b_{\lambda_1,\lambda_2;\lambda_1-2,\lambda_2}\tilde{p}_{\lambda_1-2,\lambda_2}(x) + b_{\lambda_1,\lambda_2;\lambda_1-1,\lambda_2+1}\tilde{p}_{\lambda_1-1,\lambda_2+1}(x) \end{aligned} \tag{8.34}$$

where the matrices $(a^{(1)}_{\lambda_1,\lambda_2;\mu_1,\mu_2})$, $(a^{(2)}_{\lambda_1,\lambda_2;\mu_1,\mu_2})$, and $(b_{\lambda_1,\lambda_2;\mu_1,\mu_2})$ define the bounded self-adjoint operators A_1, A_2, B on $L^2(\mathbf{Z}^2_{\geq 0})$ respectively.

We have the connection formulae between the $\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}(x) = \tilde{p}^{\gamma}_{\lambda_1,\lambda_2}(x)$ and $\tilde{p}^{\alpha,\beta,\gamma+1}_{\lambda_1,\lambda_2}(x) = \tilde{p}^{\gamma+1}_{\lambda_1,\lambda_2}(x)$ as follows.

$$\begin{aligned} \tilde{p}^{\gamma}_{\lambda_1,\lambda_2}(x) &= \tilde{p}^{\gamma+1}_{\lambda_1,\lambda_2}(x) + \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2-1}\tilde{p}^{\gamma+1}_{\lambda_1-1,\lambda_2-1}(x) \\ &+ \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2+1}\tilde{p}^{\gamma+1}_{\lambda_1-1,\lambda_2+1}(x) \\ &+ \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2}\tilde{p}^{\gamma+1}_{\lambda_1-1,\lambda_2}(x) + \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-2,\lambda_2}\tilde{p}^{\gamma+1}_{\lambda_1-2,\lambda_2}(x) \end{aligned} \tag{8.35}$$

We put $\tilde{R}_{\lambda_1,\lambda_2/\mu_1,\mu_2}$ to be 0 for $(\lambda_1,\lambda_2) < (\mu_1,\mu_2)$. We denote by $\tilde{R}$ the lower triangular matrix $\{\tilde{R}_{\lambda_1,\lambda_2/\mu_1,\mu_2}\}_{\lambda_1,\lambda_2/\mu_1,\mu_2}$ thus obtained.

The matrix R is then defined by (6.19).

Let the operators A_1', A_2', B' be the corresponding operators for $\gamma+1$ in place of γ. Then the following formulae of LR-transforms similar to (6.19) and (6.20) hold.

$$B = R \cdot {}^tR \tag{8.36}$$
$$A_1' = R^{-1} \cdot A_1 \cdot R \tag{8.37}$$
$$A_2' = R^{-1} \cdot A_2 \cdot R \tag{8.38}$$

These are equivalent to the idetities (8.18),(8.19) and (8.20) respectively.

The elements of $\tilde{R}$ can be evaluated in a remarkable way.

Proposition 8

$$\begin{aligned} &\tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2+1} \\ &= -\frac{(\lambda_1-\lambda_2-1)(\lambda_1-\lambda_2)}{(\gamma+\lambda_1-\lambda_2-\frac{1}{2})(\gamma+\lambda_1-\lambda_2+\frac{1}{2})} \\ &\tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2} \\ &= \frac{4(\lambda_1-\lambda_2)(\alpha-\beta)(\alpha+\beta)(\alpha+\beta+\lambda_1+\lambda_2+1)}{(\alpha+\beta+2\lambda_2)(\alpha+\beta+2\lambda_2+2)(\alpha+\beta+2\gamma+2\lambda_1+1)(\alpha+\beta+2\gamma+2\lambda_1+1)} \\ &\tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2-1} \\ &= -\frac{4\lambda_2(\alpha+\lambda_2)(\beta+\lambda_2)(\alpha+\beta+\lambda_2)}{(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{1}{2})(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{3}{2})} \\ &\cdot \frac{(\alpha+\beta+\lambda_1+\lambda_2)(\alpha+\beta+\lambda_1+\lambda_2+1)}{(\alpha+\beta+2\lambda_2-1)(\alpha+\beta+2\lambda_2)^2(\alpha+\beta+2\lambda_2+1)} \end{aligned}$$

$$R_{\lambda_1,\lambda_2/\lambda_1-2,\lambda_2}$$

$$= \frac{4(\lambda_1-\lambda_2-1)(\lambda_1-\lambda_2)(\alpha+\beta+\lambda_1+\lambda_2)(\alpha+\beta+\lambda_1+\lambda_2+1)}{(\gamma+\lambda_1-\lambda_2-\frac{1}{2})(\gamma+\lambda_1-\lambda_2+\frac{1}{2})(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{1}{2})(\alpha+\beta+\gamma+\lambda}$$

$$\cdot \frac{(\alpha+\gamma+\lambda_1+\frac{1}{2})(\beta+\gamma+\lambda_1+\frac{1}{2})(\gamma+\lambda_1+\frac{1}{2})(\alpha+\beta+\gamma+\lambda_1+\frac{1}{2})}{(\alpha+\beta+2\gamma+2\lambda_1)(\alpha+\beta+2\gamma+2\lambda_1+1)^2(\alpha+\beta+2\gamma+2\lambda_1+2)}$$

The operator $B_+ = 1 - A_1 + A_2$ corresponding to the shift $\alpha \to \alpha+1$ is defined by

$$B_+ : \varphi(u,v) \to (1-u+v)\varphi(u,v)$$

Its connection formula for $\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}(x) = \tilde{p}^{\alpha}_{\lambda_1,\lambda_2}(x)$ and $\tilde{p}^{\alpha+1,\beta,\gamma}_{\lambda_1,\lambda_2}(x) = \tilde{p}^{\alpha+1}_{\lambda_1,\lambda_2}(x)$ is given by

$$\tilde{p}^{\alpha}_{\lambda_1,\lambda_2}(x) = \tilde{p}^{\alpha+1}_{\lambda_1,\lambda_2}(x) + \tilde{R}_{\lambda_1,\lambda_2/\lambda_1,\lambda_2-1}\tilde{p}^{\alpha+1}_{\lambda_1,\lambda_2-1}(x)$$
$$+ \quad \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2}\tilde{p}^{\alpha+1}_{\lambda_1-1,\lambda_2}(x) + \tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2-1}\tilde{p}^{\alpha+1}_{\lambda_1-1,\lambda_2-1}(x)$$

where

$$\tilde{R}_{\lambda_1,\lambda_2/\lambda_1,\lambda_2-1} = -2\frac{\lambda_2(\beta+\lambda_2)}{(\alpha+\beta+2\lambda_2)(\alpha+\beta+2\lambda_2+1)}$$

$$\tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2+1}$$

$$= -2\frac{(\lambda_1-\lambda_2)(2\gamma+\lambda_1-\lambda_2)}{(\gamma+\lambda_1-\lambda_2+\frac{1}{2})(\gamma+\lambda_1-\lambda_2-\frac{1}{2})}$$

$$\cdot \frac{(\gamma+\lambda_1+\frac{1}{2})(\beta+\gamma+\lambda_1+\frac{1}{2})}{(\alpha+\beta+2\gamma+2\lambda_1+1)(\alpha+\beta+2\gamma+2\lambda_1+2)}$$

$$\tilde{R}_{\lambda_1,\lambda_2/\lambda_1-1,\lambda_2-1}$$

$$= 4\frac{\lambda_2(\beta+\lambda_2)(\gamma+\frac{1}{2})}{(\alpha+\beta+2\lambda_2)(\alpha+\beta+2\lambda_2+1)(\alpha+\beta+2\gamma+2\lambda_1+1)}$$

$$\cdot \frac{(\beta+\gamma+\lambda_1+\frac{1}{2})(\alpha+\beta+2\gamma+\lambda_1+\lambda_2+1)(\alpha+\beta+\lambda_1+\lambda_2+1)}{(\alpha+\beta+2\gamma+2\lambda_1+2)(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{1}{2})(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{3}{2})}$$

One can obtain a similar formula for the operator $B_- = 1 + A_1 + A_2$ induced by the shift $\beta \to \beta+1$, seeing that $\tilde{p}^{\alpha,\beta,\gamma}_{\lambda_1,\lambda_2}(x) = (-1)^{\lambda_1+\lambda_2}\tilde{p}^{\beta,\alpha,\gamma}_{\lambda_1,\lambda_2}(-x)$.

We have the corresponding LR-transforms

$$1 \mp A_1 + A_2 = R \cdot {}^tR,$$
$$A_1' = R^{-1} \cdot A_1 \cdot R$$
$$A_2' = R^{-1} \cdot A_2 \cdot R$$

respectively.

Details of these formulae will be discussed later.

Remark 3 *It seems interesting to generalize our observations to more general case. For example, Koornwinder polynbomials have been generalized to BC-type orthogonal polynomials by Heckman-Opdam. Prof. K.Kadell has discussed the connection relations (8.15)-(8.16) in the case of Selberg-Jack polynomials. One may ask if simple product formulae like in Proposition 8 will be given in these cases.*

References

[1] G.Andrews, R.Askey and R.Roy. *Special Functions,* Cambridge UP. 2000.

[2] K.Aomoto. *Density matrices for Eigenfunction Expansions and LR-Transforms,* preprint, 2000.

[3] K.Aomoto, *A Formula of Eigenfunction Expansions I, Case of Asymptotic Trees; II Exterior Dirichlet Problem in a Lattice,* Proc. Japan Acad., 61, Ser.A, No1(1985) 11-14; 157-160.

[4] K.Aomoto, *Self-adjointness and limit pointness for adjacency operators on a tree,* J. d'Anal. Math., 53(1989), 219-232.

[5] R.Askey and J.A.Wilson *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials,* Mem. Amer. Math. Soc.,1985..

[6] J.M.Berezanski, *Expansions in eigenfunctions of self-adjoint operators,* AMS Transl., 17(1968).

[7] M.Case and M.Kac, *A discrete version of the inverse scattering problem,* J.Math. phys.,14(1973),594-603

[8] J. F.van Diejen. *Properties of Some Families of Hypergeometric Orthogonal Polynomials in Several Variables,* Transact. AMS. , 351(1999), 233-270

[9] S.N.Elaydi, *An Introduction to Difference Equations,* Springer, 1996.

[10] H.Flaschka, *The Toda Lattice II, Inverse scattering solution,* Prog. Theoret. Phys., 51(1974), 703-716.

[11] I.M.Gelfand and B.M.Levitan, *On the determination of a differential equation from its spectral function* AMS Transl.(2),1, 253-304(1955) .

[12] G.J.Heckman and E.M.Opdam . *Root System and Hypergeometric Functions, I* Compo. Math., 64(1987), 329-352.

[13] S.Kakei, *Dressing method and the coupled KP hierachy* preprint, 1999.

[14] Y.Kato and K.Aomoto, *Jacobi-Perron algorithmus, bi-orthogonal polynomials and inverse scattering problems* Publ. R.I.M.S., 20(1984), 635-658.

[15] K.Kadell. *The Selberg-Jack Symmetric Functions,* Ad in Math., 130(1997), 33-102.

[16] K.Kadell. *The Dyson Polynomials,* mimeographed note, 2000 .

[17] J.Kaneko. *Selberg Integrals and Hypergeometric Functions Associated with Jack Polynomials,* SIAM J. Math. Anal., 24(1993), 1086-1110.

[18] T.H.Koornwinder. *Orthogonal Polynomials in Two Variables Which are Eigenfunctions of Two Algebraically Indepependent Partial Differential Operators I,II,* Proc. Kon. Ned. Akad. v.Wet., Amsterdam, Series A, 77(1974), 48-58, 59-66.

[19] J.Moser *Integrable Hamiltonian systems and spectral theory* Pisa, 1981.

[20] P.van Moerbeke and D.Mumford *The spectrum od difference operators and algebraic curves,* Acta Math., 143, 93-154(1979).

[21] A.Mukaihira and Y.Nakamura, *Schur flow for orthogonal polynomials on the unit circle and its integrable discretization,* preprint, 1999.

[22] A.Mukaihira and Y.Nakamura, *Integrable discretization of the modified KdV equations and applications,* Inverse Problems, 16(2000), 413-424.

[23] Y.Nakamura, *A tau-function for the finite Toda molecule, and information spaces,* in Symplectic Problem and Quantization, Contemp. Math., 17(1993), 205-211.

[24] Y.Nakamura and Y.Kodama, *Moment problem of Hamburger, hierachies of integrable systems, and positivity of tau-functions,* Acta Appli. Math., 39(1995), 435-443.

[25] H.Rutishauser, *Der Quotienten-Differenzen-Algorithmus,* Zeit. und angew. Math. Phys.,5(1954),233-251.

[26] H.Rutishauser, *Anwendungen des Quotienten-Differenzen-Algorithmus,* Zeit. und angew. Math. Phys.,5(1954),496-508

[27] H.Rutishauser, *Une methode pour la determination des valeurs propres d'une matrice,* C.R.Acad.Sci.Paris 240(1955), 34-36

[28] Jiro Sekiguchi. *Zonal Spherical Functions on Some Symmetric Spaces* Pub. RIMS, 12 Suppl.,(1977), 455-464.

[29] B. Simon *The classical moment problem as a self-adjoint finite difference operator,* Adv. Math., 137(1998), 82-203.

[30] V.Spiridonov and A.Zhedanov. *Discrete Darboux Transformations, the Discrete Time and Toda Lattice and the Askey-Wilson Polynomials* Methods and Appli. of Anal., 2(4), 1995, 369-398.

[31] G.Szego. *Orthogonal Polynomials,* AMS 1975.

[32] M.Toda, *Non-linear lattice waves,*(*in Japanese*) Iwanami, 1978.

[33] L.Vretare. *Formulas for Elementary Spherical Functions and Generalized Jacobi Polynomials,* SIAM J. Math. Anal., 15(1984) 805-833.

[34] G.Wilson *Infinite-dimensional Lie groups and algebraic geometry in soliton theory,* Phil. Trans. R. Soc. Lond.A315, 393-404(1985)

[35] V.E.Zakhalov and A.B.Shabat, *A scheme for integrating the non-linear equations of mathematical physics by the method of the inverse scattering problem I,* Funct Anal and Its Appli., 8(1974),226-235.