数理解析研究所講究録 1260 巻 2002 年 172-187

Coherent States and Some Topics in Quantum Information Theory

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概要

In the first half we make a general review of coherent states and generalized coherent ones based on Lie algebras su(2) and su(1,1). In the second half we make a review of recent developments of both swap of coherent states and cloning of coherent states which are main subjects in Quantum Information Theory.

1 Introduction

The purpose of this paper is to introduce several basic theorems of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1), and to give some applications of them to Quantum Information Theory.

In the first half we make a general review of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1).

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [1] and its references, or the book [2]. They also play an important one in mathematical physics, see the book [3]. For example, they are very useful in performing stationary phase approximations to path integral, [4], [5], [6].

In the latter half we apply a method of generalized coherent states to some important topics in Quantum Information Theory, in particular, swap of coherent states and cloning of coherent ones.

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Quantum Information Theory is one of most exciting fields in modern physics or mathematical physics. It is mainly composed of three subjects

Quantum Computation, Quantum Cryptgraphy and Quantum Teleportation .

See for example [7], [8], [9] or [10], [11]. Coherent states or generalized coherent states also play an important role in it.

We construct the swap operator of coherent states by making use of a generalized coherent operator based on su(2) and moreover show an "imperfect cloning" of coherent states, and last present some related problems.

2 Coherent and Generalized Coherent Operators Revisited

We make a some review of general theory of both a coherent operator and generalized coherent ones based on Lie algebras $su(1,1)$ and $su(2)$.

2.1 Coherent Operator

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger \ , \ [N, a] = -a \ , \ [a^\dagger, a] = -\mathbf{1} \ . \tag{1}$$

Let $\mathcal{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle |\ n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , N|n\rangle = n|n\rangle \tag{2}$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \ . \tag{3}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn} \ , \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1} \ . \tag{4}$$

Let us state coherent states. For the normalized state $|z\rangle \in \mathcal{H}$ for $z \in \mathbf{C}$ the following three conditions are equivalent :

$$\text{(i)} \quad a|z\rangle = z|z\rangle \quad \text{and} \quad \langle z|z\rangle = 1 \tag{5}$$

$$\text{(ii)} \quad |z\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle = \mathrm{e}^{-|z|^2/2} e^{za^\dagger}|0\rangle \tag{6}$$

$$\text{(iii)} \quad |z\rangle = \mathrm{e}^{za^\dagger - \bar{z}a}|0\rangle. \tag{7}$$

In the process from (6) to (7) we use the famous elementary Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^{A}e^{B} \tag{8}$$

whenever $[A,[A,B]] = [B,[A,B]] = 0$, see [1]. This is the key formula.

Definition The state $|z\rangle$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following partition (resolution) of unity.

$$\int_{\mathbf{C}} \frac{[d^2z]}{\pi}|z\rangle\langle z| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbf{1}\ , \tag{9}$$

where we have put $[d^2z] = d(\mathrm{Re}z)d(\mathrm{Im}z)$ for simplicity.

Since the operator

$$D(z) = e^{za^{\dagger}-\bar{z}a} \quad \text{for} \quad z \in \mathbf{C} \tag{10}$$

is unitary, we call this a coherent (displacement) operator. For these operators the following property is crucial :

$$D(z+w) = e^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\ D(z)D(w) \quad \text{for} \quad z,\ w \in \mathbf{C}. \tag{11}$$

From this we have a well-known commutation relation

$$D(z)D(w) = e^{z\bar{w}-\bar{z}w}\ D(w)D(z). \tag{12}$$

Here we once more list the disentangling formula of $D(z)$ for the latter convenience :

$$e^{za^{\dagger}-\bar{z}a} = e^{-\frac{1}{2}|z|^2}e^{za^{\dagger}}e^{-\bar{z}a} = e^{\frac{1}{2}|z|^2}e^{-\bar{z}a}e^{za^{\dagger}} \tag{13}$$

2.2 Generalized Coherent Operator Based on $su(1,1)$

Let us state generalized coherent operators and states based on $su(1,1)$. Let $\{k_+, k_-, k_3\}$ be a Weyl basis of Lie algebra $su(1,1) \subset sl(2,\mathbf{C})$,

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\quad k_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Then we have

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3. \tag{15}$$

We note that $(k_+)^{\dagger} = -k_-$.

Next we consider a spin K (> 0) representation of $su(1,1) \subset sl(2,\mathbf{C})$ and set its generators $\{K_+, K_-, K_3\}$ ($(K_+)^{\dagger} = K_-$ in this case),

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \tag{16}$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K,n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and whose actions are

$$\begin{aligned} K_+|K,n\rangle &= \sqrt{(n+1)(2K+n)}|K,n+1\rangle, \\ K_-|K,n\rangle &= \sqrt{n(2K+n-1)}|K,n-1\rangle, \\ K_3|K,n\rangle &= (K+n)|K,n\rangle, \end{aligned} \tag{17}$$

where $|K,0\rangle$ is a normalized vacuum ($K_-|K,0\rangle = 0$ and $\langle K,0|K,0\rangle = 1$). We have written $|K,0\rangle$ instead of $|0\rangle$ to emphasize the spin K representation, see [4]. From (17), states $|K,n\rangle$ are given by

$$|K,n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}}|K,0\rangle, \tag{18}$$

where $(a)_n$ is the Pochammer's notation

$$(a)_n \equiv a(a+1)\cdots(a+n-1). \tag{19}$$

These states satisfy the orthogonality and completeness conditions

$$\langle K,m|K,n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |K,n\rangle\langle K,n| = \mathbf{1}_K. \tag{20}$$

Now let us consider a generalized version of coherent states :
Definition We call a state

$$|z\rangle = \mathrm{e}^{zK_+ - \bar{z}K_-}|K,0\rangle \quad \text{for} \quad z \in \mathbf{C}. \tag{21}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(1,1)$ in our terminology).
This is the extension of (7). See the book [3].
Then the partition of unity corresponding to (9) is

$$\begin{aligned} &\int_{\mathbf{C}} \frac{2K-1}{\pi} \frac{\tanh(|z|)[d^2z]}{\left(1-\tanh^2(|z|)\right)|z|}|z\rangle\langle z| \\ &= \int_{\mathrm{D}} \frac{2K-1}{\pi} \frac{[d^2\zeta]}{(1-|\zeta|^2)^2}|\zeta\rangle\langle\zeta| = \sum_{n=0}^{\infty} |K,n\rangle\langle K,n| = \mathbf{1}_K, \end{aligned} \tag{22}$$

where

$$\mathbf{C} \rightarrow \mathrm{D} : z \mapsto \zeta = \zeta(z) \equiv \frac{\tanh(|z|)}{|z|}z \quad \text{and} \quad |\zeta\rangle \equiv \left(1-|\zeta|^2\right)^K \mathrm{e}^{\zeta K_+}|K,0\rangle. \tag{23}$$

In the process of the proof we use the disentangling formula :

$$\mathrm{e}^{zK_+ - \bar{z}K_-} = \mathrm{e}^{\zeta K_+}\mathrm{e}^{\log(1-|\zeta|^2)K_3}\mathrm{e}^{-\bar{\zeta}K_-} = \mathrm{e}^{-\bar{\zeta}K_-}\mathrm{e}^{-\log(1-|\zeta|^2)K_3}\mathrm{e}^{\zeta K_+}. \tag{24}$$

This is also the key formula for generalized coherent operators. See [3] or [14].

Here let us construct an example of this representation. First we assign

$$K_+ \equiv \frac{1}{2}\left(a^\dagger\right)^2 \ , \ K_- \equiv \frac{1}{2}a^2 \ , \ K_3 \equiv \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \ , \tag{25}$$

then it is easy to check

$$[K_3, K_+] = K_+ \ , \ [K_3, K_-] = -K_- \ , \ [K_+, K_-] = -2K_3 \ . \tag{26}$$

That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1,1)$with spin $K = 1/4$ and $3/4$, [3]. Now we also call an operator

$$S(z) = e^{\frac{1}{2}\{z(a^\dagger)^2 - \bar{z}a^2\}} \quad \text{for} \quad z \in \mathbf{C} \tag{27}$$

the squeezed operator, see the papers in [1] or the book [3].

2.3 Generalized Coherent Operator Based on $su(2)$

Let us state generalized coherent operators and states based on $su(2)$. Let $\{j_+, j_-, j_3\}$ be a Weyl basis of Lie algebra $su(2) \subset sl(2, \mathbf{C})$,

$$j_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

Then we have

$$[j_3, j_+] = j_+, \quad [j_3, j_-] = -j_-, \quad [j_+, j_-] = 2j_3. \tag{29}$$

We note that $(j_+)^\dagger = j_-$.

Next we consider a spin J (> 0) representation of $su(2) \subset sl(2, \mathbf{C})$ and set its generators $\{J_+, J_-, J_3\}$ $((J_+)^\dagger = J_-)$,

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{30}$$

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which $\{J_+, J_-, J_3\}$ act is $\mathcal{H}_J \equiv \{|J, n\rangle | 0 \le n \le 2J\}$ and whose actions are

$$\begin{aligned} J_+|J,n\rangle &= \sqrt{(n+1)(2J-n)}|J,n+1\rangle, \\ J_-|J,n\rangle &= \sqrt{n(2J-n+1)}|J,n-1\rangle, \\ J_3|J,n\rangle &= (-J+n)|J,n\rangle, \end{aligned} \tag{31}$$

where $|J,0\rangle$ is a normalized vacuum ($J_-|J,0\rangle = 0$ and $\langle J,0|J,0\rangle = 1$). We have written $|J,0\rangle$ instead of $|0\rangle$ to emphasize the spin J representation, see [4]. From (31), states $|J,n\rangle$ are given by

$$|J,n\rangle = \frac{(J_+)^n}{\sqrt{n!\,{}_{2J}P_n}}|J,0\rangle. \tag{32}$$

These states satisfy the orthogonality and completeness conditions

$$\langle J,m|J,n\rangle=\delta_{mn},\quad \sum_{n=0}^{2J}|J,n\rangle\langle J,n|\ =\mathbf{1}_J. \tag{33}$$

Now let us consider a generalized version of coherent states :
Definition We call a state

$$|z\rangle=\mathrm{e}^{zJ_+-\bar{z}J_-}|J,0\rangle\quad\text{for}\quad z\in\mathbf{C}. \tag{34}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(2)$ in our terminology).
This is the extension of (7). See the book [3].

Then the partition of unity corresponding to (9) is

$$\begin{aligned}&\int_{\mathbf{C}}\frac{2J+1}{\pi}\frac{\tan(|z|)[d^2z]}{(1+\tan^2(|z|))\,|z|}|z\rangle\langle z|\\ =&\int_{\mathbf{C}}\frac{2J+1}{\pi}\frac{[d^2\zeta]}{(1+|\eta|^2)^2}|\eta\rangle\langle\eta|=\sum_{n=0}^{2J}|J,n\rangle\langle J,n|\ =\mathbf{1}_J,\end{aligned} \tag{35}$$

where

$$\mathbf{C}\rightarrow\mathbf{C}:z\mapsto\eta=\eta(z)\equiv\frac{\tan(|z|)}{|z|}z\quad\text{and}\quad|\eta\rangle\equiv\left(1+|\eta|^2\right)^{-J}\mathrm{e}^{\eta J_+}|J,0\rangle. \tag{36}$$

In the process of the proof we use the disentangling formula :

$$\mathrm{e}^{zJ_+-\bar{z}J_-}=\mathrm{e}^{\eta J_+}\mathrm{e}^{\log(1+|\eta|^2)J_3}\mathrm{e}^{-\bar{\eta}J_-}=\mathrm{e}^{-\bar{\eta}J_-}\mathrm{e}^{-\log(1+|\eta|^2)J_3}\mathrm{e}^{\eta J_+}. \tag{37}$$

This is also the key formula for generalized coherent operators.

2.4 Schwinger's Boson Methhod

Here let us construct the spin K and J representations by making use of Schwinger's boson method.

Next we consider the system of two-harmonic oscillators. If we set

$$a_1=a\otimes 1,\ {a_1}^\dagger=a^\dagger\otimes 1;\ a_2=1\otimes a,\ {a_2}^\dagger=1\otimes a^\dagger, \tag{38}$$

then it is easy to see

$$[a_i,a_j]=[{a_i}^\dagger,{a_j}^\dagger]=0,\ [a_i,{a_j}^\dagger]=\delta_{ij},\quad i,j=1,2. \tag{39}$$

We also denote by $N_i={a_i}^\dagger a_i$ number operators.

Now we can construct representation of Lie algebras $su(2)$ and $su(1,1)$ making use of Schwinger's boson method, see [4], [5]. Namely if we set

$$su(2): \quad J_+ = {a_1}^\dagger a_2,\ J_- = {a_2}^\dagger a_1,\ J_3 = \frac{1}{2}\left({a_1}^\dagger a_1 - {a_2}^\dagger a_2\right), \tag{40}$$

$$su(1,1): \quad K_+ = {a_1}^\dagger {a_2}^\dagger,\ K_- = a_2 a_1,\ K_3 = \frac{1}{2}\left({a_1}^\dagger a_1 + {a_2}^\dagger a_2 + 1\right), \tag{41}$$

then we have

$$su(2): \quad [J_3, J_+] = J_+,\ [J_3, J_-] = -J_-,\ [J_+, J_-] = 2J_3, \tag{42}$$

$$su(1,1): \quad [K_3, K_+] = K_+,\ [K_3, K_-] = -K_-,\ [K_+, K_-] = -2K_3. \tag{43}$$

In the following we define (unitary) generalized coherent operators based on Lie algebras $su(2)$ and $su(1,1)$.

Definition We set

$$su(2): \quad U_J(z) = e^{z{a_1}^\dagger a_2 - \bar{z}{a_2}^\dagger a_1} \quad \text{for } z \in \mathbf{C}, \tag{44}$$

$$su(1,1): \quad U_K(z) = e^{z{a_1}^\dagger {a_2}^\dagger - \bar{z}a_2 a_1} \quad \text{for } z \in \mathbf{C}. \tag{45}$$

For the details of $U_J(z)$ and $U_K(z)$ see [3] and [4].
Here let us ask a question. What is a relation between (27) and (45) of generalized coherent operators based on $su(1.1)$? The answer is given by the following :

Formula We have

$$W(-\frac{\pi}{4})S_1(z)S_2(-z)W(-\frac{\pi}{4})^{-1} = U_K(z), \tag{46}$$

where $S_j(z) = (27)$ with a_j instead of a.
Namely, $U_K(z)$ is given by "rotating" the product $S_1(z)S_2(-z)$ by $W(-\frac{\pi}{4})$. This is an interesting relation. The proof is relatively easy, see [13] or [11].

Before closing this section let us make some mathematical preliminaries for the latter sections. We have easily

$$U_J(t)a_1U_J(t)^{-1} = cos(|t|)a_1 - \frac{t sin(|t|)}{|t|}a_2, \tag{47}$$

$$U_J(t)a_2U_J(t)^{-1} = cos(|t|)a_1 + \frac{\bar{t} sin(|t|)}{|t|}a_2, \tag{48}$$

so the map $(a_1, a_2) \longrightarrow (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1})$ is

$$(U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1}) = (a_1, a_2)\begin{pmatrix} cos(|t|) & \frac{\bar{t}sin(|t|)}{|t|} \\ -\frac{t sin(|t|)}{|t|} & cos(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} cos(|t|) & \frac{\bar{t}sin(|t|)}{|t|} \\ -\frac{t sin(|t|)}{|t|} & cos(|t|) \end{pmatrix} \in SU(2).$$

On the other hand we have easily

$$U_K(t)a_1U_K(t)^{-1} = cosh(|t|)a_1 - \frac{tsinh(|t|)}{|t|}a_2^\dagger, \tag{49}$$

$$U_K(t)a_2^\dagger U_K(t)^{-1} = cosh(|t|)a_2^\dagger - \frac{\bar{t}sinh(|t|)}{|t|}a_1, \tag{50}$$

so the map $(a_1, a_2^\dagger) \longrightarrow (U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^\dagger U_K(t)^{-1})$ is

$$(U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^\dagger U_K(t)^{-1}) = (a_1, a_2^\dagger)\begin{pmatrix} cosh(|t|) & -\frac{\bar{t}sinh(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & cosh(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} cosh(|t|) & -\frac{\bar{t}sinh(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & cosh(|t|) \end{pmatrix} \in SU(1,1).$$

3 Some Topics in Quantum Information Theory

In this section we don't introduce a general theory of quantum information theory (see for example [8]), but focus our attension to special topics of it, that is,

- swap of coherent states
- cloning of coherent states

Because this is just a good one as examples of applications of coherent and generalized coherent states and our method developed in the following may open a new possibility. First let us define a swap operator :

$$S : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}, \quad S(a\otimes b) = b\otimes a \quad \text{for any } a, b \in \mathcal{H} \tag{51}$$

where $\mathcal{H}$ is the Fock space in Section 2.
It is not difficult to construct this operator in a universal manner, see [11] ; Appendix C. But for coherent states we can construct a better one by making use of generalized coherent operators in the preceding section.

Next let us introduce no cloning theorem, [17]. For that we define a cloning (copying) operator C which is unitary

$$C : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}, \quad C(h\otimes|0\rangle) = h\otimes h \quad \text{for any } h \in \mathcal{H}\ . \tag{52}$$

It is very known that there is no cloning theorem
"No Cloning Theorem" We have no C above.

The proof is very easy (almost trivial). Because $2h = h + h \in \mathcal{H}$ and C is a linear operator, so

$$C(2h \otimes |0\rangle) = 2C(h \otimes |0\rangle). \tag{53}$$

The LHS of (53) is

$$C(2h \otimes |0\rangle) = 2h \otimes 2h = 4(h \otimes h),$$

while the RHS of (53)

$$2C(h \otimes |0\rangle) = 2(h \otimes h).$$

This is a contradiction. This is called no cloning theorem.
Let us return to the case of coherent states. For coherent states $|\alpha\rangle$ and $|\beta\rangle$ the superposition $|\alpha\rangle + |\beta\rangle$ is no longer a coherent state, so that coherent states may not suffer from the theorem above.

Problem Is it possible to clone coherent states ?

At this stage it is not easy, so we will make do with approximating it (imperfect cloning in our terminology) instead of making a perfect cloning.
We write notations once more.

$$\begin{array}{rlll} \text{Coherent States} & |\alpha\rangle = D(\alpha)|0\rangle & \text{for} & \alpha \in \mathbf{C} \\ \text{Squeezed–like States} & |\beta\rangle = S(\beta)|0\rangle & \text{for} & \beta \in \mathbf{C} \end{array}$$

3.1 Some Useful Formulas

We list and prove some useful formulas in the following. Now we prepare some parameters α, ϵ, κ in which ϵ, κ are free ones, while α is unknown one in the cloning case. Let us unify the notations as follows.

$$\alpha : \text{(unknown)} \quad \alpha = |\alpha|e^{i\chi}, \tag{54}$$

$$\epsilon : \text{known} \quad \epsilon = |\epsilon|e^{i\phi}, \tag{55}$$

$$\kappa : \text{known} \quad \kappa = |\kappa|e^{i\delta}, \tag{56}$$

Let us start.
(i) First let us calculate

$$S(\epsilon)D(a)S(\epsilon)^{-1}. \tag{57}$$

For that we show

$$S(\epsilon)aS(\epsilon)^{-1} = cosh(|\epsilon|)a - e^{i\phi}sinh(|\epsilon|)a^{\dagger}. \tag{58}$$

Proof is as follows. For $X = (1/2)\{\epsilon(a^\dagger)^2 - \bar{\epsilon}a^2\}$ we have easily $[X, a] = -\epsilon a^\dagger$ and $[X, a^\dagger] = -\bar{\epsilon}a$, so

$$\begin{aligned} S(\epsilon)aS(\epsilon)^{-1} &= e^X a e^{-X} = a + [X, a] + \frac{1}{2!}[X, [X, a]] + \frac{1}{3!}[X, [X, [X, a]]] + \cdots \\ &= a - \epsilon a^\dagger + \frac{|\epsilon|^2}{2!}a - \frac{\epsilon|\epsilon|^2}{3!}a^\dagger + \cdots \\ &= \left\{1 + \frac{|\epsilon|^2}{2!} + \cdots\right\} a - \frac{\epsilon}{|\epsilon|}\left\{|\epsilon| + \frac{|\epsilon|^3}{3!} + \cdots\right\} a^\dagger \\ &= cosh(|\epsilon|)a - \frac{\epsilon sinh(|\epsilon|)}{|\epsilon|}a^\dagger = cosh(|\epsilon|)a - e^{i\phi} sinh(|\epsilon|)a^\dagger. \end{aligned}$$

From this it is easy to check

$$\begin{aligned} S(\epsilon)D(\alpha)S(\epsilon)^{-1} &= D\left(\alpha S(\epsilon)a^\dagger S(\epsilon)^{-1} - \bar{\alpha}S(\epsilon)aS(\epsilon)^{-1}\right) \\ &= D\left(cosh(|\epsilon|)\alpha + e^{i\phi} sinh(|\epsilon|)\bar{\alpha}\right). \end{aligned} \tag{59}$$

Therefore

$$S(\epsilon)D(\alpha)S(\epsilon)^{-1} = \begin{cases} D(e^{|\epsilon|}\alpha) & \text{if } \phi = 2\chi \\ D(e^{-|\epsilon|}\alpha) & \text{if } \phi = 2\chi + \pi \end{cases} \tag{60}$$

By making use of this formula we can change a scale of α.

(ii) Next le us calculate

$$S(\epsilon)S(\alpha)S(\epsilon)^{-1}. \tag{61}$$

From the definition

$$S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\epsilon)\exp\left\{\frac{1}{2}\left(\alpha(a^\dagger)^2 - \bar{\alpha}a^2\right)\right\} S(\epsilon)^{-1} \equiv e^{Y/2}$$

where

$$Y = \alpha\left(S(\epsilon)a^\dagger S(\epsilon)^{-1}\right)^2 - \bar{\alpha}\left(S(\epsilon)aS(\epsilon)^{-1}\right)^2.$$

From (58) and after some calculations we have

$$\begin{aligned} Y &= \left\{cosh^2(|\epsilon|)\alpha - e^{2i\phi} sinh^2(|\epsilon|)\bar{\alpha}\right\}(a^\dagger)^2 - \left\{cosh^2(|\epsilon|)\bar{\alpha} - e^{-2i\phi} sinh^2(|\epsilon|)\alpha\right\} a^2 \\ &+ \frac{(-e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha})}{2} sinh(2|\epsilon|)(a^\dagger a + a a^\dagger) \\ &= \left\{cosh^2(|\epsilon|)\alpha - e^{2i\phi} sinh^2(|\epsilon|)\bar{\alpha}\right\}(a^\dagger)^2 - \left\{cosh^2(|\epsilon|)\bar{\alpha} - e^{-2i\phi} sinh^2(|\epsilon|)\alpha\right\} a^2 \\ &+ (-e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha}) sinh(2|\epsilon|)(a^\dagger a + \frac{1}{2}) \quad (\Longleftarrow [a, a^\dagger] = 1), \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}Y &= \left\{cosh^2(|\epsilon|)\alpha - e^{2i\phi} sinh^2(|\epsilon|)\bar{\alpha}\right\} K_+ - \left\{cosh^2(|\epsilon|)\bar{\alpha} - e^{-2i\phi} sinh^2(|\epsilon|)\alpha\right\} K_- \\ &+ (-e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha}) sinh(2|\epsilon|)K_3 \end{aligned} \tag{62}$$

with $\{K_+, K_-, K_3\}$ in (25). This is our formula.
Now

$$-e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} = |\alpha|(-e^{-i(\phi-\chi)} + e^{i(\phi-\chi)}) = 2i|\alpha| sin(\phi-\chi),$$

so if we choose $\phi = \chi$, then $e^{2i\phi}\bar{\alpha} = e^{2i\chi}e^{-i\chi}|\alpha| = \alpha$ and

$$cosh^2(|\epsilon|)\alpha - e^{2i\phi} sinh^2(|\epsilon|)\bar{\alpha} = \left(cosh^2(|\epsilon|) - sinh^2(|\epsilon|)\right)\alpha = \alpha$$

, and finally

$$Y = \alpha(a^\dagger)^2 - \bar{\alpha}a^2.$$

That is,

$$S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\alpha) \Longleftrightarrow S(\epsilon)S(\alpha) = S(\alpha)S(\epsilon).$$

The operators $S(\epsilon)$ and $S(\alpha)$ commute if the phases of ϵ and α coincide.

(iii) Third formula is : For $V(t) = e^{itN}$ where $N = a^\dagger a$ (a number operator)

$$V(t)D(\alpha)V(t)^{-1} = D(e^{it}\alpha). \tag{63}$$

The proof is as follows.

$$V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha V(t)a^\dagger V(t)^{-1} - \bar{\alpha}V(t)aV(t)^{-1}\right).$$

It is easy to see

$$\begin{aligned} V(t)aV(t)^{-1} &= e^{itN}ae^{-itN} = a + [itN, a] + \frac{1}{2!}[itN,[itN,a]] + \cdots \\ &= a + (-it)a + \frac{(-it)^2}{2!}a + \cdots = e^{-it}a. \end{aligned}$$

Therefore we obtain

$$V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha e^{it}a^\dagger - \bar{\alpha}e^{-it}a^\dagger\right) = D(e^{it}\alpha).$$

This formula is often used as follows.

$$|\alpha\rangle \longrightarrow V(t)|\alpha\rangle = V(t)D(\alpha)V(t)^{-1}V(t)|0\rangle = D(e^{it}\alpha)|0\rangle = |e^{it}\alpha\rangle, \tag{64}$$

where we have used

$$V(t)|0\rangle = |0\rangle$$

becase $N|0\rangle = 0$. That is, we can add a phase to α by making use of this formula.

(iv) Fourth formula is : Let us calculate the following

$$U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = U_J(t)e^{\left\{\frac{\alpha}{2}(a_1^\dagger)^2 - \frac{\bar{\alpha}}{2}(a_1)^2 + \frac{\beta}{2}(a_2^\dagger)^2 - \frac{\bar{\beta}}{2}(a_2)^2\right\}}U_J(t)^{-1} = e^X \tag{65}$$

$$X = \frac{\alpha}{2}(U_J(t)a_1^\dagger U_J(t)^{-1})^2 - \frac{\bar{\alpha}}{2}(U_J(t)a_1 U_J(t)^{-1})^2 + \frac{\beta}{2}(U_J(t)a_2^\dagger U_J(t)^{-1})^2 - \frac{\bar{\beta}}{2}(U_J(t)a_2 U_J(t)^{-1})^2.$$

From (47) and (48) we have

$$\begin{aligned} X = &\frac{1}{2}\left\{cos^2(|t|)\alpha + \frac{t^2 sin^2(|t|)}{|t|^2}\beta\right\}(a_1^\dagger)^2 - \frac{1}{2}\left\{cos^2(|t|)\bar{\alpha} + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\bar{\beta}\right\}a_1^2 \\ &+ \frac{1}{2}\left\{cos^2(|t|)\beta + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\alpha\right\}(a_2^\dagger)^2 - \frac{1}{2}\left\{cos^2(|t|)\bar{\beta} + \frac{t^2 sin^2(|t|)}{|t|^2}\bar{\alpha}\right\}a_2^2 \\ &+ (\beta t - \alpha\bar{t})\frac{sin(2|t|)}{2|t|}a_1^\dagger a_2^\dagger - (\bar{\beta}\bar{t} - \bar{\alpha}t)\frac{sin(2|t|)}{2|t|}a_1 a_2. \end{aligned} \tag{66}$$

If we set

$$\beta t - \alpha\bar{t} = 0 \Longleftrightarrow \beta t = \alpha\bar{t}, \tag{67}$$

then it is easy to check

$$cos^2(|t|)\alpha + \frac{t^2 sin^2(|t|)}{|t|^2}\beta = \alpha, \quad cos^2(|t|)\beta + \frac{\bar{t}^2 sin^2(|t|)}{|t|^2}\alpha = \beta,$$

so, in this case,

$$X = \frac{1}{2}\alpha(a_1^\dagger)^2 - \frac{1}{2}\bar{\alpha}a_1^2 + \frac{1}{2}\beta(a_2^\dagger)^2 - \frac{1}{2}\bar{\beta}a_2^2 .$$

Therefore

$$U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = S_1(\alpha)S_2(\beta). \tag{68}$$

That is, $S_1(\alpha)S_2(\beta)$ commutes with $U_J(t)$ under the condition (67).

3.2 Swap of Coherent States

The purpose of this section is to construct a swap operator satifying

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle. \tag{69}$$

Let us remember $U_J(\kappa)$ once more

$$U_J(\kappa) = e^{\kappa a_1^\dagger a_2 - \bar{\kappa} a_1 a_2^\dagger} \quad \text{for} \quad \kappa \in \mathbf{C}.$$

We note an important property of this operator :

$$U_J(\kappa)|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle. \tag{70}$$

The construction is as follows.

$$\begin{aligned} U_J(\kappa)|\alpha_1\rangle \otimes |\alpha_2\rangle &= U_J(\kappa)D(\alpha_1) \otimes D(\alpha_2)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}U_J(\kappa)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \quad \text{by} \quad (70), \end{aligned} \tag{71}$$

and

$$\begin{aligned} &U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1} = U_J(\kappa)\mathrm{exp}\left\{\alpha_1 a_1^\dagger - \bar{\alpha}_1 a_1 + \alpha_2 a_2^\dagger - \bar{\alpha}_2 a_2\right\} U_J(\kappa)^{-1} \\ &= \mathrm{exp}\left\{\alpha_1 (U_J(\kappa)a_1 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_1 U_J(\kappa)a_1 U_J(\kappa)^{-1} \right. \\ &\qquad \left. +\alpha_2 (U_J(\kappa)a_2 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_2 U_J(\kappa)a_2 U_J(\kappa)^{-1}\right\} \\ &\equiv \mathrm{exp}(X). \end{aligned} \tag{72}$$

From (47) and (48) we have

$$\begin{aligned} X = &\left\{cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right\} a_1^\dagger - \left\{cos(|\kappa|)\bar{\alpha}_1 + \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\bar{\alpha}_2\right\} a_1 \\ &+ \left\{cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right\} a_2^\dagger - \left\{cos(|\kappa|)\bar{\alpha}_2 - \frac{\kappa sin(|\kappa|)}{|\kappa|}\bar{\alpha}_1\right\} a_2, \end{aligned}$$

so

$$\begin{aligned} \mathrm{exp}(X) &= D_1\left(cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right) D_2\left(cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right) \\ &= D\left(cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\right) \otimes D\left(cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\right). \end{aligned}$$

Therefore we have from (72)

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |cos(|\kappa|)\alpha_1 + \frac{\kappa sin(|\kappa|)}{|\kappa|}\alpha_2\rangle \otimes |cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} sin(|\kappa|)}{|\kappa|}\alpha_1\rangle.$$

If we write κ as $|\kappa|\mathrm{e}^{i\delta}$, then the above formula reduces to

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |cos(|\kappa|)\alpha_1 + \mathrm{e}^{i\delta} sin(|\kappa|)\alpha_2\rangle \otimes |cos(|\kappa|)\alpha_2 - \mathrm{e}^{-i\delta} sin(|\kappa|)\alpha_1\rangle.$$

Here if we choose $sin(|\kappa|) = 1$, then

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\mathrm{e}^{i\delta}\alpha_2\rangle \otimes |-\mathrm{e}^{-i\delta}\alpha_1\rangle = |\mathrm{e}^{i\delta}\alpha_2\rangle \otimes |\mathrm{e}^{-i(\delta+\pi)}\alpha_1\rangle.$$

Now by operating the operator $V = \mathrm{e}^{-i\delta N} \otimes \mathrm{e}^{i(\delta+\pi)N}$ where $N = a^\dagger a$ from the left (see (64)) we obtain the swap

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle.$$

A comment is in order. In the formula we set $\alpha_1 = \alpha$ and $\alpha_2 = 0$, then the formula reduces to

$$U_J(\kappa)D_1(\alpha)U_J(\kappa)^{-1} = D_1(cos(|\kappa|)\alpha)D_2(-\mathrm{e}^{-i\delta} sin(|\kappa|)\alpha). \tag{73}$$

3.3 Imperfect Cloning of Coherent States

We cannot clone coherent states in a perfect manner likely

$$|\alpha\rangle \otimes |0\rangle \longrightarrow |\alpha\rangle \otimes |\alpha\rangle \quad \text{for } \alpha \in \mathbf{C}. \tag{74}$$

Then our question is : is it possible to approximate ? We show that we can at least make an "imperfect cloning" in our terminology against the statement of [18].
Let us start. The method is almost same with one in the preceding subsection, but we repeat it once more. Operating the operator $U_J(\kappa)$ on $|\alpha\rangle \otimes |0\rangle$

$$\begin{aligned}
&U_J(\kappa)|\alpha\rangle \otimes |0\rangle = U_J(\kappa)\{D(\alpha)\otimes \mathbf{1}\}\,|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha)|0\rangle \otimes |0\rangle \\
=&U_J(\kappa)D_1(\alpha)U_J(\kappa)^{-1}U_J(\kappa)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \quad \text{by (70)} \\
=&D_1(cos(|\kappa|)\alpha)D_2(-\mathrm{e}^{-i\delta}sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \quad \text{by (73)} \\
=&D_1(cos(|\kappa|)\alpha)D_2(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \\
=&\left\{D(cos(|\kappa|)\alpha)\otimes D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)\right\}|0\rangle \otimes |0\rangle.
\end{aligned}$$

Operating the operator $\mathbf{1}\otimes \mathrm{e}^{i(\delta+\pi)N}$ on the last equation

$$\begin{aligned}
&D(cos(|\kappa|)\alpha)\otimes \mathrm{e}^{i(\delta+\pi)N}D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \\
=&D(cos(|\kappa|)\alpha)\otimes \mathrm{e}^{i(\delta+\pi)N}D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)\mathrm{e}^{-i(\delta+\pi)N}\mathrm{e}^{i(\delta+\pi)N}|0\rangle \otimes |0\rangle \\
=&D(cos(|\kappa|)\alpha)\otimes \mathrm{e}^{i(\delta+\pi)N}D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha)\mathrm{e}^{-i(\delta+\pi)N}|0\rangle \otimes |0\rangle \\
=&D(cos(|\kappa|)\alpha)\otimes D(\mathrm{e}^{-i(\delta+\pi)}sin(|\kappa|)\alpha\mathrm{e}^{i(\delta+\pi)})|0\rangle \otimes |0\rangle \quad \text{by (63)} \\
=&D(cos(|\kappa|)\alpha)\otimes D(sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \\
=&|cos(|\kappa|)\alpha\rangle \otimes |sin(|\kappa|)\alpha\rangle.
\end{aligned}$$

Namely we have constructed

$$|\alpha\rangle \otimes |0\rangle \longrightarrow |cos(|\kappa|)\alpha\rangle \otimes |sin(|\kappa|)\alpha\rangle. \tag{75}$$

This is an "imperfect cloning" what we have called.

A comment is in order. The authors in [18] state that the "perfect cloning" (in their terminology) for coherent states is possible. But it is not correct as shown in [11]. Nevertheless their method is simple and very interesting, so it may be possible to modify their "proof" more subtly by making use of (60).

Problem Is it possible to make a "perfect cloning" in the sense of [18] ?

3.4 Swap of Squeezed–like States ?

We would like to construct an operator like

$$|\beta_1\rangle \otimes |\beta_2\rangle \longrightarrow |\beta_2\rangle \otimes |\beta_1\rangle. \tag{76}$$

In this case we cannot use an operator $U_J(\kappa)$. Let us explain the reason.
Similar to (71)

$$\begin{aligned} U_J(\kappa)|\beta_1\rangle \otimes |\beta_2\rangle &= U_J(\kappa)S(\beta_1) \otimes S(\beta_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)S_1(\beta_1)S_2(\beta_2)|0\rangle \otimes |0\rangle \\ &= U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle. \end{aligned} \tag{77}$$

On the other hand by (65)

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = e^X,$$

where

$$\begin{aligned} X = &\frac{1}{2}\left\{cos^2(|\kappa|)\beta_1 + \frac{\kappa^2 sin^2(|\kappa|)}{|\kappa|^2}\beta_2\right\}(a_1^\dagger)^2 - \frac{1}{2}\left\{cos^2(|\kappa|)\bar{\beta}_1 + \frac{\bar{\kappa}^2 sin^2(|\kappa|)}{|\kappa|^2}\bar{\beta}_2\right\}a_1^2 \\ &+ \frac{1}{2}\left\{cos^2(|\kappa|)\beta_2 + \frac{\bar{\kappa}^2 sin^2(|\kappa|)}{|\kappa|^2}\beta_1\right\}(a_2^\dagger)^2 - \frac{1}{2}\left\{cos^2(|\kappa|)\bar{\beta}_2 + \frac{\kappa^2 sin^2(|\kappa|)}{|\kappa|^2}\bar{\beta}_1\right\}a_2^2 \\ &+ (\beta_2\kappa - \beta_1\bar{\kappa})\frac{sin(2|\kappa|)}{2|\kappa|}a_1^\dagger a_2^\dagger - (\bar{\beta}_2\bar{\kappa} - \bar{\beta}_1\kappa)\frac{sin(2|\kappa|)}{2|\kappa|}a_1 a_2 \ . \end{aligned}$$

Here an extra term containing $a_1^\dagger a_2^\dagger$ appeared. To remove this we must set $\beta_2\kappa - \beta_1\bar{\kappa} = 0$, but in this case we meet

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = S_1(\beta_1)S_2(\beta_2)$$

by (68). That is, there is no change.
We could not construct an operator likely in the subsection 3.2 in spite of very our efforts , so we present

Problem Is it possible to find an operator such as $U_J(\kappa)$ in the preceding subsection for performing the swap ?

参考文献

[1] J. R. Klauder and Bo-S. Skagerstam (Eds) : Coherent States, World Scientific, Singapore, 1985.

[2] L. Mandel and E. Wolf : Optical Coherence and Quantum Optics, Cambridge University Press, 1995.

[3] A. Perelomov : Generalized Coherent States and Their Applications, Springer–Verlag, 1986.

[4] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Coherent states, path integral, and semiclassical approximation, J. Math. Phys., 36(1995), 3232.

[5] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Exactness in the Wentzel-Kramers-Brillouin approximation for some homogeneous spaces, J. Math. Phys., 36(1995), 4590.

[6] K. Fujii, T. Kashiwa, S. Sakoda :Coherent states over Grassmann manifolds and the WKB exactness in path integral, J. Math. Phys., 37(1996), 567.

[7] A. Hosoya : Lectures on Quantum Computation (in Japanese), 1999, Science Company (in Japan).

[8] H-K. Lo, S. Popescu and T. Spiller (eds) : Introduction to Quantum Computation and Information, 1998, World Scientific.

[9] A. Steane : Quantum Computing, Rept. Prog. Phys, 61(1998), 117.

[10] K. Fujii : Introduction to Grassmann Manifolds and Quantum Computation, quant-ph/0103011.

[11] K. Fujii : Introduction to Coherent States and Quantum Information Theory, quant-ph/0112090.

[12] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer II, quant-ph/0101102.

[13] M. G. A. Paris : Entanglement and visibility at the output of a Mach–Zehnder interferometer, quant-ph/9811078.

[14] K. Fujii and T. Suzuki : A Universal Disentangling Formula for Coherent States of Perelomov's Type, hep-th/9907049.

[15] K. Fujii : Basic Properties of Coherent and Generalized Coherent Operators Revisited, Mod. Phys. Lett. A, 16(2001), 1277, quant-ph/0009012.

[16] K. Fujii : Note on Extended Coherent Operators and Some Basic Properties, quant-ph/0009116.

[17] W.K.Wootters and W.H.Zurek : A single quantum cannot be cloned, Nature 299(1982), 802.

[18] N.D. Dass and P. Ganesh : Perfect cloning of harmonic oscillator coherent states is possible, quant-ph/0108090.