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On the Cauchy problem of the Chern-Simons-Higgs theory

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Abstract

We study the Cauchy problem for the (2+1)-dimensional relativistic abelian Chern-Simons-Higgs model. Given finite energy data, we prove the global existence and uniqueness of the solutions.

1 Introduction

Chern-Simons theories have been proposed in order to explain such physical phenomena as high-temperature superconductivity, quantum hall effect and anyon physics. The first (2+1) dimensional abelian Chern-Simons-Higgs (CSH) model was proposed by Hong, Kim and Pac [9] and Jackiw and Weinberg [10] independently.

The lagrangian density of the CSH model is given by

$$\mathcal{L}(A_\mu, \phi) = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + D_\mu\phi\overline{D^\mu\phi} - V(|\phi|^2), \tag{1.1}$$

where A_μ is a real vector field, ϕ a complex scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$, $\kappa > 0$ a Chern-Simons coupling constant, $\epsilon^{\mu\nu\rho}$ the totally skew-symmetric tensor with $\epsilon^{012} = 1$, $V(|\phi|^2)$ the Higgs potential, and $\overline{f}$ represents the complex conjugate of a complex valued function f. We are working on the Minkowski space-time $\mathbb{R}^{1+2}$ with metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$.

The corresponding Euler-Lagrange equations are

$$\begin{aligned} &F_{\mu\nu} = \frac{1}{\kappa}\epsilon_{\mu\nu\rho}J^\rho, \qquad J^\rho := 2\mathrm{Im}(\overline{\phi}D^\rho\phi), \\ &D_\mu D^\mu\phi = -\phi V'(|\phi|^2), \end{aligned} \tag{1.2}$$

and the energy density corresponding to the lagrangian density (1.1) is given by

$$\mathcal{E}(t,x) = \sum_{\mu=0}^{2} |D_\mu \phi(t,x)|^2 + V(|\phi(t,x)|^2).$$

In the case of the static configuration, Hong, Kim and Pac [9] and Jackiw and Weinberg [10] showed that the CSH model admits first-order self-dual equations if the Higgs potential takes the special form $V(|\phi|^2) = \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2$. There are three possible boundary conditions of the self-dual equations on $\mathbb{R}^2$; the topological boundary condition ($|\phi(x)| \to 1$ as $|x| \to \infty$), the nontopological one ($\phi(x) \to 0$ as $|x| \to \infty$) and the periodic one. There are several results available on the topological multivortex solution ([16], [14]), the nontopological one ([4], [13]) and the periodic ones ([3], [15], [12]). However, it is still open whether the self-dual equations are equivalent to the Euler-Lagrange equations (1.2) in a suitable sense.

In this paper, we study the full evolution problem of (1.2). We decompose A_μ into A_0 and $A = (A_j)_{j=1,2}$. Given a function $f(t,x)$ we denote by ∇f the spatial derivative. We also denote by $\Box$ the D'Alembertian operator $\Box = \partial_t^2 - \Delta = \partial_t^2 - \partial_1^2 - \partial_2^2$.

Given a vector field $\mathbf{V}$, we can decompose $\mathbf{V} = \mathbf{V}^* + \nabla\varphi$ where $\nabla \cdot \mathbf{V}^* = 0$ and φ is a scalar field. We introduce the projection operator $\mathcal{P} : \mathbf{V} \to \mathbf{V}^*$.

Notice that the system of equations (1.2) is invariant under the transform

$$A_\mu \to A_\mu + \partial_\mu \chi, \qquad \phi \to \phi e^{i\chi}.$$

Then we can choose a function χ so that A_μ satisfies a special property. In this paper we construct a solution of (1.2) assuming the Coulomb gauge condition $\nabla \cdot A = 0$. Then A_μ can be determined by the elliptic equations

$$\begin{aligned} \Delta A_0(t,x) &= (1/\kappa)(\partial_1 J_2 - \partial_2 J_1), \\ \Delta A_j(t,x) &= (1/\kappa)\epsilon_{kj}\partial_k J_0, \end{aligned} \tag{1.3}$$

where ϵ_{jk} is the skew-symmetric tensor with $\epsilon_{12} = 1$.

In view of (1.3), we assume A_0 is given by

$$A_0(t,x) = -\frac{1}{\kappa}\int_{\mathbb{R}^2} \mathbf{G}(x-y)\cdot \mathbf{J}(t,y)\,dy, \quad t \geq 0,$$

throughout this paper. Here, $\mathbf{G}(x) = \frac{1}{2\pi|x|^2}(x_2, -x_1)$ and $\mathbf{J} = (J_1, J_2)$.

Then we find that the Euler-Lagrange equations (1.2) consist of constraint equations and evolution equations for A and ϕ. We note that the constraint equations are automatically satisfied for all $t > 0$ whenever they are satisfied at $t = 0$.

Therefore we have derived the following system of evolution equations

$$
\begin{aligned}
&\partial_t A = \frac{1}{\kappa}\mathcal{P}(-J_2, J_1),\\
&\Box\phi = 2iA_0\partial_t\phi + i\phi\partial_t A_0 + A_0^2\phi - 2iA\cdot\nabla\phi - |A|^2\phi - \phi V'(|\phi|^2)
\end{aligned}
\tag{1.4}
$$

subject to the initial condition at $t = 0$

$$
\begin{aligned}
&A_j(0,\cdot) = a_j, \qquad j = 1,2\\
&\phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1
\end{aligned}
\tag{1.5}
$$

with the following constraints

$$
\partial^j a_j = 0, \qquad \partial_1 a_2 - \partial_2 a_1 = \frac{2}{\kappa}\mathrm{Im}\big[\overline{\phi_0}\phi_1 - iA_0(0,\cdot)|\phi_0|^2\big]. \tag{1.6}
$$

It can be verified that the solution of (1.4)-(1.6) also satisfy the Euler-Lagrange equation (1.2). The system (1.4)-(1.6) is the main equation to study in this paper.

We prove the global existence and uniqueness of the solution of the system (1.4)-(1.6) for the finite energy data. For this purpose, we introduce the norm

$$
\mathcal{T}^{(s)}(t) := \mathcal{T}^{(s)}(A,\phi)(t) = \sum_{j=1}^{2}\|A_j(t,\cdot)\|_{H^s} + \|\phi(t,\cdot)\|_{H^{s+1}} + \|\partial_t\phi(t,\cdot)\|_{H^s}, \quad t \geq 0
$$

and we make use of the energy estimates

$$
\begin{aligned}
&\|A(t,\cdot)\|_{H^s} \leq \mathcal{T}^{(s)}(0) + \int_0^t \|\partial_\tau A(\tau,\cdot)\|_{H^s} d\tau,\\
&\|\phi(t,\cdot)\|_{H^{s+1}} + \|\partial_t\phi(t,\cdot)\|_{H^s} \leq C_1\Big(\mathcal{T}^{(s)}(0) + \int_0^t \|\Box\phi(\tau,\cdot)\|_{H^s} d\tau\Big).
\end{aligned}
\tag{1.7}
$$

Our main result is the following.

Theorem 1.1 *Suppose that $V \in C^3(\mathbb{R}_+)$, V''' is locally Lipschitz, $V(0) = 0$, $V(s) \geq -\alpha^2 s$ and $|V''(s)| \leq C(1+s^\beta)$ for some constants $\alpha, C, \beta > 0$. Given data $a_j, \phi_1 \in H^1$ and $\phi_0 \in H^2$ satisfying the constraints (1.6), the system (1.4)-(1.6) has a unique global solution $A_j \in C([0,T]; H^1)$ and $\phi \in C([0,T]; H^2) \cap C^1([0,T]; H^1)$ for any $T > 0$.*

In our case the evolution is governed by the nonlinear wave type of equations. In [1], Bergé et al. proved the finite time blow-up of the solutions of the Cauchy problem from the non-relativistic Chern-Simons-Higgs model where the evolution is governed by the nonlinear Schrödinger type of equations.

We outline the proof of Theorem 1.1 in the following sections. The detailed proof can be found in [5].

2 Local Existence

In this section we prove the local in time existence of the solution of the system (1.4)-(1.6).

Proposition 2.1 *Suppose that $V \in C^3(\mathbb{R}_+)$ and V''' is locally Lipschitz. Given data $a_j, \phi_1 \in H^2$ and $\phi_0 \in H^3$ satisfying (1.6), there exists a $T_0 > 0$ depending only on $\mathcal{T}^{(2)}(0)$ such that the initial value problem (1.4)-(1.6) has a unique solution $A_j \in C([0,T_0]; H^2)$, and $\phi \in C([0,T_0]; H^3) \cap C^1([0,T_0]; H^2)$.*

Since the righthand side of the second equation in (1.4) contains the time derivative of A, we consider the modified problem

$$\begin{aligned} \partial_t A &= \frac{1}{\kappa}\mathcal{P}(-J_2, J_1), \\ \Box\phi &= 2iA_0\partial_t\phi + i\phi h + A_0^2\phi - 2iA\cdot\nabla\phi - |A|^2\phi - \phi V'(|\phi|^2), \end{aligned} \tag{2.1}$$

subject to the initial condition (1.5)-(1.6). Here, h is defined by

$$h(t,x) = -\frac{2}{\kappa}\int_{\mathbb{R}^2} \mathbf{G}(x-y)\cdot \mathbf{W}(t,y)\,dy$$

with the vector field $\mathbf{W} = 2\mathrm{Im}(\partial_t\bar{\phi}\nabla\phi) - \frac{1}{\kappa}|\phi|^2\mathcal{P}(-J_2, J_1) - 2\mathrm{Re}(A\bar{\phi}\partial_t\phi)$. If (A_μ, ϕ) is a solution of the modified system (2.1) then $\mathbf{W} = (1/2)\partial_t\mathbf{J} - \nabla\mathrm{Im}(\bar{\phi}\partial_t\phi)$ and hence $h = \partial_t A_0$. Then we can prove Proposition 2.1 by studying the system (2.1) together with (1.5)-(1.6).

We need the following estimates for A_0 and h and the difference estimates for $A_0 - \tilde{A}_0$ and $h - \tilde{h}$. The next two lemmas can be proved from the Calderon-Zygmund inequality.

Lemma 2.1 *If $A \in C([0,T]; H^2)$ and $\phi \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ for some $T > 0$, then there exists a constant C_p depending only on p such that for $0 \le t \le T$*

(i) $\|\nabla A_0(t,\cdot)\|_{L^p} + \|\nabla h(t,\cdot)\|_{L^p} \le C\big[1 + \mathcal{T}^{(1)}(t)\big]^5$ *for* $1 < p < \infty$,

(ii) $\|\nabla^2 A_0(t,\cdot)\|_{L^p} + \|\nabla^2 h(t,\cdot)\|_{L^p} \le C\big[1 + \mathcal{T}^{(1)}(t)\big]^4 \mathcal{T}^{(2)}(t)$ *for* $2 \le p < \infty$.

Lemma 2.2 *If $A, \tilde{A} \in C([0,T]; H^2)$ and $\phi, \tilde{\phi} \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ for some $T > 0$, then there exists a constant C_p such that for each $0 \le t \le T$*

(i) $\|\nabla(A_0 - \tilde{A}_0)(t,\cdot)\|_{L^p} + \|\nabla(h - \tilde{h})(t,\cdot))\|_{L^p}$
$\le C\big[1 + \mathcal{T}^{(1)}(t) + \tilde{\mathcal{T}}^{(1)}(t)\big]^4 \mathcal{T}^{(1)}(A - \tilde{A}, \phi - \tilde{\phi})(t)$ *for* $1 < p < \infty$,

(ii) $\|\nabla^2(A_0 - \tilde{A}_0)(t,\cdot)\|_{L^p} + \|\nabla^2(h - \tilde{h})(t,\cdot)\|_{L^p}$
$\le C\big[1 + \mathcal{T}^{(2)}(t) + \tilde{\mathcal{T}}^{(2)}(t)\big]^4 \mathcal{T}^{(2)}(A - \tilde{A}, \phi - \tilde{\phi})(t)$ *for* $2 \le p < \infty$,

where $\tilde{\mathcal{T}}^{(s)}(t) = \mathcal{T}^{(s)}(\tilde{A}, \tilde{\phi})(t)$, $s = 1, 2$.

Proof of Proposition 2.1. Let $\Lambda_T = [C(0, T; H^2)]^2 \times [C(0, T; H^3) \cap C^1(0, T; H^2)]$. It follows from the contraction mapping argument that there is a constant $T_0 > 0$ such that the initial value problem (2.1) admits a unique solution $(A, \phi) \in \Lambda_{T_0}$. Indeed, it follows from the energy estimates (1.7) and Lemma 2.1-Lemma 2.2 that if R_0 is sufficiently large and $T = T(R_0)$ is sufficiently small, then the mapping $\mathcal{F}$ from $\mathcal{B}_T = \{(A, \phi) \in \Lambda_T \mid \sup_{0 \le t \le T} \mathcal{T}^{(2)}(t) \le R_0\}$ into itself such that $(A^*, \phi^*) = \mathcal{F}(A, \phi)$ satisfies

$$\begin{aligned} \partial_t A^* &= \frac{1}{\kappa} \mathcal{P}\big(- J_2(A, \phi), J_1(A, \phi)\big), \\ \Box \phi^* &= 2iA_0 \partial_t \phi + i\phi h + A_0^2 \phi - 2iA \cdot \nabla \phi - |A|^2 \phi - \phi V'(|\phi|^2), \end{aligned}$$

subject to the initial condition (1.5)-(1.6), is a well-defined contraction mapping. $\Box$

3 Global Existence and uniqueness

In this section we denote by (A_μ, ϕ) the solution constructed in Proposition 2.1. The next two lemmas show that $\mathcal{T}^{(1)}(t)$ is uniformly bounded on each finite time interval, which in turn implies that (A_μ, ϕ) can be extended past any finite time interval.

Lemma 3.1 *If* $V(s) \ge -\alpha^2 s$ *for some constant* $\alpha > 0$, *then there exists a constant* C *such that*

$$\begin{aligned} &\|\phi(t, \cdot)\|_{H^1} + \|\partial_t \phi(t, \cdot)\|_{L^2} \le C e^{3\alpha t}, \\ &\|\nabla A_\mu(t, \cdot)\|_{L^p} \le C e^{2\alpha t}, \quad \textit{for each } 1 < p < 2. \end{aligned}$$

Lemma 3.2 *Suppose that* $V(0) = 0$, $V(s) \ge -\alpha^2 s$ *and* $|V''(s)| \le C(1 + s^\beta)$ *for some constants* $\alpha, C, \beta > 0$. *The quantity*

$$y(t) = \Big(1 + \sum_{j=1}^{2} \|D_j D_0 \phi(t, \cdot)\|_{L^2}^2 + \sum_{j,k=1}^{2} \|D_j D_k \phi(t, \cdot)\|_{L^2}^2\Big)^{1/2}$$

is uniformly bounded on each finite time interval $[0, T]$.

Since $\Delta \phi = D_1^2 \phi + D_2^2 \phi + 2i \sum_{j=1}^{2} A_j D_j \phi - |A|^2 \phi$, Lemma 3.1 and the following inequality

$$\|\psi\|_{L^p} \le C \|\psi\|_{L^2}^{2/p} \Big(\sum_{j=1}^{2} \|D_j \psi\|_{L^2} \Big)^{(p-2)/p}, \quad 2 < p < \infty \tag{3.1}$$

imply that

$$\|\phi(t, \cdot)\|_{H^2} \le C(y(t) + e^{9\alpha t}). \tag{3.2}$$

Then it follows from (1.7), (3.2), Lemma 3.1-Lemma 3.2 and the identity

$$\partial_j \partial_t \phi = D_j D_0 \phi + i\phi \partial_j A_0 + iA_0 D_j \phi + iA_j D_0 \phi - A_0 A_j \phi$$

that $\mathcal{T}^{(1)}(t)$ is uniformly bounded on each finite time interval $[0, T]$.

We now prove Lemma 3.1 and Lemma 3.2. We note that the mapping $t \mapsto E(t) := \int_{\mathbb{R}^2} \mathcal{E}(t,x)\,dx$ is constant by the law of conservation of the energy.

Proof of Lemma 3.1. Lemma 3.1 follows from (3.1), the following inequalities

$$\partial_t \|\phi\|_{L^2}^2 = 2\mathrm{Re} \int_{\mathbb{R}^2} \phi D_0 \phi \, dx \leq C\|\phi\|_{L^2}(|E(0)|^{1/2} + \alpha \|\phi\|_{L^2}),$$

$$\|\nabla A_\mu\|_{L^p} \leq C \sum_{\nu=0}^{2} \|J_\nu\|_{L^p} \leq \sum_{\nu=0}^{2} C\|D_\nu \phi\|_{L^2} \|\phi\|_{L^{\frac{2p}{2-p}}}, \quad 1 < p < 2,$$

and the identity $\partial_\mu \phi = D_\mu \phi + iA_\mu \phi$. □

Proof of Lemma 3.2. We note that $F_{\mu\nu}$ satisfies the first equation of (1.2). From Lemma 3.1 and (3.1), we obtain

$$\frac{d}{dt}[y(t)]^2 \leq C[y(t)]^2 (1 + \|\phi(t,\cdot)\|_{L^\infty}^2)^{(6\beta+8)\alpha t}.$$

It follows from (3.2) and Lemma 3.3 below that $y'(t) \leq Cy(t)(1 + \ln y(t))e^{(6\beta+15)\alpha t}$, which in turn implies that $y(t)$ is uniformly bounded on each finite time interval. □

Lemma 3.3 *(Brezis-Gallouet, [2]) There exists a constant C such that*

$$\|u\|_{L^\infty} \leq C(1 + \|u\|_{H^1})\sqrt{\ln(1 + \|u\|_{H^2})}$$

for each $u \in H^2(\mathbb{R}^2)$.

We next estimate the differences between two solutions (A_μ, ϕ) and $(\tilde{A}_\mu, \tilde{\phi})$. Let $\mathcal{T}^{(1)} = \mathcal{T}^{(1)}(A, \phi)$ and $\tilde{\mathcal{T}}^{(1)} = \mathcal{T}^{(1)}(\tilde{A}, \tilde{\phi})$.

Proposition 3.1 *Suppose that $V \in C^3(\mathbb{R}_+)$, $V(0) = 0$, $V(s) \geq -\alpha^2 s$ and $|V''(s)| \leq C(1 + s^\beta)$ for some constants $\alpha, C, \beta > 0$. Let (A_μ, ϕ) and $(\tilde{A}_\mu, \tilde{\phi})$ be two solutions of the system (1.4)-(1.6). Then there exist positive increasing functions $f, g : [0,\infty) \to [0,\infty)$ depending only on $\mathcal{T}^{(1)}$ and $\tilde{\mathcal{T}}^{(1)}$ such that*

$$\mathcal{T}^{(1)}(A - \tilde{A}, \phi - \tilde{\phi})(t) \leq f(t) e^{\int_0^t g(s)f(s)ds} \mathcal{T}^{(1)}(A - \tilde{A}, \phi - \tilde{\phi})(0)$$

for all $t > 0$.

Proof. The proof follows from (1.3) and the difference estimates

$$\begin{aligned}\mathcal{T}^{(1)}(A-\tilde{A},\phi-\tilde{\phi})(t) \leq & f(t)\Big(\mathcal{T}^{(1)}(A-\tilde{A},\phi-\tilde{\phi})(0) \\ & + \int_0^t [\|(\partial_s A-\partial_s\tilde{A})(s,\cdot)\|_{H^1} + \|(\Box\phi-\Box\tilde{\phi})(s,\cdot)\|_{H^1}]ds\Big).\end{aligned}$$

A straightforward calculation shows that

$$\|(\partial_t A-\partial_t\tilde{A})(s,\cdot)\|_{H^1} + \|(\Box\phi-\Box\tilde{\phi})(s,\cdot)\|_{H^1} \leq g(s)\mathcal{T}^{(1)}(A-\tilde{A},\phi-\tilde{\phi})(s)$$

for some positive increasing function $g : [0,\infty) \to [0,\infty)$ depending only on $\mathcal{T}^{(1)}$ and $\tilde{\mathcal{T}}^{(1)}$. Then Proposition 3.1 can be easily proved from the Gronwall inequality. $\Box$

Then Theorem 1.1 can be proved from the density argument.

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