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Inequalities Involving Unitarily Invariant Norms and Operator Monotone Functions[1]

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We consider square complex matrices. A norm $\|\cdot\|$ on the space of $n\times n$ matrices is called *unitarily invariant* if

$$\|UAV\|=\|A\| \qquad \forall A,\ \forall \text{unitary } U,V.$$

Such a norm is determined by a symmetric gauge function Φ on $\mathbb{R}^n$:

$$\|A\|=\Phi(s_1(A),\dots,s_n(A))$$

where $s_1(A)\ge s_2(A)\ge\cdots\ge s_n(A)$ are the singular values of A, that is, the eigenvalues of $|A|\equiv(A^*A)^{1/2}$.

Examples of unitarily invariant norms are:

Schatten p-norm $\|\cdot\|_p$ $(1\le p\le\infty)$:

$$\|A\|_p\equiv\{\sum_{j=1}^n s_j(A)^p\}^{1/p}.$$

Then $\|A\|_\infty=s_1(A)$ is the *spectral norm* and $\|A\|_2=\{\sum_{i,j=1}^n|a_{ij}|^2\}^{1/2}$ is the *Frobenius norm.*

Fan k-norm $\|\cdot\|_{(k)}$ $(k=1,2,\dots,n)$:

$$\|A\|_{(k)}\equiv\sum_{j=1}^k s_j(A).$$

For Hermitian matrices A,B, we write $A\ge B$ to mean that $A-B$ is positive semidefinite. In particular, $A\ge 0$ means that A is positive semidefinite.

We consider only continuous nonnegative functions on $[0,\infty)$. $f(t)$ is called *operator monotone* if

$$A\ge B\ge 0 \implies f(A)\ge f(B).$$

[1]This paper appeared in Linear Algebra Appl. 341(2002) 151-169.

Here $f(A)$ is defined by the usual functional calculus via the spectral decomposition of A.

Examples of operator monotone functions are:

$$t^p \ (0 < p \leq 1), \quad \log(t+1)$$

1. Convexity of certain functions involving unitarily invariant norms

Theorem 1. Given matrices $A, B \geq 0$, $\forall X$, real number $r > 0$, and any unitarily invariant norm, the function

$$\phi(t) = \| \, |A^t X B^{1-t}|^r \, \| \cdot \| \, |A^{1-t} X B^t|^r \, \|$$

is convex on the interval $[0,1]$ and attains its minimum at $t = 1/2$. Consequently, it is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.

Corollary 2. For $0 \leq t \leq 1$,

$$\begin{aligned} \| \, |A^{1/2} X B^{1/2}|^r \, \|^2 &\leq \| \, |A^t X B^{1-t}|^r \, \| \cdot \| \, |A^{1-t} X B^t|^r \, \| \\ &\leq \| \, |AX|^r \, \| \cdot \| \, |XB|^r \, \| \end{aligned}$$

Note that this interpolates the known matrix Cauchy-Schwarz inequality

$$\| \, |A^{1/2} X B^{1/2}|^r \, \|^2 \leq \| \, |AX|^r \, \| \cdot \| \, |XB|^r \, \|.$$

Corollary 3. Let A, B be positive definite and X be arbitrary. For every $r > 0$ and every unitarily invariant norm, the function

$$g(s) = \| \, |A^s X B^s|^r \, \| \cdot \| \, |A^{-s} X B^{-s}|^r \, \|$$

is convex on $(-\infty, \infty)$, attains its minimum at $s = 0$, and hence it is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

The case $r = 1, X = B = I$ (the identity matrix) of this result says that the condition number

$$c(A^s) \equiv \|A^s\| \cdot \|A^{-s}\|$$

is increasing in $s > 0$, which is due to A. W. Marshall and I. Olkin (1965).

2. Norm inequalities for operator monotone functions with applications

A norm on $n \times n$ matrices is said to be *normalized* if $\|\mathrm{diag}(1, 0, \ldots, 0)\| = 1$.

All the Fan k-norms ($k = 1, \ldots, n$) and Schatten p-norms ($1 \leq p \leq \infty$) are normalized.

Theorem 4. Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\|\cdot\|$ be a normalized unitarily invariant norm. Then for every matrix A,

$$f(\|A\|) \leq \|f(|A|)\|.$$

This inequality is reversed when the norm is normalized in another way.

Theorem 5. Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\|\cdot\|$ be a unitarily invariant norm with $\|I\| = 1$. Then for every matrix A,

$$f(\|A\|) \geq \|f(|A|)\|.$$

Given a unitarily invariant norm $\|\cdot\|$, for $p > 0$ define

$$\|X\|^{(p)} \equiv \|\,|X|^p\,\|^{1/p}.$$

Then it is known that when $p \geq 1$, $\|\cdot\|^{(p)}$ is also a unitarily invariant norm.

Corollary 6. Let $\|\cdot\|$ be a normalized unitarily invariant norm. Then for any matrix A, the function $p \mapsto \|A\|^{(p)}$ is decreasing on $(0, \infty)$ and

$$\lim_{p \to \infty} \|A\|^{(p)} = \|A\|_\infty.$$

The above limit formula remains valid without the normalization condition on $\|\cdot\|$.

We denote by $A \vee B$ the supremum of $A, B \geq 0$: $A \vee B = \lim_{p \to \infty} \{(A^p + B^p)/2\}^{1/p}$.

Theorem 7. Let A, B be positive semidefinite. For every unitarily invariant norm, the function $p \mapsto \|(A^p + B^p)^{1/p}\|$ is decreasing on $(0, 1]$. For every normalized unitarily invariant norm, the function $p \mapsto \|A^p + B^p\|^{1/p}$ is decreasing on $(0, \infty)$ and

$$\lim_{p\to\infty} \|A^p + B^p\|^{1/p} = \|A \vee B\|_\infty .$$

The above limit formula remains valid without the normalization condition.

3. Norm inequalities of Hölder and Minkowski types

Theorem 8. Let $1 \le p, q \le \infty$ with $p^{-1} + q^{-1} = 1$. For all matrices A, B, C, D and every unitarily invariant norm,

$$2^{-|\frac{1}{p}-\frac{1}{2}|}\|C^*A + D^*B\| \le \|\,|A|^p + |B|^p\,\|^{1/p} \cdot \|\,|C|^q + |D|^q\,\|^{1/q} .$$

Moreover, the constant $2^{-|\frac{1}{p}-\frac{1}{2}|}$ is best possible.

Theorem 9. Let $1 \le p < \infty$. For any A_i, B_i $(i = 1, 2)$ and every unitarily invariant norm,

$$2^{-|\frac{1}{p}-\frac{1}{2}|}\|\,|A_1 + A_2|^p + |B_1 + B_2|^p\,\|^{1/p}$$

$$\le \|\,|A_1|^p + |B_1|^p\,\|^{1/p} + \|\,|A_2|^p + |B_2|^p\,\|^{1/p} .$$

Main Ingredients of the Proofs

• *Integral representation*: A nonnegative operator monotone function $f(t)$ on $[0, \infty)$ is represented as

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t} d\mu(s)$$

where $\alpha, \beta \ge 0$ and $\mu(\cdot)$ is a positive measure on $[0, \infty)$.

• *Dual norm*: Given a norm $\|\cdot\|$ on $n \times n$ matrices, the dual norm of $\|\cdot\|$ with respect to the Frobenius inner product is

$$\|A\|^D \equiv \max\{|\mathrm{tr}\, AX^*| : \|X\| = 1\} .$$

If $\|\cdot\|$ is a unitarily invariant norm and $A \geq 0$, then by the duality theorem we have

$$\|A\| = \max\left\{\operatorname{tr} AB : B \geq 0, \|B\|^D = 1\right\}.$$

• *Theorem* [conjectured by F. Hiai and proved by by T. Ando and X. Zhan, Math. Ann. 315 (1999)]: Let $A, B \geq 0$, and $\|\cdot\|$ be a unitarily invariant norm. If $f(t)$ nonnegative operator monotone on $[0, \infty)$, then

$$\|f(A+B)\| \leq \|f(A) + f(B)\|.$$

If $g(t)$ is strictly increasing on $[0, \infty)$ with $g(0) = 0$, $g(\infty) = \infty$ and the inverse function g^{-1} on $[0, \infty)$ is operator monotone, then

$$\|g(A+B)\| \geq \|g(A) + g(B)\|.$$