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On Inherited Properties and Scalarization Algorithms for Set-Valued Maps*

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1. Introduction

This paper consists of two parts which are several inherited properties of set-valued maps and scalarization algorithms for their maps.

Firstly, we present certain results on inherited properties on convexity and semicontinuity. Convexity and lower semicontinuity of real-valued functions are useful properties for analysis of optimization problems, and they are dual concepts to concavity and upper semicontinuity, respectively. These properties are related to the total ordering of R^n. We consider certain generalizations and modifications of convexity and semicontinuity for set-valued maps in a topological vector space with respect to a cone preorder in the target space, which have motivated by [3, 4] and studied in [1] for generalizing the classical Fan's inequality. These properties are inherited by special scalarizing functions;

$$\inf\{h_C(x,y;k) : y \in F(x)\} \tag{1.1}$$

and

$$\sup\{h_C(x,y;k) : y \in F(x)\} \tag{1.2}$$

where $h_C(x,y;k) = \inf\{t : y \in tk - C(x)\}$, $C(x)$ is a closed convex cone with nonempty interior, x and y are vectors in two topological vector spaces E, Y, and $k \in \mathrm{int}C(x)$. Note that $h_C(x,\cdot;k)$ is positively homogeneous and subadditive for every fixed $x \in X$ and $k \in \mathrm{int}C(x)$, and moreover $-h_C(x,-y;k) = \sup\{t : y \in tk + C(x)\}$.

Secondly, we develop computational procedures how to calculate the values of scalarizing functions (1.1) and (1.2). In order to find solutions in multicriteria situations, we use some types of scalarization algorithms such as positive linear functionals and Tchebyshev scalarization. The function $h_C(x,y;k)$ is regarded as a generalization of the Tchebyshev scalarization. By using the function, we give four types of characterizations of set-valued maps.

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2. Inherited Properties of Set-Valued Maps

The aim of this section is to investigate how the property of cone-convexity [resp., cone-concavity] is inherited into scalarizing functions (1.1) and (1.2) from set-valued maps, and how the property of cone-semicontinuity is inherited into such scalarizing functions from set-valued maps. Let E and Y be topological vector spaces and F, $C : E \to 2^Y$ two multivalued mapping. Denote $B(x) = (\mathrm{int}C(x)) \cap (2S \setminus \overline{S})$ (an open base of $\mathrm{int}C(x)$), where S is a neighborhood of 0 in Y. To avoid confusion for properties of convexity, we consider the constant case of $C(x) = C$ (a convex cone) and its base $B(x) = B$ firstly and $h_C(x, y; k) = h_C(y; k) := \inf\{t : y \in tk - C\}$. We observe the following four types of scalarizing functions:

$$\varphi_C^F(x; k) := \sup_{y \in F(x)} h_C(y; k), \qquad \psi_C^F(x; k) := \inf_{y \in F(x)} h_C(y; k);$$

$$-\psi_C^{-F}(x; k) = \sup_{y \in F(x)} -h_C(-y; k), \qquad -\varphi_C^{-F}(x; k) = \inf_{y \in F(x)} -h_C(-y; k).$$

The first and fourth functions have symmetric properties and then results for the fourth function $-\varphi_C^{-F}(x; k)$ can be easily proved by those for the first function $\varphi_C^F(x; k)$. Similarly, the results for the third function $-\psi_C^{-F}(x; k)$ can be deduced by those for the second function $\psi_C^F(x; k)$. By using these four functions we measure each image of set-valued map F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image.

To begin with, we recall some kinds of convexity for set-valued maps.

Definition 2.1. A multifunction $F : E \to 2^Y$ is called *C-quasiconvex*, if the set $\{x \in E : F(x) \cap (a - C) \neq \emptyset\}$ is convex for every $a \in Y$. If $-F$ is C-quasiconvex, then F is said to be C-quasiconcave, which is equivalent to $(-C)$-quasiconvex mapping.

Definition 2.2. A multifunction $F : E \to 2^Y$ is called (in the sense of [4, Definition 3.6])

(a) *type-*(v) *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$;

(b) *type-*(iii) *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$.

If $-F$ is type-(v) [resp. type-(iii)] C-properly quasiconvex, then F is said be type-(v) [resp. type-(iii)] C-properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] $(-C)$-properly quasiconvex mapping.

Definition 2.3. A multifunction $F : E \to 2^Y$ is called *type-*(v) *C-naturally quasiconvex*, if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ there exists $\mu \in [0, 1]$ such that $F(\lambda x_1 + (1 - \lambda)x_2) \subset \mu F(x_1) + (1 - \mu)F(x_2) - C$. If $-F$ is type-(v) C-naturally quasiconvex, then F is said to be type-(v) C-naturally quasiconcave, which is equivalent to type-(v) $(-C)$-naturally quasiconvex mapping.

Theorem 2.1. (Inherited convexity (1))

(i) *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-properly quasiconvex, then the function $\inf_{k\in B} \varphi_C^F(x;k) = \inf_{k\in B} \sup_{y\in F(x)} h_C(y;k)$ is quasiconvex, and especially $\varphi_C^F(x;k)$ is also quasiconvex;*

(ii) *If the multifunction $F : E \to 2^Y$ is type-*(iii) *C-properly quasiconcave, then the function $\varphi_C^F(x;k) = \sup_{y\in F(x)} h_C(y;k)$ is quasiconcave;*

(iii) *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-properly quasiconcave, then the function $\psi_C^F(x;k) = \inf_{y\in F(x)} h_C(y;k)$ is quasiconcave;*

(iv) *If the multifunction $F : E \to 2^Y$ is type-*(iii) *C-properly quasiconvex, then the function $\psi_C^F(x;k) = \inf_{y\in F(x)} h_C(y;k)$ is quasiconvex.*

Proof. To prove (i), by definition, for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$ we have either $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$. Assume that $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$. Then

$$\begin{aligned}
f_1(\lambda x_1 + (1-\lambda)x_2) &:= \inf_{k\in B} \sup \{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2)\} \\
&\leq \inf_{k\in B} \sup \{h_C(y;k) \mid y \in F(x_1) - C\} \\
&= \inf_{k\in B} \sup_{\substack{y\in F(x_1)\\ c\in C}} h_C(y-c;k) \\
&\leq \inf_{k\in B} \sup_{\substack{y\in F(x_1)\\ c\in C}} (h_C(y;k) + h_C(-c;k)) \quad \text{(by subadditivity of } h_C(\cdot;k)) \\
&\leq \inf_{k\in B} \sup_{y\in F(x_1)} h_C(y;k) \\
&= f_1(x_1) \\
&\leq \max\{f_1(x_1), f_1(x_2)\}.
\end{aligned}$$

Analogously we can prove the other case when $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$.

Next, to prove (ii), we assume that for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$ F satisfies either $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) - C$ or $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C$. Assume that $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) - C$, then

$$\begin{aligned}
\varphi_C^F(x_1;k) &= \sup\{h_C(y;k) \mid y \in F(x_1)\} \\
&\leq \sup\{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2) - C\} \\
&= \sup_{\substack{y\in F(\lambda x_1+(1-\lambda)x_2)\\ c\in C}} h_C(y-c;k) \\
&\leq \sup_{\substack{y\in F(\lambda x_1+(1-\lambda)x_2)\\ c\in C}} (h_C(y;k) + h_C(-c;k)) \quad \text{(by subadditivity of } \mathrm{h_C}(\cdot;\mathrm{k})) \\
&\leq \sup_{y\in F(\lambda x_1+(1-\lambda)x_2)} h_C(y;k) \\
&= \varphi_C^F(\lambda x_1 + (1-\lambda)x_2;k),
\end{aligned}$$

and hence

$$\min\left\{\varphi_C^F(x_1;k), \varphi_C^F(x_2;k)\right\} \leq \varphi_C^F(\lambda x_1 + (1-\lambda)x_2;k).$$

Analogously we can prove the other case when $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C$.

To prove (iii), we assume that for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$ F satisfies either $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) + C$ or $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) + C$. Assume that $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) + C$. Then

$$\begin{aligned}
\psi_C^F(\lambda x_1 + (1-\lambda)x_2; k) &= \inf\{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2)\} \\
&\geq \inf\{h_C(y;k) \mid y \in F(x_1) + C\} \\
&= \inf_{\substack{y \in F(x_1) \\ c \in C}} h_C(y+c;k) \\
&\geq \inf_{\substack{y \in F(x_1) \\ c \in C}} (h_C(y;k) - h_C(-c;k)) \quad \text{(by subadditivity of } \mathrm{h_C(\cdot;k)}) \\
&\geq \inf_{y \in F(x_1)} h_C(y;k) \\
&= \psi_C^F(x_1;k) \\
&\geq \min\left\{\psi_C^F(x_1;k), \psi_C^F(x_2;k)\right\},
\end{aligned}$$

and hence

$$\min\left\{\psi_C^F(x_1;k), \psi_C^F(x_2;k)\right\} \leq \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k).$$

Analogously we can prove the other case when $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) + C$.

At last, to prove (iv), we assume that for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$ $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$. Assume that $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$. Then

$$\begin{aligned}
\psi_C^F(x_1;k) &= \inf\{h_C(y;k) \mid y \in F(x_1)\} \\
&\geq \inf\{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2) + C\} \\
&= \inf_{\substack{y \in F(\lambda x_1 + (1-\lambda)x_2) \\ c \in C}} h_C(y+c;k) \\
&\geq \inf_{\substack{y \in F(\lambda x_1 + (1-\lambda)x_2) \\ c \in C}} (h_C(y;k) - h_C(-c;k)) \quad \text{(by subadditivity of } \mathrm{h_C(\cdot;k)}) \\
&\geq \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)} h_C(y;k) \\
&= \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k),
\end{aligned}$$

and hence

$$\psi_C^F(\lambda x_1 + (1-\lambda)x_2; k) \leq \max\left\{\psi_C^F(x_1;k), \psi_C^F(x_2;k)\right\}.$$

Analogously we can prove the other case when $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$. ∎

Corollary 2.1.

(i) *If $F : X \to 2^Y$ is type-*(v) *C-properly quasiconcave, then the function*

$$f_2(x) := \sup_{k \in B}(-\varphi_C^{-F}(x;k)) = \sup_{k \in B} \inf\{-h_C(-y;k) \mid y \in F(x)\}$$

is quasiconcave, and especially $-\varphi_C^{-F}(x;k)$ is also quasiconcave;

(ii) *If $F : X \to 2^Y$ is type-*(iii) *C-properly quasiconvex, then the function*

$$-\varphi_C^{-F}(x;k) = \inf\{-h_C(-y;k) \mid y \in F(x)\}$$

is quasiconvex for any $k \in \operatorname{int} C$;

(iii) *If $F : X \to 2^Y$ is type-*(v) *C-properly quasiconvex, then the function*

$$-\psi_C^{-F}(x;k) = \sup\{-h_C(-y;k) \mid y \in F(x)\}$$

is quasiconvex for any $k \in \operatorname{int} C$;

(iv) *If $F : X \to 2^Y$ is type-*(iii) *C-properly quasiconcave, then the function*

$$-\psi_C^{-F}(x;k) = \sup\{-h_C(-y;k) \mid y \in F(x)\}$$

is quasiconcave for any $k \in \operatorname{int} C$.

Theorem 2.2. (Inherited convexity (2)) *If the multifunction $F : E \to 2^Y$ is C-quasiconvex, then for every $k \in B$ the function*

$$\psi_C^F(x;k) = \inf\{h_C(y;k) \mid y \in F(x)\}$$

is quasiconvex.

Proof. By the definition of ψ_C^F, for every $\varepsilon > 0$ and $x_1, x_2 \in X$ there exist $z_i \in F(x_i), t_i \in R$ such that for each $i = 1, 2$ $z_i - t_i k \in -C$ and $t_i < \psi_C^F(x_i;k) + \varepsilon$. Since $s_1 k - C \subset s_2 k - C$ for $s_1 \leq s_2 (s_1, s_2 \in R)$, we have

$$z_i \in t_i k - C \subset \max\{t_1, t_2\}\, k - C.$$

Hence, by the C-quasiconvex of F, for every $\lambda \in [0,1]$ there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y \in \max\{t_1, t_2\}\, k - C$. which means

$$\begin{aligned} h_C(y;k) &\leq \max\{t_1, t_2\} \\ &< \max\left\{\psi_C^F(x_1;k), \psi_C^F(x_2;k)\right\} + \varepsilon. \end{aligned}$$

Therefore, we have

$$\psi_C^F(\lambda x_1 + (1-\lambda)x_2;k) = \inf\{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2)\},$$

and since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\psi_C^F(\lambda x_1 + (1-\lambda)x_2;k) \leq \max\left\{\psi_C^F(x_1;k), \psi_C^F(x_2;k)\right\}.$$

∎

Corollary 2.2. *If $F : X \to 2^Y$ is C-quasiconcave, then for every $k \in B$ the function*

$$-\psi_C^{-F}(x;k) = \sup\{-h_C(-y;k) \mid y \in F(x)\}$$

is quasiconcave.

Theorem 2.3. (Inherited convexity (3)) *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-naturally quasiconvex, then for every $k \in \operatorname{int} C$ the function*

$$\varphi_C^F(x;k) = \sup\{h_C(y;k) \mid y \in F(x)\}$$

is quasiconvex.

Proof. By definition, for every $x_1, x_2 \in X$ and every $\lambda \in (0,1)$ we have

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \mu F(x_1) + (1-\mu)F(x_2) - C.$$

$$\begin{aligned}
\varphi_C^F(\lambda x_1 + (1-\lambda)x_2; k) &:= \sup\{h_C(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2)\} \\
&\leq \sup\{h_C(y;k) \mid y \in \mu F(x_1) + (1-\mu)F(x_2) - C\} \\
&= \sup_{\substack{y_1 \in F(x_1) \\ y_2 \in F(x_2) \\ c \in C}} h_C(\mu y_1 + (1-\mu)y_2 - c; k) \\
&\leq \sup_{\substack{y_1 \in F(x_1) \\ y_2 \in F(x_2) \\ c \in C}} (h_C(\mu y_1; k) + h_C((1-\mu)y_2; k) + h_C(-c; k)) \\
&\leq \sup_{\substack{y_1 \in F(x_1) \\ y_2 \in F(x_2)}} (\mu h_C(y_1; k) + (1-\mu)h_C(y_2; k)) \\
&\leq \mu \sup_{y_1 \in F(x_1)} h_C(y_1; k) + (1-\mu) \sup_{y_2 \in F(x_2)} h_C(y_2; k) \\
&= \mu \varphi_C^F(x_1; k) + (1-\mu)\varphi_C^F(x_2; k) \\
&\leq \max\left\{\varphi_C^F(x_1; k), \varphi_C^F(x_2; k)\right\}.
\end{aligned}$$

∎

Corollary 2.3. *If $F : E \to 2^Y$ is type-(v) C-naturally quasiconcave, then for every $k \in \operatorname{int} C$ the function*

$$-\varphi_C^{-F}(x; k) = \inf\{-h_C(-y; k) \mid y \in F(x)\}$$

is quasiconcave.

Next, we proceed to observe another inherited property on set-valued maps. We introduce two types of cone-semicontinuity of set-valued mappings, which are regarded as extensions of the ordinary lower semicontinuity for real-valued functions; see [3].

Definition 2.4. Let $\hat{x} \in E$. The multifunction F is called $C(\hat{x})$-*upper semicontinuous* at x_0, if for every $y \in C(\hat{x}) \cup (-C(\hat{x}))$ such that $F(x_0) \subset y + \text{int}C(\hat{x})$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \text{int}C(\hat{x})$ for every $x \in U$.

Definition 2.5. Let $\hat{x} \in E$. The multifunction F is called $C(\hat{x})$-*lower semicontinuous* at x_0, if for every open V such that $F(x_0) \cap V \neq \emptyset$, there exists an open $U \ni x_0$ such that $F(x) \cap (V + \text{int}C(\hat{x})) \neq \emptyset$ for every $x \in U$.

Remark 2.1. In the two definitions above, the notions for single-valued functions are equivalent to the ordinary notion of lower semicontinuity of real-valued ones, whenever $Y = R$ and $C = [0, \infty)$. When the cone $C(\hat{x})$ consists only of the zero of the space, the notion in Definition 2.5 coincides with that of lower semicontious set-valued mapping. Moreover, it is equivalent to the cone-lower semicontinuity defined in [3], based on the fact that $V + \text{int}C(\hat{x}) = V + C(\hat{x})$; see [5, Theorem 2.2].

Proposition 2.1 ([1, Proposition 3.1]) *If for some $x_0 \in E$, $A \subset \text{int}C(x_0)$ is a compact subset and multivalued mapping $W(\cdot) := Y \setminus \{\text{int}C(\cdot)\}$ has a closed graph, then there exists an open set $U \ni x_0$ such that $A \subset C(x)$ for every $x \in U$. In particular C is lower semicontinuous.*

We shall say that (F, X), where X is a subset of E, has property (P), if

(P) *for every $x \in X$ there exists an open $U \ni x$ such that the set $F(U \cap X)$ is precompact in Y, that is, $\overline{F(U \cap X)}$ is compact.*

Theorem 2.4. (Inherited semicontinuity (1); see [1, Lemma 3.1]) *Suppose that multifunction $W : E \to 2^Y$ defined as $W(x) = Y \setminus \text{int}C(x)$ has a closed graph. If the multifunction F is $(-C(x))$-upper semicontinuous at x for each $x \in E$, then the function $f_1|_X$ (the restriction of*

$$f_1(x) := \inf_{k \in B(x)} \sup_{y \in F(x)} h(k, x, y).$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P). If the mapping C is constant-valued, then f_1 is upper semicontinuous.

Theorem 2.5. (Inherited semicontinuity (2); see [1, Lemma 3.3]) *Suppose that multifunction $W : E \to 2^Y$ defined as $W(x) = Y \setminus \text{int}C(x)$ has a closed graph. If the multifunction F is $-C(x)$-lower semicontinuous at x for each $x \in E$, then the function $f_2|_X$ (the restriction of*

$$f_2(x) := \inf_{k \in B(x)} \inf_{y \in F(x)} h(k, x, y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P). If the mapping C is constant-valued, then f_2 is upper semicontinuous.

Other results on inherited semicontinuity are observed in Lemmas 3.1 and 3.2 of [2].

3. Scalarization Algorithms for Set-Valued Maps

To give computational procedures how to calculate the values of $\inf\{h_C(x, y; k) : y \in F(x)\}$ and $\sup\{h_C(x, y; k) : y \in F(x)\}$ practically, we restrict finite dimensional cases ($Y = R^p$) and we consider the constant case of $C(x) = C$ (a convex cone) and $h_C(x, y; k) = h_C(y; k) := \inf\{t : y \in tk - C\}$.

Tchebyshev scalarization is one of the main tools in the multiobjective optimization problem. In this paper we consider four kinds of scalarizations φ_C^F, ψ_C^F, $-\psi_C^{-F}$, $-\varphi_C^{-F}$ for some multiobjective optimization problems. They are regarded as generalization of Tchebyshev scalarization. Our proposed algorithm is based on some properties stated in the previous section, basically those of positively homogeneous and subadditive play key roles.

Moreover, if the set-valued image $F(x)$ is a simplex (a convex hull generated by finite vectors), called "polyhedron," such as $\mathbf{co}\{y_1, \dots, y_m\}$, we obtain

$$h_C(y; k) = \max\left\{\frac{y^{(1)}}{k^{(1)}}, \dots, \frac{y^{(p)}}{k^{(p)}}\right\}, -h_C(-y; k) = \min\left\{\frac{y^{(1)}}{k^{(1)}}, \dots, \frac{y^{(p)}}{k^{(p)}}\right\},$$

and then we can calculate

$$\varphi_C^F(x;k) = \max_j \max_i \frac{y_j^{(i)}}{k^{(i)}}, \qquad \psi_C^F(x;k) := \min_j \max_i \frac{y_j^{(i)}}{k^{(i)}},$$

$$-\psi_C^{-F}(x;k) = \max_j \min_i \frac{y_j^{(i)}}{k^{(i)}}, \qquad -\varphi_C^{-F}(x;k) = \min_j \min_i \frac{y_j^{(i)}}{k^{(i)}}.$$

References

[1] P. Gr. Georgiev and T. Tanaka, *Vector-valued set-valued variants of Ky Fan's inequality*, Journal of Nonlinear and Convex Analysis, **1(3)**, pp.245–254, 2000.

[2] P. Gr. Georgiev and T. Tanaka, *Fan's inequality for set-valued maps*, Nonlinear Analysis Theory, Methods and Applications, **47(1)**, pp.607–618, 2001.

[3] Y. Kimura, K. Tanaka, and T. Tanaka, *On semicontinuity of set-valued maps and marginal functions*, 181–188 in Nonlinear Anlysis and Convex Analysis —Proceedings of the International Conference (W. Takahashi and T. Tanaka, eds.), World Scientific, 1999.

[4] D. Kuroiwa, T. Tanaka, and T.X.D. Ha, *On cone convexity of set-valued maps*, Nonlinear Analysis, TMA, **30(3)**, 1487–1496 (1997).

[5] T. Tanaka and D. Kuroiwa, *Another observation on conditions assuring* $\mathrm{int}A + B = \mathrm{int}(A+B)$, Appl. Math. Lett., **7 (1)**, 19–22 (1994).