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Variational Inequalities and Nonlinear Semi-groups Applied to Certain Nonlinear Problems for the Stokes Equation

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1 Introduction

The purpose of this paper is to present some result of our recent study on the stationary and non-stationary Stokes equations under the nonlinear boundary or interface conditions of friction type.

The method of analysis is based on the theory of variational inequalities, a modern branch of the variational calculus which the late Prof. Tosio Kato liked, as well as on the theory of nonlinear semi-groups to which he contributed much by developing the pioneering work by Y. Kōmura in 1967.

The consequence is the strong solvability (i.e., the unique existence of the L^2-strong solution) of the initial value problem for the Stokes equation under the above-mentioned nonlinear boundary/interface conditions. There are various kinds of boundary/interface conditions, we shall describe our analysis mostly for the case of Leak-BCF and of Leak-ICF which means the boundary condition and the interface condition of friction type respectively. Although we shall formulate specifically Leak-BCF soon, let us say in short with Leak-BCF that this is a boundary condition for the fluid motion such that leak or penetration of the fluid through the boundary can take place when the relevant stress on the boundary reaches a threshold in its magnitude, while no leak occurs as long as the stream is gentle and the stress is small. The other types of boundary conditions of friction type, particularly, Slip-BCF can be dealt with similarly or more simply.

In this paper we shall confine our attention to the theoretical aspects of the study, while introduction of the boundary/interface conditions seems to be effective in modelling and simulating some flow phenomena arising from applications, like flow in a drain with its bottom covered by sherbet of mud and like flow through a tight sieve.

To fix the idea, we describe here our target problem for the case of Leak-BCF in an exterior domain Ω in R^3, with smooth compact boundary Γ, since the case of a bounded flow region is theoretically simpler. The flow velocity and pressure will be denoted by u and p, respectively.

We expect a possible leak through Γ but for simplicity we exclude the possibility of the slip along Γ when we impose Leak-BCF on Γ. Thus our Leak-BCF includes the non-slip condition

$$u_t = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where u_t means the tangential component of u. Incidentally, u_n means the normal component of u on the boundary, i. e., $u_n = u \cdot n$, where n stands for the unit outer normal. The crucial part of our Leak-BCF is the following leak condition which involves a given positive function g on Γ :

$$-\sigma_n \in g\partial|u_n| \qquad \text{on } \Gamma. \tag{1.2}$$

Here $\sigma_n = \sigma_n(u, p)$ is the normal component of the stress on the boundary, and $\partial|\cdot|$ means the sub-differential of the absolute value function of real numbers. Actually, for any $x \in R$, the sub-differential $\partial|x|$ is given explicitly as

$$\partial|x| = \begin{cases} \text{the closed interval } [-1, 1], & (x = 0), \\ 1, & (x > 0), \\ -1, & (x < 0). \end{cases} \tag{1.3}$$

We note $\partial|x|$ is multi-valued at $x = 0$. Also we recall

$$\sigma_n = \sigma(u, p)_n = -p + 2\nu\, n \cdot e(u)n, \tag{1.4}$$

where ν is the viscosity and n the outer unit normal to the boundary, and $e(u)$ means the strain rate tensor $e(u) = (e_{ij}(u))$:

$$e_{ij} = e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right).$$

The given function g is assumed to be continuous for simplicity. It is called the barrier function for the leak, which determined the threshold for the occurrence of the leak. This role of g can be read off when we re-write (1.2) to the following system of conditions:

$$|\sigma_n(u, p)| \leq g \quad \text{on } \Gamma, \tag{1.5}$$

and

$$\begin{cases} |\sigma_n| < g \implies u_n = 0, \\ |\sigma_n| = g \implies \begin{cases} u_n = 0 \text{ or } u_n \neq 0, \\ u_n \neq 0 \implies -\sigma_n = g\dfrac{u_n}{|u_n|}. \end{cases} \end{cases} \tag{1.6}$$

Our target problem is the initial boundary value problem, Leak-IVP, for $\{u, p\}$which consists of the above mentioned Leak-BCF, the initial condition

$$u(0) = u(0, \cdot) = a \qquad \text{in } \Omega \tag{1.7}$$

and the Stokes equation

$$\frac{\partial u}{\partial t} = \nu\Delta u - \nabla p, \quad \operatorname{div} u = 0 \qquad \text{in } [0,\infty)\times\Omega. \tag{1.8}$$

Another target problem, i.e., the initial value problem with the leak interface condition of friction type, Leak-ICF, will be described as we proceed.

Finally, the content of this paper has been mostly adapted from the author's previous presentations ([4, 5, 6]) but it is re-organized in view of his forthcoming paper ([7]).

2 Preliminaries

Here we prepare further symbols, assumptions and (seemingly well-known) facts which we shall make use of later.

2.1 Modification of Leak-IVP

As long as we are concerned only with the solvability of Leak-IVP in the exterior domain Ω, it is theoretically convenient to reduce the Stokes equation to a modified form below by means of the transformation $u = e^t v$ (and then writing u for v), since the equations and boundary conditions are positively homogeneous.

$$\frac{\partial u}{\partial t} + u = \nu\Delta u - \nabla p, \qquad \operatorname{div} u = 0. \tag{2.1}$$

The target problem Leak-IVP with (1.8) replaced by (2.1) will be denoted by m-Leak-IVP, with which we shall deal from now on. The boundary value problem for stationary flows of m-Leak-IVP is the following m-Leak-BVP:

m-Leak-BVP Find $\{u,p\}$which satisfies the modified steady Stokes equation

$$-\nu\Delta u + u + \nabla p = f, \qquad \operatorname{div} u = 0, \tag{2.2}$$

and is subject to Leak-BCF, i.e., (1.1) and (1.2). ∎

In m-Leak-BVP, the external force f is assumed to be in $L^2(\Omega)$. The inner product and norm in $L^2(\Omega)$ will be simply denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. Also symbols for usual Sobolev spaces will be made use of, for instance, $H^1(\Omega)$, $H^{1/2}(\Gamma)$. We shall put

$$a(u,v) = (u,v) + 2\nu\sum_{i,j=1}^{3}\int_\Omega e_{ij}(u)e_{ij}(v)dx \tag{2.3}$$

for any $u, v \in H^1(\Omega)$. The quadratic form $a(\cdot,\cdot)$ is continuous over $H^1(\Omega)$. Moreover, it

Lemma 2.1 *(Korn's inequality)* *There exits positive (domain) constants* c_0, c_1 *such that*

$$c_0\|u\|^2_{H^1(\Omega)} \le a(u,u) \le c_1\|u\|^2_{H^1(\Omega)} \quad (\forall u \in H^1(\Omega)). \tag{2.4}$$

We put

$$\begin{aligned} H^1_0(\Omega) &= \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma\}, \\ H^{1,bs}_0(\Omega) &= \{u \in H^1_0(\Omega);\ \operatorname{supp} u \text{ is bounded.}\}, \\ H^1_\sigma(\Omega) &= \{u \in H^1(\Omega);\ \operatorname{div} u = 0\}, \\ H^1_{0,\sigma}(\Omega) &= \{u \in H^1_0(\Omega);\ \operatorname{div} u = 0\}, \end{aligned} \tag{2.5}$$

where $\operatorname{supp} u$ means the (essential) support of u.

In our variational arguments below we use the following classes of admissible functions:

$$\begin{aligned} K &= \{u \in H^1(\Omega);\ u_t = 0 \text{ on } \Gamma\}, \\ K^{bs} &= \{u \in K;\ \operatorname{supp} u \text{ is bounded.}\}, \\ K_\sigma &= \{u \in H^1_\sigma(\Omega);\ u_t = 0 \text{ on } \Gamma\}, \\ K^{bs}_\sigma &= \{u \in K_\sigma;\ \operatorname{supp} u \text{ is bounded.}\} \end{aligned} \tag{2.6}$$

Furthermore, in dealing with the stress component on Γ, we need

$$\begin{aligned} Y &= \text{the scalar } H^{1/2}(\Gamma), \\ Y_0 &= \{\eta \in Y\,;\ \int_\Gamma \eta d\Gamma = 0\}, \\ Z &= \{\zeta \in \text{ the vector } H^{1/2}(\Gamma)\,; \zeta_t = 0,\ \zeta_n = \eta \quad (\eta \in Y)\,\}, \\ Z_0 &= \{\zeta \in \text{ the vector } H^{1/2}(\Gamma)\,; \zeta_t = 0,\ \zeta_n = \eta \quad (\eta \in Y_0)\,\}. \end{aligned} \tag{2.7}$$

We state here the following facts which are known or can be easily shown:

Lemma 2.2 *Let* $D(\bar\Omega)$ *be the set of smooth vector functions with compact supports in* $\bar\Omega$ *and let* $D_\sigma(\bar\Omega)$ *be the set of smooth solenoidal (i.e., divergence-free) vector functions with compact supports in* $\bar\Omega$. *Then* $D(\bar\Omega)$*is dense in* $H^1(\Omega)$ *and* $D_\sigma(\bar\Omega)$ *is dense in* $H^1_\sigma(\Omega)$. ▮

Lemma 2.3 *If* $\zeta \in Z$, *then it can be extended to a function in* K^{bs}. *And if* $\zeta \in Z_0$, *then it can be extended to a function in* K^{bs}_σ. *Namely,*

$$\begin{aligned} Z &= \{v|_\Gamma\,;\ v \in K\} = \{v|_\Gamma\,;\ v \in K^{bs}\} \\ Z_0 &= \{v|_\Gamma\,;\ v \in K_\sigma\} = \{v|_\Gamma\,;\ v \in K^{bs}_\sigma\}. \end{aligned} \tag{2.8}$$ ▮

2.2 Weak solutions of the Stokes equation

We give here necessary comments concerning the weak formulation of steady Stokes equation, although we state it actually for the modified Stokes equation (2.2).

Definition 2.1 *$u \in H^1_\sigma(\Omega)$ is a weak solution of (2.2) for given $f \in L^2(\Omega)$ if the following identity holds true:*

(2.9) $$a(u, \varphi) = (f, \varphi) \quad (\forall \varphi \in H^1_{0,\sigma}(\Omega)).$$

The following lemma is known:

Lemma 2.4 *Let u be a weak solution of (2.2). Then there exists a scalar function $p \in L^2_{loc}(\Omega)$ such that*

(2.10) $$a(u, \varphi) - (p, \operatorname{div} \varphi) = (f, \varphi) \quad (\forall \varphi \in H_0^{1,bs}(\Omega)).$$

p is uniquely determined except for an arbitrary additive constant for each u, and is called the pressure associated with u. ∎

Definition 2.2 *The couple $\{u, p\}$, where u is a weak solution of (2.2) and p is its associate pressure, is again called a weak solution of (2.2). In this sense, the identity (2.10) is the defining condition for $u \in H^1_\sigma$, $p \in L^2_{loc}(\Omega)$ to be the weak solution of (2.2).* ∎

2.3 Stress components of weak solutions

When $\{u, p\}$is a weak solution, its stress component $\sigma_n = \sigma_n(u, p)$ can be defined by virtue of the (modified) weak Stokes equation (2.10) as an element in $H^{-1/2}(\Gamma)$, although the trace of $e(u)$ or of p onto Γ cannot be defined in general. To this end , we firstly note that if $\{u, p\}$were a smooth classical solution, then the following identity should hold true:

(2.11) $$\int_\Gamma \sigma_n \cdot \varphi_n \, d\Gamma = a(u, \varphi) - (p, \operatorname{div} \varphi) - (f, \varphi) \qquad (\forall \varphi \in K^{bs}).$$

Now, suppose that $\{u, p\}$is a weak solution of (2.2), and take $\eta \in Y$. Then let $\zeta \in Z$ be a vector function defined on Γ as in (2.7): $\zeta_t = 0, \zeta_n = \eta$. Furthermore, by $\varphi_\eta \in K^{bs}$ be any extension of ζ over to Ω such that $\varphi_\eta|_\Gamma = \zeta$ and $\varphi_\eta \in K^{bs}$. Then we define a linear Functional $\Sigma_n[\cdot]$ on Y by setting as

(2.12) $$\Sigma_n[\eta] = a(u, \varphi_\eta) - (p, \operatorname{div} \varphi_\eta) - (f, \varphi_\eta).$$

$\Sigma_n[\eta]$ is well-defined, since the right-hand side above does not depend on the way of extension from $\eta \in Y$ to $\varphi_\eta \in K^{bs}$, as is verified by means of (2.10). Also, the value of the right-hand side of (2.12) is seen to depend continuously on η in the $H^{1/2}(\Gamma)-$topology.

Thus $\Sigma_n \in H^{-1/2}(\Gamma)$. Noting that in the smooth case, Σ_n is represented by the function σ_n as the left-hand side of (2.11), we write in place of $\Sigma_n[\varphi_n|_\Gamma]$

$$\int_\Gamma \sigma_n \cdot \varphi_n d\Gamma$$

when this can be understood. In this sense, for any weak solution $\{u, p\}$we can write (2.11) for all $\varphi \in K^{bc}$.

Finally, if η is in Y_0 and if φ_η is an extension of ζ with $\zeta_t = 0, \zeta_n = \eta$ over to Ω such that $\varphi_\eta \in K_\sigma$, then we have

$$\int_\Gamma \sigma_n \cdot \eta d\Gamma = a(u, \varphi_\eta) - (f, \varphi_\eta), \qquad (\eta \in Y_0). \tag{2.13}$$

3 Variational Inequalities for m-Leak-BVP

In order to analyze m-Leak-BVP, we introduce following variational inequalities, m-Leak-VI:

m-Leak-VI Find $u \in K_\sigma$ and $p \in L^2_{loc}(\Omega)$ such that

$$a(u, v-u) - (p, \operatorname{div}(v-u) + j(v) - j(u) \geq (f, v-u) \quad (\forall v \in K^{bs}), \tag{3.1}$$

where

$$j(v) = \int_\Gamma g|v_n|d\Gamma \qquad (\forall v \in K). \qquad \blacksquare \tag{3.2}$$

If $\{u, p\}$ is a solution of m-Leak-VI, then we have

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad (\forall v \in K_\sigma). \tag{3.3}$$

This can be verified by means of Lemma 2.2. Furthermore, if $\{u, p\}$is a solution of m-Leak-VI, then the couple is a weak solution of (2.2). To see this, we take an arbitrary $\varphi \in H_0^{1,bs}(\Omega)$ and put $v = u \pm \varphi$. Again by virtue of Lemma 2.2, we see that this v can be substituted into (3.1), which yields

$$\pm a(u, \varphi) \mp (p, \operatorname{div} \varphi) \geq \pm(f, \varphi) \quad (\forall \varphi \in H_0^{1,bs}(\Omega)),$$

which is nothing but (2.10). Consequently, we can re-write (3.1) by means of (2.11) as

$$\int_\Gamma \sigma_n \cdot (v-u)_n d\Gamma + j(v) - j(u) \geq 0 \quad (\forall v \in K^{bs} \text{ and equivalently } \forall v \in K). \tag{3.4}$$

At this point, let us confirm the definition of weak solution of m-Leak-BVP.

Definition 3.1 $\{u, p\}$*is a weak solution of m-Leak-BVP if the following conditions are all satisfied;*

(i) $u \in K_\sigma$ *and* $p \in L^2_{loc}(\Omega)$.

(ii) $\{u,p\}$*is a weak solution of (2.2).*

(iii) *The non-slip boundary condition (1.1) is satisfied in the trace sense, and the leak condition (1.2) holds true almost everywhere on* Γ.

By **m-Leak-WBVP**, *we denote the problem to seek a weak solution* $\{u,p\}$*of m-Leak-BVP for given* f. ❙

We note that the last condition in (iii) above requests particularly that σ_n which is originally in $H^{-1/2}(\Gamma)$ turns out to be a bounded function subject to (1.5) almost everywhere on Γ.

3.1 Theorems for m-Leak-VI

We claim

Theorem 3.1 *m-Leak-VI and m-Leak-WBVP are equivalent.* ❙

Before proving the theorem, we prepare

Lemma 3.1 *The leak condition (1.2) is equivalent to the following set of conditions*

$$|\sigma_n| \leq g, \quad \sigma_n \cdot u_n + g|u_n| = 0 \quad \textit{on } \Gamma. \tag{3.5}$$ ❙

Proof of the Lemma.

In fact, (3.5) follows immediately from (1.5) and (1.6). Conversely, by means of (3.5) we have for any real number x

$$\begin{aligned} g|x| \; &- \; g|u_n| + \sigma_n \cdot (x - u_n) \\ &= \; g|x| + \sigma_n \cdot x - (g|u_n| + \sigma_n \cdot u_n) \\ &= \; g|x| + \sigma_n \cdot x \geq 0, \end{aligned} \tag{3.6}$$

which implies (1.2) in virtue of the definition of the sub-differential. Q.E.D.

Proof of Theorem 3.1

Suppose that $\{u,p\}$is a weak solution of m-Leak-BVP. We have only to prove the inequality (3.4). From (1.2) we have

$$g|v_n| - g|u_n| \geq -\sigma_n \cdot (v-u)_n \text{ a.e. on } \Gamma \qquad (\forall v \in K^{bs}).$$

Integrating the inequality above, we get to

$$j(v) - j(u) \geq -\int_\Gamma \sigma_n \cdot (v-u)_n d\Gamma,$$

which is nothing but (3.4). Thus $\{u,p\}$solves m-Leak-VI.

Conversely, let us suppose that $\{u,p\}$is a solution of m-Leak-VI. Already we have seen that $\{u,p\}$is a weak solution of (2.2). It remains to prove the leak condition (1.2). From (3.4), we have

$$(3.7)\qquad -\int_\Gamma \sigma_n\cdot(v-u)_n d\Gamma \le j(v)-j(u) \le \int_\Gamma g|(v-u)_n|d\Gamma \quad (\forall v\in K^{bs}).$$

Namely, we have

$$(3.8)\qquad -\int_\Gamma \sigma_n\cdot\eta d\Gamma \le \int_\Gamma g|\eta|d\Gamma \qquad (\forall\eta\in Y).$$

This inequality hold true if we replace η by $-\eta$. Hence we have

$$(3.9)\qquad \left|\int_\Gamma \sigma_n\cdot\eta d\Gamma\right| \le \int_\Gamma g|\eta|d\Gamma \qquad (\forall\eta\in Y).$$

Here we make a duality argument. Actually, let us consider the Banach space M of L^1- type over Γ with the weighted measure $gd\Gamma$, i.e., with the norm

$$(3.10)\qquad \|\eta\|_M = \int_\Gamma g|\eta|d\Gamma.$$

(3.9) means that σ_n defines a linear functional on $Y\subset M$ with its functional norm bounded by 1. Since Y is dense in M, σ_n can be viewed as an element in the dual space M^* of M. As a matter of fact, M^* is an $L^\infty-$type space with its norm defined by

$$(3.11)\qquad \|\eta\|_{M^*} = \text{ ess. sup}_{s\in\Gamma}\frac{|\eta(s)|}{g(s)}.$$

Therefore, σ_n turns out to be a bounded function on Γ subject to (1.5). We are now going to show the second equality in (3.5). Coming back to (3.7), we put $v=0$ there, obtaining

$$-\int_\Gamma \sigma_n\cdot u_n d\Gamma - \int_\Gamma g|u_n|d\Gamma \ge 0,$$

which leads to

$$\int_\Gamma (\sigma_n\cdot u_n + g|u_n|)d\Gamma = 0,$$

with the aid of (1.5), and leads furthermore to the second equality of (3.5) in the a.e. sense on Γ. Thus we have shown that $\{u,p\}$is a solution of m-Leak-WBVP, which completes the proof of Theorem 3.1. Q.E.D.

We proceed to one of our main theorems, by claiming

Theorem 3.2 *m-Leak-VI has a solution* $\{u,p\}$, *of which* u *is unique but* p *is unique except for an additive constant. The range of the additive constant to* p *is limited to* $\{0\}$ *or to a finite closed interval. So does m-Leak-WBVP.*

Proof of Theorem 3.2

Uniqueness Argument. Let $\{u_i,p_i\}$ be solutions of m-Leak-VI ($i=1,2$). Then by (3.3) we have

$$\begin{aligned} a(u_1,u_2-u_1) &+ j(u_2)-j(u_1) \ge (f,u_2-u_1),\\ a(u_2,u_1-u_2) &+ j(u_1)-j(u_2) \ge (f,u_1-u_2), \end{aligned}$$

since $\operatorname{div} u_1 = 0, \operatorname{div} u_2 = 0$. Adding these two inequalities, we have $a(u_2 - u_1, u_2 - u_1) \leq 0$, which gives $u_2 - u_1 = 0$ by Lemma 2.1 (Korn's inequality). After obtaining the uniqueness of u, it is easy to see the uniqueness of p in $L^2_{loc}(\Omega)/R$. Then the range of the additive constant can be examined through (1.2).

Existence Proof. We have to start from the following variational inequalities with in solenoidal functions.

m-Leak-VI$_\sigma$

Find $u \in K_\sigma$ such that

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad (\forall v \in K_\sigma). \tag{3.12}$$

The existence of the solution u of m-Leak-VI$_\sigma$ can be shown by a standard argument in the theory of variational inequalities. Then in the same way as before, we can verify that u is a weak solution of (2.2) and see that there exists an associated pressure p. We fix this p. $\{u, p\}$may not satisfy (1.2) but we can use (2.11) for $\sigma_n(u, p)$. If $v \in K_\sigma$, then we have by (2.11) and (3.12)

$$\begin{aligned} \int_\Gamma \sigma_n \cdot (v-u)_n d\Gamma &= a(u, v-u) - (p, \operatorname{div}(v-u)) - (f, v-u) \\ &= a(u, v-u) - (f, v-u) \\ &\geq -j(v) + j(u). \end{aligned} \tag{3.13}$$

Hence we have

$$\int_\Gamma \sigma_n \cdot (v-u)_n d\Gamma + j(v) - j(u) \geq 0 \quad (\forall v \in K_\sigma). \tag{3.14}$$

Partly repeating the argument in the proof of the preceding theorem, we deduce

$$\left| \int_\Gamma \sigma_n \cdot \eta d\Gamma \right| \leq \int_\Gamma g|\eta| d\Gamma \quad (\forall \eta \in Y_0), \tag{3.15}$$

in consideration that $(v-u)_n$ ranges over Y_0 on Γ as v ranges over K_σ Here, we have to note that Y_0 is not dense in the L^1-type Banach space M introduced in the proof of the previous theorem. We can, however, regard σ_n as a linear functional defined on the subspace Y_0 of M, and its functional norm is bounded by 1. At this point, we apply the Hahn-Banach theorem and see that there exist an element λ^* of the dual space M^* such that

$$\langle \lambda^*, \eta \rangle = \langle \sigma_n, \eta \rangle \qquad (\forall \eta \in Y_0), \tag{3.16}$$

and

$$\|\lambda^*\|_{M^*} \leq 1. \tag{3.17}$$

From (3.17), we see that λ^* is a bounded function on Γ and is subject to

$$|\lambda^*| \leq g \qquad \text{a.e. on } \Gamma. \tag{3.18}$$

On the other hand, (3.16) implies

$$\lambda - \sigma_n = -k^* \tag{3.19}$$

for some constant k^*. Let us put $p^* = p + k^*$. Then we have

$$\lambda^* = \sigma_n(u,p) - k^* = \sigma_n(u,p^*),$$

and also in view of (3.18)

$$|\sigma_n(u,p^*)| \le g \quad \text{a.e. on } \Gamma. \tag{3.20}$$

Furthermore, we can write (3.14) for $\{u,p^*\}$ as

$$\int_\Gamma \sigma_n^* \cdot (v-u)_n d\Gamma + j(v) - j(u) \ge 0 \quad (\forall v \in K_\sigma). \tag{3.21}$$

From (3.20) and (3.21) with $v = 0$, we can deduce for $\sigma_n^* = \sigma_n(u,p^*)$

$$\sigma_n^* \cdot u_n + g|u_n| = 0,$$

in a parallel way as in the proof of preceding theorem. Thus we have shown that $\{u,p^*\}$ satisfies (3.5) and is a solution of m-Leak-VI and so of m-Leak-WBVP. Q.E.D.

4 Leak-IVP

We study the solvability of Leak-IVP through that of m-Leak-IVP. In doing so we shall rely on the generation theorem in the nonlinear semigroup theory. In short, this theorem tells us that the initial value problem is nicely solvable (in an abstract sense to be specified below), if it is generated by the minus of a maximal monotone (m-monotone) operator A in a Hilbert space X. Here we should note that A is possibly multi-valued.

4.1 Monotone operators

Let us recall some fundamental concepts for our later use.

Definition 4.1 *A multi-valued operator A in Hilbert space X is monotone (or accretive) if*

$$(f_1 - f_2, u_1 - u_2) \ge 0 \quad (\forall u_1, u_2 \in D(A), \quad \forall f_1 \in Au_1, \forall f_2 \in Au_2), \tag{4.1}$$

where $D(A)$ is the domain of definition of A. ∎

The following definition is concerned with the maximality of monotone property.

Definition 4.2 *A monotone operator A is a maximal monotone (or m-accretive) operator, if*

$$R(I+A) \equiv \text{Range of } (I+A) = X. \tag{4.2}$$ ∎

As for a monotone operator, the condition (4.2) is equivalent to

(4.3) $$R(I+\lambda A)=X,$$

for all $\lambda > 0$ or for some λ. If A is a maximal monotone operator, then the subset Au is a non-empty closed convex set in X for each $u \in D(A)$, which enables us to make the following definition.

Definition 4.3 *Let A be a maximal monotone operator. Then its canonical restriction A^0 is defined by assigning as A^0u the element with the smallest norm in Au.* ▮

Sometimes, one prefers the following terminology:

Definition 4.4 *An operator B in X is dissipative if $-B$ is monotone, and is maximal dissipative if $-B$ is maximal monotone.* ▮

We shall make use of the following well-known facts concerning an evolution equation (evolution condition) with a maximal dissipative operator as its generator.
abst-IVP (abstract IVP):

Let A be a maximal monotone operator and let a be an element in X. The abst-IVP is to find $u = u(t)$ which is an $X-$valued absolutely continuous function on $[0, +\infty)$ such that the evolution condition

(4.4) $$\frac{du}{dt} \in -Au(t) \quad (\text{ a.e.}t),$$

and the initial condition

(4.5) $$u(0) = a$$

hold true. ▮

Then the following theorem is known:

Theorem 4.1 *The abst-IVP is uniquely solvable if $a \in D(A)$. Moreover, the solution $u(t) \in D(A)$ for every t, and it satisfies*

(4.6) $$\frac{d^+u}{dt} = -A^0u(t) \quad (\forall t \in [0, +\infty)).$$

4.2 Stokes operator under Leak-BCF

Having m-Leak-IVP in our mind, we define the modified Stokes operator with the boundary condition Leak-BCF (which corresponds to "the Stokes operator + I ") as follows. The basic Hilbert space X is $L^2(\Omega)$. Then the modified Stokes operator A is defined as

Definition 4.5 *The domain of definition $D(A)$ of the modified Stokes operator A is given by*

$$D(A) = \{u \in K_\sigma\,;\exists p, \exists f \text{ such that } u \text{ is a solution of m-Leak-VI}\}, \tag{4.7}$$

and for each $u \in D(A)$ we define the set Au by

(4.8) $f \in Au \Leftrightarrow u$ *is the solution of m-Leak-VI for some p and for the very f.* ∎

Then A is easily verified to be monotone. In fact, let $\{u_i, p_i\}$ be the solution of m-Leak-VI for f_i, $(i = 1, 2)$. Then we have

$$\begin{aligned} a(u_1, u_2 - u_1) &+ j(u_2) - j(u_1) \geq (f_1, u_2 - u_1), \\ a(u_2, u_1 - u_2) &+ j(u_1) - j(u_2) \geq (f_2, u_1 - u_2), \end{aligned}$$

since $\operatorname{div} u_1 = 0, \operatorname{div} u_2 = 0$. Adding these two inequalities, we have $a(u_2 - u_1, u_2 - u_1) \leq (f_1 - f_2, u_2 - u_1)$, which gives (4.1) by virtue of the non-negative property of $a(u, u)$.

Moreover, A is maximal monotone. This can be confirmed easily by repeating the relevant argument in the preceding section or by making use of a known theorem (e.g., Brezis [1]) which can be applied when Range of A is the whole space and $a(u, u) \geq c_0 \|u\|^2$ holds true with some positive domain constant c_0.

Thus we have

Theorem 4.2 *The modified Stokes operator A with Leak-BCF is a maximal monotone operator.* ∎

Consequently, the generation theorem in the nonlinear semigroup theory can be applied to yield the desired solvability of m-Leak-IVP and so that of Leak-IVP.

Theorem 4.3 *If $a \in D(A)$, then m-Leak-IVP is solvable uniquely and strongly in the sense stated in Theorem 4.1.* ∎

Remark 1 By making use of those theorems in the NSG theory which are concerned with generators of the sub-differential type, we can relax the condition on the initial value above so that $a \in K_\sigma$ is sufficient instead of the condition $a \in D(A)$. (see, Brezis [1], Fujita [7]). ∎

Remark 2 The equation (4.6) implies that with some pressure p

$$\frac{d^+u}{dt} + u = \nu\Delta u - \nabla p \text{ in } \Omega \tag{4.9}$$

holds true for every t. At this stage, however, we know only that the distribution $\nu\Delta u - \nabla p$ turns out to be in $L^2(\Omega)$. In order to obtain more regularity like Δu, $\nabla p \in L^2(\Omega)$, we would need a little more smoothness assumption on g, and also the regularity theorem due to N. Saito [17].

5 Leak Interface Conditions

In this section we sketch our result on the Stokes flow under an interface condition of friction type for the case of a bounded flow region Ω. The methods of analysis are quite parallel to those for the previous target problems.

5.1 Target problems with Leak-ICF

As to the geometry, however, we assume that our entire (spatial) flow region, where the velocity u and pressure p are considered, is a bounded domain Ω in R^3 with its smooth boundary Γ. Moreover, we assume that Ω is divided transversally into two sub-domains Ω_i, $(i = 1,2)$ by an interface S. In each sub-domain, Ω_i, $\{u,p\}$ is assumed to satisfy the Stokes equation. We confine our attention to the interface condition to be imposed on S, while we impose the Dirichlet boundary condition on Γ, i.e.,

(5.1) $$u = 0 \quad \text{on } \Gamma,$$

for the sake of simplicity. Before describing our leak interface condition, Leak-ICF, let us specify our notation a little more.

When $h = h(x)$ is a vector function or a scalar function defined on Ω, its restriction on $\Omega_i, (i = 1,2)$ will be denoted by h^i.

By Leak-IFC we mean the following set of conditions: firstly, we require the non-slip property:

(5.2) $$u_t^1 = u_t^2 = 0 \quad \text{on } S,$$

secondly, the continuity of normal component of velocity is assumed, i.e.,

(5.3) $$u_l^1 = u_l^2 \quad \text{on } S.$$

Here l is the unit normal to S directed from Ω_1 to Ω_2, and u_l^1, u_l^2 are the components of u^1, u^2 along l. Recalling that we generally denote by n the outer unit normal to the boundary of the domain of our concern, we note

$$u_l^1 = u_n^1, \qquad u_l^2 = -u_n^2.$$

Thirdly, as the crucial part of Leak-ICF, we impose the following leak condition which again involves a given positive continuous function g on S and the notation of sub-differential:

(5.4) $$-\delta \equiv -\delta(u,p) \in \partial g|u_l| \quad \text{on } S.$$

Here δ is the difference of the 'normal' stresses on the both sides of S and is expressed as

(5.5) $$\delta = \sigma_l(u^1,p^1) - \sigma_l(u^2,p^2).$$

In fact, the l–component of stress is expressed as

(5.6) $$\sigma_l = -p\,l \cdot n + l \cdot e(u)n.$$

The condition (5.4) can be re-written as the previous case of (1.2). For instance, it is equivalent to

(5.7) $$\begin{cases} |\delta| & \leq \; g, \\ \delta \cdot u_l + g|u_l| & = \; 0. \end{cases}$$

Our target problem for the steady flow is now stated:

Leak-ICF-BVP

For given f, find $\{u, p\}$which satisfy the steady Stokes equation in Ω_1 and Ω_2 together with the Dirichlet boundary condition on Γ and Leak-ICF on S. ▌

In dealing with the initial value problem for non-stationary flows, we again assume the absence of the external force:

Leak-ICF-IVP

For given initial value a, find $\{u, p\}$which satisfy the (non-stationary) Stokes equation in Ω_1 and Ω_2 together with the Dirichlet boundary condition on Γ, Leak-ICF on S and the initial condition. ▌

5.2 Analysis by Variational Inequalities

The method of analysis is in parallel to the previous one, being based on variational inequalities. This time, however, we can simply put

(5.8) $$a(u, v) = 2\nu \sum_{i,j=1}^{3} \int_\Omega e_{ij}(u)e_{ij}(v)dx,$$

keeping the validity of Korn's inequality, in virtue of the Dirichlet boundary condition (e.g., see Ciarlet[3], Horgan[11]).

The classes of admissible functions are now defined as

(5.9) $$\begin{aligned} K &= \{u \in H_0^1(\Omega); u_t = 0 \text{ on } S, \}, \\ K_\sigma &= \{u \in K\,; \operatorname{div} u = 0 \text{ in } \Omega\}. \end{aligned}$$

Also the definition of the barrier functional j is renewed.

(5.10) $$j(v) = \int_S g|v_l|dS \qquad (\forall v \in K).$$

We state the formulation of Leak-ICF-BVP in variational inequalities.

Leak-ICF-VI Find $u \in K_\sigma$ and $p \in L^2(\Omega)$ such that

(5.11) $$a(u, v-u) - (p, \operatorname{div}(v-u) + j(v) - j(u) \geq (f, v-u) \quad (\forall v \in K).$$ ▌

We skip an explicit definition, but the weak formulation Leak-ICF-WBVP of Leak-ICF-BVP could be understood. As before we can show the following theorems:

Theorem 5.1 *Leak-ICF-VI and Leak-ICF-WBVP are equivalent.* ▮

Theorem 5.2 *Leak-ICF-VI has a solution* $\{u, p\}$*and so does Leak-ICF-WBVP. The velocity part* u *of the solution is unique. The pressure part* p *of the solution is unique except for an additive step function* $k\chi_1 + (k+c)\chi_2$*, where* χ_i $(i = 1, 2)$ *is the characteristic function of* Ω_i*, and where the value of the constant* k *is arbitrary, but the range of the constant* c *is limited to* $\{0\}$ *or to a finite closed interval.* ▮

5.3 Leak-ICF-IVP

The L^2−strong solvability of Leak-ICF-IVP similar to the previous case in §4 is again an immediate outcome of the NSG theory, when we define the Stokes operator A under Leak-ICF properly so that A is an maximal monotone operator in $X = L^2(\Omega)$. This is achieved by setting

(5.12) $$D(A) = \{u \in K_\sigma; \exists p, \exists f,\ u \text{ is a solution of Leak-ICF-VI}\},$$

and

(5.13) $f \in Au \Leftrightarrow u$ is the solution of Leak-ICF-VI for some p and for the very f.

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