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On the classification of R-operators

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Abstract

We generalized and classified the R-operators which satisfy the quantum Yang-Baxter equation on a function space. To classify the R-operators, we gave the complete classification of the meromorphic solutions of the functional equations which are necessary and sufficient conditions for the R-operator to satisfy the quantum Yang-Baxter equation. Most of the solutions were expressed in terms of the elliptic, trigonometric and rational functions.

1 Introduction

Much attention has been directed to the R-operators, the solutions of the (quantum) Yang-Baxter equation on a function space.

Definition 1.1 (R-operator). For $x_1, x_2, \ldots, x_n \in \mathbb{C}$ and $r > 0$, define the sets $C(x_1, r)(\subseteq \mathbb{C})$ and $C((x_1, x_2, \ldots, x_n), r)(\subseteq \mathbb{C}^n)$ by

$$C(x_1, r) = \{x \in \mathbb{C} \,;\, |x - x_1| < r\},$$

$$C((x_1, x_2, \ldots, x_n), r) = C(x_1, r) \times C(x_2, r) \times \cdots \times C(x_n, r).$$

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Let functions A and B be meromorphic on $C((0,0),r)$. For a function f meromorphic on $C((0,0),r/2)$, we define the function $(R(u)f)(z_1,z_2)$ meromorphic on $C(0,r) \times C((0,0),r/2)(\ni (u,z_1,z_2))$ as

$$(R(u)f)(z_1,z_2) = A(u,z_1-z_2)f(z_1,z_2) - B(u,z_1-z_2)f(z_2,z_1). \qquad (1)$$

We call this operator $R(u)$ the R-operator.

After the discovery of the elliptic, trigonometric and rational R-operators [10], [11], [12], the various properties which the elliptic R-operator satisfies have been investigated. We obtained Belavin's R-matrix [1] from the elliptic R-operator by restricting the domain of the modified version of the elliptic R-operator to some finite-dimensional subspace [2]. The elliptic R-operator had the vertex-IRF (Interaction–Round–a–Face) correspondence [8] as Belavin's R-matrix, and we constructed the generalized Ruijsenaars operators, the commuting difference operators [7], from the elliptic R-operator and related boundary K-operators [6].

Komori [5] and the author [9] have given classifications of the R-operators. In [9], the function $A(u,x)$ in the definition of the R-operators (1) did not, however, depend on the spectral parameter u.

The aim of this article is to classify the R-operators (1) satisfying the Yang-Baxter equation below, which is a generalization of [9].

Proposition 1.1. *For any function f meromorphic on $C((0,0,0),r/2)$, a necessary and sufficient condition for the functions $R_{12}(u)R_{13}(u+v)R_{23}(v)f$ and $R_{23}(v)R_{13}(u+$*

$v)R_{12}(u)f$ meromorphic on $C((0,0,0,0,0),r/2)$ to satisfy the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v)f = R_{23}(v)R_{13}(u+v)R_{12}(u)f \tag{2}$$

is that the meromorphic functions A and B satisfy the following functional equations:

$$\begin{aligned} & B(v, z_2 - z_3)A(u, z_1 - z_2)A(u+v, z_1 - z_3) \\ =& B(v, z_2 - z_3)A(u+v, z_1 - z_2)A(u, z_1 - z_3), \end{aligned} \tag{3}$$

$$\begin{aligned} & B(u, z_1 - z_2)A(u+v, z_2 - z_3)B(v, z_1 - z_3) \\ =& A(v, z_2 - z_3)B(u+v, z_1 - z_3)B(u, z_3 - z_2) \\ +& B(v, z_2 - z_3)B(u+v, z_1 - z_2)A(u, z_2 - z_3), \end{aligned} \tag{4}$$

$$\begin{aligned} & A(u, z_1 - z_2)B(u+v, z_1 - z_3)A(v, z_2 - z_1) \\ +& B(u, z_1 - z_2)B(u+v, z_2 - z_3)B(v, z_1 - z_2) \\ =& A(v, z_2 - z_3)B(u+v, z_1 - z_3)A(u, z_3 - z_2) \\ +& B(v, z_2 - z_3)B(u+v, z_1 - z_2)B(u, z_2 - z_3) \end{aligned} \tag{5}$$

on $C((0,0,0,0,0),r/2)$. Here the operators $R_{12}(u)$, $R_{13}(u)$ and $R_{23}(u)$ are defined as follows:

$$(R_{12}(u)f)(z_1, z_2, z_3) = A(u, z_1 - z_2)f(z_1, z_2, z_3) - B(u, z_1 - z_2)f(z_2, z_1, z_3),$$

$$(R_{13}(u)f)(z_1, z_2, z_3) = A(u, z_1 - z_3)f(z_1, z_2, z_3) - B(u, z_1 - z_3)f(z_3, z_2, z_1),$$

$$(R_{23}(u)f)(z_1, z_2, z_3) = A(u, z_2 - z_3)f(z_1, z_2, z_3) - B(u, z_2 - z_3)f(z_1, z_3, z_2).$$

In order to classify the R-operator satisfying the Yang-Baxter equation (2), it suffices to give the complete classification of the meromorphic solutions A and B of the functional equations (3), (4) and (5) because of Proposition 1.1.

Theorem 1.2. *The meromorphic solutions A and B of the equations* (3), (4) *and* (5) *defined on $C((0,0),r)$ are one of the following*:

0. *trivial case* A *is arbitrary*, $B \equiv 0$.

$A \equiv 0$, $B(u,x) = \exp(F(x)u)G(u)$ *on* $C(0,r) \times C(0,r_1)$.

$(0 < r_1 \leq r)$

1. *generic case* $A(u,x) = b(u)\widetilde{A}(x)$, $B(u,x) = b(u)\widetilde{B}(u,x)$.

1 − 1. *elliptic*

$$\widetilde{A}(x) = h(x)\frac{\sigma(x+s;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(s;\tau_1,\tau_2)},$$
$$\widetilde{B}(u,x) = \exp(\rho ux)\frac{\sigma(x+au;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(au;\tau_1,\tau_2)}.$$

$(a,\tau_1,\tau_2 \in \mathbb{C}\setminus\{0\}, \mathrm{Im}\tau_2/\tau_1 > 0, s \in \mathbb{C}\setminus(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2), \rho \in \mathbb{C})$

1 − 2. *trigonometric*

$$\widetilde{A}(x) = \begin{cases} h(x)\frac{\sinh\frac{x+s}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{s}{\lambda}}, \\ h(x)\frac{1}{\sinh\frac{x}{\lambda}}, \end{cases} \quad \widetilde{B}(u,x) = \begin{cases} \exp(\rho ux)\frac{\sinh\frac{x+au}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{au}{\lambda}}, \\ \exp(\rho ux)\frac{\exp(\pm\frac{x}{\lambda})}{\sinh\frac{x}{\lambda}}. \end{cases}$$

$(a,\lambda \in \mathbb{C}\setminus\{0\}, s \in \mathbb{C}\setminus\mathbb{Z}\pi\sqrt{-1}\lambda, \rho \in \mathbb{C})$

1 − 3. *rational*

$$\widetilde{A}(x) = \begin{cases} h(x)\frac{x+s}{xs}, \\ h(x)\frac{1}{x}, \end{cases} \quad \widetilde{B}(u,x) = \begin{cases} \exp(\rho ux)\frac{x+au}{axu}, \\ \exp(\rho ux)\frac{1}{x}. \end{cases}$$

$(a,s \in \mathbb{C}\setminus\{0\}, \rho \in \mathbb{C})$

2. *singular case* $A(u,x) = c_1 b(u)h(x),\ B(u,x) = c_2 b(u)\exp(\rho ux)\frac{1}{u}.$

$(\rho \in \mathbb{C}, c_1, c_2 \in \mathbb{C} \setminus \{0\})$

Here the function F is holomorphic on $C(0,r_1)$, the functions $G(\not\equiv 0)$, $b(\not\equiv 0)$ and $\widetilde{A}$ are meromorphic on $C(0,r)$, the function h is meromorphic on $C(0,r)$ satisfying the relation $h(x)h(-x) = 1$ and the function $\sigma(x) = \sigma(x;\tau_1,\tau_2)$ is the Weierstrass sigma function.

$$\sigma(x;\tau_1,\tau_2) = x \prod_{\omega=m_1\tau_1+m_2\tau_2} \{(1-\frac{x}{\omega})\exp(\frac{x}{\omega}+\frac{1}{2}(\frac{x}{\omega})^2)\},$$

where (m_1,m_2) in the product above runs over all the elements in $\mathbb{Z}^2$ except $(0,0)$.

We prove the lemma below by the three term identity of the Weierstrass sigma function [3, page 377], [13, page 461].

Lemma 1.3. *The functions A and B in* Theorem 1.2 *satisfy the equations* (3), (4) *and* (5).

Hence the list in Theorem 1.2 gives the complete classification of the R-operators (1) satisfying the Yang-Baxter equation (2).

This article is organized as follows: Section 2 describes the results obtained in [9] briefly. We make the best use of these results (Theorem 2.1) when we solve the functional equations (3), (4) and (5). In Section 3, we give the proof of Theorem 1.2, which is the main result in this article.

2 Review of the classification of R-operators

In this section, we briefly review the paper [9].

Let functions $\widehat{A}(x)$ and $\widehat{B}(u,x)$ be meromorphic on $C(0,r)$ and $C((0,0),r)$, respectively. For a function f meromorphic on $C((0,0),r/2)$, we define the function $(\widehat{R}(u)f)(z_1,z_2)$ meromorphic on $C(0,r)\times C((0,0),r/2)(\ni(u,z_1,z_2))$ as

$$(\widehat{R}(u)f)(z_1,z_2)=\widehat{A}(z_1-z_2)f(z_1,z_2)-\widehat{B}(u,z_1-z_2)f(z_2,z_1). \tag{6}$$

In [9], we gave the complete classification of the operator $\widehat{R}(u)$ by virtue of [4].

Theorem 2.1. *The complete list of the operator $\widehat{R}(u)$ satisfying the Yang-Baxter equation* (2) *is as follows*:

0. *trivial case* $\widehat{A}(x)$ *is arbitrary*, $\widehat{B}(u,x)\equiv 0$.

$\widehat{A}(x)\equiv 0,\ \widehat{B}(u,x)=\exp(F(x)u)G(u)$ *on* $C(0,r)\times C(0,r_1)$.

$(0<r_1\leq r)$

1. *generic case*

1 − 1. *elliptic*

$$\widehat{A}(x)=c\cdot h(x)\frac{\sigma(x+s;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(s;\tau_1,\tau_2)},$$

$$\widehat{B}(u,x)=c\exp(\rho ux)\frac{\sigma(x+au;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(au;\tau_1,\tau_2)}.$$

$(a,c,\tau_1,\tau_2\in\mathbb{C}\setminus\{0\},\mathrm{Im}\tau_2/\tau_1>0,s\in\mathbb{C}\setminus(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2),\rho\in\mathbb{C})$

1 − 2. *trigonometric*

$$\widehat{A}(x)=\begin{cases}c\cdot h(x)\frac{\sinh\frac{x+s}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{s}{\lambda}},\\ c\cdot h(x)\frac{1}{\sinh\frac{x}{\lambda}},\end{cases}$$

$$\widehat{B}(u,x)=\begin{cases} c\exp(\rho ux)\frac{\sinh\frac{x+au}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{au}{\lambda}}, \\ c\exp(\rho ux)\frac{\exp(\pm\frac{x}{\lambda})}{\sinh\frac{x}{\lambda}}. \end{cases}$$

$(a, c, \lambda \in \mathbb{C}\setminus\{0\}, s \in \mathbb{C}\setminus\mathbb{Z}\pi\sqrt{-1}\lambda, \rho \in \mathbb{C})$

1 − 3. *rational* $\widehat{A}(x)=\begin{cases} c\cdot h(x)\frac{x+s}{xs}, \\ c\cdot h(x)\frac{1}{x}, \end{cases}$ $\widehat{B}(u,x)=\begin{cases} c\exp(\rho ux)\frac{x+au}{axu}, \\ c\exp(\rho ux)\frac{1}{x}. \end{cases}$

$(a, c, s \in \mathbb{C}\setminus\{0\}, \rho \in \mathbb{C})$

2. *singular case* $\widehat{A}(x)=c_1h(x),\ \widehat{B}(u,x)=c_2\exp(\rho ux)\frac{1}{u}.$

$(\rho \in \mathbb{C}, c_1, c_2 \in \mathbb{C}\setminus\{0\})$

Here the function F is holomorphic on $C(0,r_1)$, the function $G(\not\equiv 0)$ is meromorphic on $C(0,r)$ and the function h is meromorphic on $C(0,r)$ satisfying the relation $h(x)h(-x)=1$.

Remark . We correct the condition of the functions α, φ and ψ in Theorem II.1 in the paper [9] as follows: The functions α, φ and ψ are all meromorphic on the disk $C(0,r')$ for some $r'>0$.

3 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, the complete classification of the meromorphic solutions of the functional equations (3), (4) and (5) by means of Theorem 2.1.

From Proposition 1.1, we see that any A and $B \equiv 0$ solve the functional equations (3), (4) and (5).

If $A \equiv 0$, then the R-operator $R(u)$ is $(R(u)f)(z_1, z_2) = -B(u, z_1 - z_2)f(z_2, z_1)$. This operator is nothing but the operator $\widehat{R}(u)$ in the equation (6) with $\widehat{A} \equiv 0$. So we have already solved the functional equations (3), (4) and (5) with $A \equiv 0$ in Theorem 2.1.

In the sequel, we assume $A \not\equiv 0$ and $B \not\equiv 0$.

Then the equation (3) implies

$$A(u, x)A(v, y) = A(v, x)A(u, y) \tag{7}$$

on $C((0, 0, 0, 0), r)$.

Lemma 3.1. *There exists functions b and $\widetilde{A}$ meromorphic on $C(0, r)$ such that $A(u, x) = b(u)\widetilde{A}(x)$.*

Proof. By $A \not\equiv 0$, there exists $r_1 > 0$ and $(u_1, x_1) \in C((0, 0), r)$ such that the function A is holomorphic on $C((u_1, x_1), r_1)$ and that $A(u, x) \neq 0$ for all $(u, x) \in C((u_1, x_1), r_1)$. From the equation (7),

$$\frac{A(u, x)}{A(v, x)} = \frac{A(u, y)}{A(v, y)} \tag{8}$$

for all $u, v \in C(u_1, r_1)$ and all $x, y \in C(x_1, r_1)$. The equation above means that the function $A(u, x)/A(v, x)$ holomorphic on $C((u_1, u_1, x_1), r_1)$ does not depend on the variable x. We define the function f meromorphic on $C((0, 0), r)$ by

$$f(u, v) = \frac{A(u, x_1)}{A(v, x_1)}$$

because the function $A(u, x_1)$ is meromorphic and not identically zero on $C(0, r)$ (See Lemma III.5 in the paper [9].). We note that the function f is holomorphic on $C((u_1, u_1), r_1)$. By the equation (8) and the definition of the function f,

$$A(u, x) = f(u, u_1)A(u_1, x) \tag{9}$$

for all $(u, x) \in C((u_1, x_1), r_1)$. From Lemma III.5 in the paper [9], the functions $f(u, u_1)$ and $A(u_1, x)$ are meromorphic on $C(0, r)$, and consequently, the function $f(u, u_1)A(u_1, x)$ in the right hand side of the equation (9) is meromorphic on $C((0, 0), r)$. Define the functions b and $\widetilde{A}$ meromorphic on $C(0, r)$ by

$$\begin{cases} b(u) = f(u, u_1), \\ \widetilde{A}(x) = A(u_1, x). \end{cases}$$

With the aid of the identity theorem for the meromorphic functions,

$$A(u, x) = b(u)\widetilde{A}(x)$$

on $C((0, 0), r)$, which is the desired result. □

We note that $b \not\equiv 0$ because $A \not\equiv 0$. By means of Proposition 1.1, the operator $(1/b(u))R(u)$ also satisfies the Yang-Baxter equation. If we put $\widetilde{B}(u, x) := (1/b(u))B(u, x)$,

$$((1/b(u))R(u)f)(z_1, z_2) = \widetilde{A}(z_1 - z_2)f(z_1, z_2) - \widetilde{B}(u, z_1 - z_2)F(z_2, z_1),$$

that is to say, the operator $(1/b(u))R(u)$ is the operator $\widehat{R}(u)$ in the equation (6). From Theorem 2.1, we deduce Theorem 1.2.

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