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Gevrey Regularity of Solutions of Semilinear Hypoelliptic Equations on the Plane

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§1. Introduction.

In this note we discuss the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations of Grushin's type on $\mathbb{R}^2$. Most of the results will appear in [1]. Some results are new and they are presented here for the first time. We confine ourself with consideration of a model equation. Precisely, we will consider the following equation

(1) $$G_{k,\lambda}f+\Psi\Big(x,y,f,\frac{\partial f}{\partial x},x^k\frac{\partial f}{\partial y}\Big)=0 \text{ in a domain } \Omega\subset\mathbb{R}^2,$$

where

$$G_{k,\lambda}=\frac{\partial^2}{\partial x^2}+x^{2k}\frac{\partial^2}{\partial y^2}+i\lambda x^{k-1}\frac{\partial}{\partial y}$$

with $(x,y)\in\Omega\subset\mathbb{R}^2,\lambda\in\mathbb{C},i=\sqrt{-1}$ and $k\in\mathbb{Z}_+,\Omega$ is a bounded domain in $\mathbb{R}^2$. Let us define the following quantities

$$R=\left(x^{k+1}+u^{k+1}\right)^2+(k+1)^2(y-v)^2, p=\frac{4x^{k+1}u^{k+1}}{R},$$

$$A_+=x^{k+1}+u^{k+1}+i(k+1)(y-v), A_-=x^{k+1}+u^{k+1}-i(k+1)(y-v),$$

$$M=A_+^{-\frac{k+\lambda}{2k+2}}A_-^{-\frac{k-\lambda}{2k+2}},$$

here we take $z_1^{z_2}=e^{z_2\ln z_1}$ for $z_1,z_2\in\mathbb{C}$ and if $z_1=re^{i\varphi},-\pi<\varphi\leq\pi$ then $\ln z_1=\ln r+i\varphi$. First, we will find the uniform fundamental solution of $G_{k,\lambda}$, that is

$$G_{k,\lambda}F_{k,\lambda}(x,y,u,v)=\delta(x-u,y-v),$$

in the following form

$$F_{k,\lambda}(x,y,u,v)=F(p)M.$$

After some computations we arrive at

$$G_{k,\lambda}F_{k,\lambda}=16(k+1)^2u^{2k+2}x^{2k}\Big[\left(u^{k+1}-x^{k+1}\right)^2+(k+1)^2(y-v)^2\Big]\times$$
$$\times MR^{-3}F''(p)++4(k+1)x^{k-1}u^{k+1}[k\left(x^{2k+2}+u^{2k+2}+(k+1)^2(y-v)^2\right)-$$
$$-(6k+4)x^{k+1}u^{k+1}]MR^{-2}F'(p)++(\lambda^2-k^2)x^{k-1}u^{k+1}MR^{-1}F(p).$$

[1] N. M. Tri, To appear in J. Math. Sci. Univ. Tokyo.

Therefore, if $F(p)$ satisfies the following hypergeometric equation

(2) $$p(1-p)F''(p)+[c-(1+a+b)p]F'(p)-abF(p)=0,$$

with $a=\frac{k+\lambda}{2k+2}, b=\frac{k-\lambda}{2k+2}, c=\frac{k}{k+1}$, then formally we will have

$$G_{k,\lambda}F_{k,\lambda}=0.$$

The general solutions of (2) are

$$F(p)=C_1F\Big(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\Big)+C_2p^{\frac{1}{k+1}}F\Big(\frac{k+2+\lambda}{2k+2},\frac{k+2-\lambda}{2k+2},\frac{k+2}{k+1},p\Big),$$

where $F(a,b,c,p)$ is the Gauss hypergeometric function and C_1, C_2 are some complex constants [2].

§2. Case k is odd.

Since k is odd, we note that $0\le p\le 1$. Moreover, $p=1$ if and only if $x=\pm u\neq 0, y=v$. If $u=0, v=0$ then $p=0$; therefore, from the result of [3]

$$G_{k,\lambda}F\Big(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\Big)M=-\frac{2^{2+\frac{1}{k+1}}\pi\Gamma(\frac{k}{k+1})}{\Gamma(\frac{k+\lambda}{2k+2})\Gamma(\frac{k-\lambda}{2k+2})}\delta(x,y)$$

we should choose

$$C_1=-\frac{\Gamma\Big(\frac{k+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k-\lambda}{2k+2}\Big)}{2^{2+\frac{1}{k+1}}\pi\Gamma\Big(\frac{k}{k+1}\Big)}.$$

If $u\neq 0$ then the singularities of $F_{k,\lambda}(x,y,u,v)$ will be located at the one of $F(p)$. On the other hand, $F(p)$, with $0\le p\le 1$, has singularity only when $p=1$. As $p\to 1$ we have the following asymptotic expansions (see [2])

$$F\Big(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\Big)=-\frac{\Gamma\Big(\frac{k}{k+1}\Big)}{\Gamma\Big(\frac{k+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k-\lambda}{2k+2}\Big)}\log(1-p)+O(1),$$

$$F\Big(\frac{k+2+\lambda}{2k+2},\frac{k+2-\lambda}{2k+2},\frac{k+2}{k+1},p\Big)=-\frac{\Gamma\Big(\frac{k+2}{k+1}\Big)}{\Gamma\Big(\frac{k+2+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k+2-\lambda}{2k+2}\Big)}\log(1-p)+O(1).$$

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[2] H. Bateman, and A. Erdelyi, 1953, vol I, p. 74.
[3] N. M. Tri, J. Math. Sci. Univ. Tokyo, vol. 6, 1999, pp. 437-452.

We expect that $F_{k,\lambda}(x,y,u,v)$ has singularity only when $x = u, y = v$. Since $p^{\frac{1}{k+1}} = (4R^{-1})^{\frac{1}{k+1}}xu \to -1$ when $(x,y) \to (-u,v)$, we should choose

$$C_2 = -\frac{\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)}$$

such that $F(p)$ has no singularity at $x = -u, y = v$. Note that the following conditions

$$\lambda \neq \pm[2N(k+1)+k], \lambda \neq \pm[2N(k+1)+k+2], \tag{3}$$

where N is a non-negative integer, guarantee that $C_1, C_2 < \infty$ and hence $F(p)$ has logarithm growth (if $u \neq 0$) at $(x,y) = (u,v)$.

Definition. *The parameter λ is called admissible if λ satisfies the condition (3).*

Therefore, if λ is admissible then we expect that the function $F(p)M$, or

$$F_{k,\lambda}(x,y,u,v) = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)F\left(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)A_+^{\frac{k+\lambda}{2k+2}}A_-^{\frac{k-\lambda}{2k+2}}} - \frac{xu\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)F\left(\frac{k+2+\lambda}{2k+2},\frac{k+2-\lambda}{2k+2},\frac{k+2}{k+1},p\right)}{2^{2-\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)A_+^{\frac{k+2+\lambda}{2k+2}}A_-^{\frac{k+2-\lambda}{2k+2}}},$$

will be our desired uniform fundamental solution. Indeed, we have

Theorem 1. *Assume that λ is admissible. Then*

$$G_{k,\lambda}F_{k,\lambda}(x,y,u,v) = \delta(x-u, y-v).$$

Remark 1. A similar expression for $F_{k,0}$ is also given in [4].

Let us denote $X_1' = \frac{\partial}{\partial u} - iu^k\frac{\partial}{\partial v}, X_2' = \frac{\partial}{\partial u} + iu^k\frac{\partial}{\partial v}$, and $G_{k,\lambda}' = X_2'X_1' + i(\lambda + k)u^{k-1}\frac{\partial}{\partial v}$. Noting that $F_{k,\lambda}(x,y,u,v) = F_{k,-\lambda}(u,v,x,y)$, from Theorem 1 we can easily deduce

[4] R. Beals, Journées Équations aux dérivées partielles, Saint-Jean-de-Monts, 1998, pp. I1–I0

Proposition 1 (Representation formula). *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary, $f \in C^2(\bar{\Omega})$ and λ is admissible then we have*

$$(4)\quad f(x,y) = \int_{\Omega} F_{k,\lambda}(x,y,u,v) G'_{k,\lambda} f(u,v)dudv-$$
$$-\int_{\partial\Omega} F_{k,\lambda}(x,y,u,v) B'_1(f(u,v),k,-\lambda)ds + \int_{\partial\Omega} f(u,v) B'_2(F_{k,\lambda}(x,y,u,v),k)ds,$$

where

$$B'_1(f(u,v),k,-\lambda) = \left(\nu_1 - iu^k\nu_2\right) X'_2 f(u,v) - i(-\lambda+k)u^{k-1}\nu_2 f(u,v),$$
$$B'_2(F_{k,\lambda}(x,y,u,v),k) = \left(\nu_1 + iu^k\nu_2\right) X'_1 F_{k,\lambda}(x,y,u,v),$$

and $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector on $\partial\Omega$.

Now, we re-state a well-known theorem on hypoellipticity of $G_{k,\lambda}$ as follows

Theorem 2. *$G_{k,\lambda}$ is hypoelliptic if and only if the hypergeometric equation (2) has no bounded solution on the interval* $[0,1]$.

Proof. Here, with the help of $F_{k,\lambda}$, we give a proof, which is alternative to a well-known classical proof based on the theory of pseudo-differential operators. Suppose that $f \in C^2(\bar{\Omega})$ and $G_{k,\lambda}f(x,y) = h(x,y)$ where $h \in C^\infty(\bar{\Omega})$. Then we can express f through h as in (4), with $G'_{k,\lambda}f(u,v)$ replaced by $h(u,v)$. It is clear that the boundary integrals give $C^\infty(\Omega)$ functions. For the volume integral, we see that $\frac{\partial F_{k,\lambda}}{\partial y} = -\frac{\partial F_{k,\lambda}}{\partial v}$. Therefore, by integration by parts, we can differentiate the integral in x one time and in y as many times as we want to. And the resulting functions are continuous. We will complete the proof if we are able to show that if $f \in C^{n-1}(\Omega)$ then $f \in C^n(\Omega)$ for every positive integer n. This is the case because we already have $\frac{\partial^n f}{\partial y^n}, \frac{\partial^n f}{\partial y^{n-1}\partial x}$ and $\frac{\partial^{\alpha+\beta} u}{\partial y^\alpha \partial x^\beta}, \alpha+\beta \le n-1$ belong to $C(\Omega)$ from the above argument and assumption. We have to show that $\frac{\partial^n u}{\partial y^{n-2}\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n} \in C(\Omega)$. Suppose that all the derivatives $\frac{\partial^n f}{\partial y^n}, \frac{\partial^n f}{\partial y^{n-1}\partial x}, \ldots, \frac{\partial^n f}{\partial y^{n-j}\partial x^j}, 1 \le j \le n-1$ are continuous. We shall prove that $\frac{\partial^n f}{\partial y^{n-j-1}\partial x^{j+1}} \in C(\Omega)$. Indeed, we have

$$(5)\qquad \frac{\partial^2 f}{\partial x^2} = h - x^{2k}\frac{\partial^2 f}{\partial y^2} - i\lambda x^{k-1}\frac{\partial f}{\partial y}.$$

Therefore, differentiating $\frac{\partial^{n-2}}{\partial y^{n-j-1}\partial x^{j-1}}$ both sides of (5) gives

$$\frac{\partial^n f}{\partial y^{n-j-1}\partial x^{j+1}} = \frac{\partial^{n-2} h}{\partial y^{n-j-1}\partial x^{j-1}} -$$
$$- \sum_{i=0}^{j} \binom{j}{i} 2k(2k-1)\cdots(2k-i+1)x^{2k-i}\frac{\partial^{n-i} f}{\partial y^{n-j+1}\partial x^{j-i-1}} -$$
$$- i\lambda \sum_{i=0}^{j} \binom{j}{i} (k-1)(k-2)\cdots(k-i)x^{k-i-1}\frac{\partial^{n-i-1} f}{\partial y^{n-j}\partial x^{j-i-1}} \in C(\Omega).\square$$

Actually, a more detailed examination of the proof of Theorem 2 would show that the integral operators

$$K: \ h \longrightarrow K(h)(x,y) = \int_\Omega F_{k,\lambda}(x,y,u,v)h(u,v)dudv,$$
$${}^tK: \ h \longrightarrow {}^tK(h)(x,y) = \int_\Omega F_{k,\lambda}(u,v,x,y)h(u,v)dudv$$

map $C_0^\infty(\Omega)$ into $C^\infty(\Omega)$. In other words, K and ${}^tK$ are separately regular. Since $F_{k,\lambda}$ is a C^∞ function in the complement of the diagonal of $\Omega \times \Omega$, we conclude that K and ${}^tK$ are very regular.

Next, we introduce some notations

$$\Xi_t = \{(\alpha,\beta,\gamma) \in \mathbb{Z}_+^3 : \alpha+\beta \le t, kt \ge \gamma \ge \alpha + (1+k)\beta - t\}.$$

For a function $f(x,y)$ on $\mathbb{R}^2$, we write $\partial_1^\alpha f, \partial_2^\beta f, \partial_{1,2}^{\alpha,\beta} f$, ${}_\gamma\partial_{\alpha,\beta} f$ for $\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}$, $\frac{\partial^\beta f(x,y)}{\partial y^\beta}$, $\frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, $x^\gamma \frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, respectively. For $m \in \mathbb{Z}^+$, let us denote by $\mathbb{H}_{loc}^m(\Omega)$ the space of all function $f \in L_{loc}^2(\Omega)$ such that for any compact K of Ω we have $\sum_{(\alpha,\beta,\gamma)\in\Xi_m} \|{}_\gamma\partial_{\alpha,\beta} f\|_{L^2(K)} < \infty$. Now we are in a position to formulate the main theorem of this section.

Theorem 3. *Assume that $m \ge 2k^2+6k+5$. Let f be a $\mathbb{H}_{loc}^m(\Omega)$ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f.*

Proof. The proof of Theorem 3 consists of Theorem 4 and Theorem 5 below. The proof follows the scheme : $f \in \mathbb{H}_{loc}^m \Longrightarrow f \in C^\infty(\Omega) \Longrightarrow f \in A(\Omega)$. □

Theorem 4. *Let Ψ be a C^∞–function of its arguments and $m \ge 2k^2+6k+5$. Assume that $f \in \mathbb{H}_{loc}^m(\Omega)$ is a solution of the equation (1) then $f \in C^\infty(\Omega)$.*

Proof. Theorem 4 can be proved with the help of Proposition 2 . □

Proposition 2. *Let $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbb{H}^m_{loc}(\Omega)$. Then $\Psi(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}) \in \mathbb{H}^{m-1}_{loc}(\Omega)$.*

Next, put $r_0 = 2k+2$. For $r \in \mathbb{Z}_+$ let Γ_r denote the set of pairs of multi-indices (α, β) such that $\Gamma_r = \Gamma_r^1 \cup \Gamma_r^2$ where

$$\Gamma_r^1 = \{(\alpha, \beta) : \alpha \leq r_0, 2\alpha + \beta \leq r\}, \Gamma_r^2 = \{(\alpha, \beta) : \alpha \geq r_0, \alpha + \beta \leq r - r_0\}.$$

Define the following norm

$$|f, \Omega|_r = \max_{(\alpha,\beta)\in\Gamma_r} |\partial_1^\alpha \partial_2^\beta f, \Omega| + \max_{\substack{(\alpha,\beta)\in\Gamma_r \\ \alpha\geq 1,\beta\geq 1}} \max_{(x,y)\in\bar{\Omega}} |\partial_1^{\alpha+2} \partial_2^\beta f|,$$

where $|f, \Omega| = \max_{(x,y)\in\bar{\Omega}} \left(|f| + \left|\frac{\partial f}{\partial x}\right| + \left|x^k \frac{\partial f}{\partial y}\right|\right)$.

Theorem 5. *Let f be a C^∞ solution of the equation (1) and $\Psi \in G^s$. Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f.*

Proof. Theorem 5 can be proved with the help of Proposition 3, Corollary 1, Lemmas 2-4. □

Proposition 3. *Assume that $\Psi \in G^s$. Then there exist constants C, D such that for every $H_0 \geq 1, H_1 \geq C H_0^{2k+3}$ if*

$$|f, \Omega|_d \leq H_0 H_1^{(d-r_0-2)} (d - r_0 - 2)!^s, \quad 0 \leq d \leq N+1, r_0 + 2 \leq N$$

then

$$\max_{(x,y)\in\bar{\Omega}} \left|\partial_1^\alpha \partial_2^\beta \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right)\right| \leq D H_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Corollary 1. *Under the same hypotheses of Proposition 3 with $d \leq N+1$ replaced by $d \leq N$, then*

$$\max_{x\in\bar{\Omega}} \left|\partial_1^\alpha \partial_2^\beta \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right)\right| \leq D\left(|f, \Omega|_{N+1} + H_0 H_1^{N-r_0-1} (N - r_0 - 1)!^s\right)$$

for every $(\alpha, \beta) \in \Gamma_{N+1}$.

Since $G_{k,\lambda}$ is elliptic if $x \neq 0$, it suffices to consider the case $(0,0) \in \Omega$ and Ω is a small neighborhood of $(0,0)$. Let us define the distance

$$\rho((u, v), (x, y)) = \begin{cases} \max\left\{|x^{k+1} - u^{k+1}|, (k+1)|y - v|\right\}, & \text{for } xu \geq 0 \\ \max\left\{x^{k+1} + u^{k+1}, (k+1)|y - v|\right\}, & \text{for } xu \leq 0. \end{cases}$$

For two sets S_1, S_2, the distance between them is defined as

$$\rho(S_1, S_2) = \inf_{(x,y)\in S_1,(u,v)\in S_2} \rho((x,y),(u,v)).$$

Let $V^T (T \leq 1)$ be the cube with edges of size (in the ρ metric) $2T$ which are parallel to the coordinate axes and centered at $(0,0)$. Denote by V_δ^T the sub-cube which is homothetic with V^T and such that the distance between its boundary and the boundary of V^T is δ. We shall prove by induction that if T is small enough then there exist constants H_0, H_1 with $H_1 \geq CH_0^{2k+3}$ such that

$$|f, V_\delta^T|_n \leq H_0 \quad \text{for} \quad 0 \leq n \leq 6k+4, \tag{6}$$

and

$$|f, V_\delta^T|_n \leq H_0 \Big(\frac{H_1}{\delta}\Big)^{n-r_0-2} (n-r_0-2)!^s := Q_{n-1} \tag{7}$$

for $n \geq 6k+4$, and δ sufficiently small. Hence the desired conclusion follows. (6) follows easily from the C^∞ smoothness assumption on f. Assume that (7) holds for $n = N$. We shall prove it for $n = N+1$. Put $\delta' = \delta(1-1/N), \delta'' = \delta(1-4/N)$. Fix $(x,y) \in V_\delta^T$ and then define $\sigma = \rho((x,y), \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}(x,y)$ denote the cube with center at (x,y) and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes, and $S_{\tilde{\sigma}}(x,y)$ the boundary of $V_{\tilde{\sigma}}(x,y)$. Note that $\sigma \geq \delta$, and $V_{\tilde{\sigma}}(x,y) \subset V_{\delta'}^T$. Let $E_1, E_3 (E_2, E_4)$ be edges of $S_{\tilde{\sigma}}(x,y)$ which are parallel to $Ox (Oy)$ respectively. We have to estimate $\max_{(x,y)\in V_\delta^T} |{}_\gamma\partial_{\alpha,\beta}(\partial_1^{\alpha_1}\partial_2^{\beta_1} f)|$ for all $(\alpha,\beta,\gamma) \in \Xi_1, (\alpha_1,\beta_1) \in \Gamma_{N+1}$, and $\max_{(x,y)\in V_\delta^T} |(\partial_1^{2+\alpha_1}\partial_2^{\beta_1} f)|$ for all $(\alpha_1,\beta_1) \in \Gamma_{N+1}, \alpha_1 \geq 1, \beta_1 \geq 1$. Let us abbreviate $\frac{\partial^\alpha}{\partial u^\alpha}, \frac{\partial^\beta}{\partial v^\beta}, \frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}$ as $\partial_u^\alpha, \partial_v^\beta, \partial_u^\alpha\partial_v^\beta$, respectively.

Lemma 2. *Assume that* $(\alpha,\beta,\gamma) \in \Xi_1$ *and* $(\alpha_1,\beta_1) \in \Gamma_{N+1}$. *Then if* $\alpha_1 \geq 1, \beta_1 \geq 1$ *there exists a constant* C *such that*

$$\max_{(x,y)\in V_\delta^T} |{}_\gamma\partial_{\alpha,\beta}(\partial_1^{\alpha_1}\partial_2^{\beta_1} f(x,y))| \leq C\Big(T^{\frac{1}{k+1}}|f, V_{\delta'}^T|_{N+1} + Q_N\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\Big).$$

Lemma 3. *Assume that* $(\alpha,\beta,\gamma) \in \Xi_1$. *Then there exists a constant* C *such that*

$$\max_{(x,y)\in V_\delta^T} |{}_\gamma\partial_{\alpha,\beta}(\partial_2^{N+1} f(x,y))| \leq C\Big(T^{\frac{1}{k+1}}|f, V_{\delta''}^T|_{N+1} + Q_N\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\Big).$$

Lemma 4. *Assume that* $(\alpha,\beta,\gamma)\in\Xi_1$. *Then there exists a constant* C *such that*

$$\max_{(x,y)\in V_\delta^T}\left|\,{}_\gamma\partial_{\alpha,\beta}(\partial_1^{N-r_0+1}f(x,y))\right|\le C\left(T^{\frac{1}{k+1}}|f,V_{\delta'}^T|_{N+1}+Q_N\left(T^{\frac{1}{k+1}}+\frac{1}{H_1}\right)\right).$$

Lemma 5. *Assume that* $(\alpha_1,\beta_1)\in\Gamma_{N+1}\backslash\Gamma_N,\alpha_1\ge 1,\beta_1\ge 1$. *Then there exists a constant* C *such that*

$$\max_{(x,y)\in V_\delta^T}\left|\,(\partial_1^{\alpha_1+2}\partial_2^{\beta_1}f(x,y))\right|\le C\left(T^{\frac{1}{k+1}}|f,V_{\delta''}^T|_{N+1}+Q_N\left(T^{\frac{1}{k+1}}+\frac{1}{H_1}\right)\right).$$

§3. Case k is even.

A. First, we consider the case $\lambda = 2N(k+1)$, where N is an integer. In this case we will prove a similar result as in §2 by establishing the explicit uniform fundamental solutions of $G_{k,2N(k+1)}$. Let us maintain the notations used for $p, A_+, A_-, M, F_{k,\lambda},\ldots$ from the very beginning of the paper (now, of course, with an even k). If $(u,v)\ne(0,v)$ is fixed then the real parts of A_+,A_- change sign when (x,y) passes through $(-u,v)$. Therefore, $M=A_+^{-\frac{k+\lambda}{2k+2}}A_-^{-\frac{k-\lambda}{2k+2}}$ may have singularities alone the half-line (x,v) with $x\le -u$ for an arbitrary complex number λ. But if $\lambda=2N(k+1)$ then it is not difficult to see that $M=A_+^{-\frac{k+\lambda}{2k+2}}A_-^{-\frac{k-\lambda}{2k+2}}$ is smooth alone the half-line (x,v) with $x<-u$, that is $M(\cdot,\cdot,u,v)\in C^\infty(\mathbb{R}^2\backslash\{(u,v),(-u,v)\})$. Moreover, when k is even and $u\ne 0$ we have $-\infty\le p\le 1$. More precisely, $p\to 1$ when $(x,y)\to(u,v)$, and $p\to-\infty$ when $(x,y)\to(-u,v)$. If $N<0$ and $p\to-\infty$ then from the asymptotic expansions of hypergeometric functions (see [2], p. 63) we should choose the expressions for constants C_1,C_2 as in the beginning of the paper (with λ replaced by $2N(k+1)$). And we will have $F_{k,2N(k+1)}(\cdot,\cdot,u,v)\in C^\infty(\mathbb{R}^2\backslash(u,v))$, with

$$F_{k,2N(k+1)}(-u,v,u,v)=0.$$

Similar conclusions hold for $F_{k,2N(k+1)}(x,y,u,v)$ when $N>0$. If $N=0$ then $F_{k,0}(\cdot,\cdot,u,v)\in C^\infty(\mathbb{R}^2\backslash(u,v))$, with

$$F_{k,0}(-u,v,u,v)=-\frac{\cot\frac{k\pi}{2k+2}}{4u^k}.$$

Theorem 6. *Let $\Psi \in G^s$. Assume that $m \geq 2k^2 + 6k + 5, \lambda = 2N(k+1)$, and f is a $\mathrm{H}^m_{loc}(\Omega)$ solution of the equation (1). Then $f \in G^s$. In particular, if Ψ is analytic in its arguments then so is f.*

Proof. Almost all the arguments used for the case when k is odd can be applied here. Therefore, we only give the sketch of the proof. Instead of the distance ρ in §2 we use the following metric

$$\tilde{\rho}\big((u,v),(x,y)\big) = \max\left\{|x^{k+1} - u^{k+1}|, (k+1)|y - v|\right\}.\square$$

B. In this sub-section we will present some computations for finding the fundamental solutions of $G_{k,\lambda}$ with source at the origin $(0,0)$ for λ other than the values $2N(k+1)$ considered in sub-section **A**. Make the following change of variables

$$x = \rho|\sin\theta|^{\frac{1}{k+1}}\operatorname{sign}(\sin\theta), y = \frac{\rho^{k+1}}{k+1}\cos\theta, \theta \in (-\pi,\pi).$$

Then $G_{k,\lambda}$ will be transformed into

$$\operatorname{sign}(\sin\theta)|\sin\theta|^{\frac{k-1}{k+1}}\Bigg(\sin\theta\frac{\partial^2}{\partial\rho^2} + (k+1)^2\rho^{-2}\sin\theta\frac{\partial^2}{\partial\theta^2} +$$
$$(i\lambda\cos\theta + (k+1)\sin\theta)\rho^{-1}\frac{\partial}{\partial\rho} + (k+1)\rho^{-2}(k\cos\theta - i\lambda\sin\theta)\frac{\partial}{\partial\theta}\Bigg).$$

If we seek the fundamental solution in the form $F_{k,\lambda}(x,y) = \rho^{-k}F(\theta)$ then $F(\theta)$ must satisfy the following equation

$$\begin{aligned}&(k+1)^2\sin\theta F''(\theta) + (k+1)(k\cos\theta - i\lambda\sin\theta)F'(\theta) - \\ &- ik\lambda\cos\theta F(\theta) = 0.\end{aligned} \tag{8}$$

The general solutions of (8) are

$$F(\theta) = \left(C_3 + C_4\int_0^\theta |\sin s|^{-\frac{k}{k+1}}e^{-\frac{i\lambda s}{k+1}}ds\right)e^{\frac{i\lambda\theta}{k+1}},$$

where C_3 and C_4 are some complex constants. Among all these solutions, we are interested in finding a non-trivial periodic solution. When $\lambda = 2N(k+1)$ – this case was considered in sub-section **A** – the periodic solution is $F(\theta) = e^{\frac{i\lambda\theta}{k+1}}$, and the function $F_{k,\lambda}(x,y) = \rho^{-k}F(\theta)$ serves as a fundamental solution. When

$\lambda = (2N+1)(k+1)$ then the periodic solution again is $F(\theta) = e^{\frac{i\lambda\theta}{k+1}}$. But in this case, we have $F_{k,\lambda}(x,y) = \rho^{-k}F(\theta)$ is a non-smooth solution of the equation $G_{k,\lambda}f(x,y) = 0$ (see [3]); hence, hypoellipticity for $G_{k,\lambda}$ fails in this case. If $\lambda \neq 2N(k+1)$ and $\lambda \neq (2N+1)(k+1)$ then we should choose

$$C_3 = \frac{iC_4\left(e^{\frac{i\pi\lambda}{k+1}}\int_0^{\pi} |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}}ds + e^{-\frac{i\pi\lambda}{k+1}}\int_{-\pi}^{0} |\sin s|^{-\frac{k}{k+1}} e^{-\frac{i\lambda s}{k+1}}ds\right)}{2\sin\frac{\pi\lambda}{k+1}}$$

to obtain the only periodic solution. In this case, the function $F_{k,\lambda}(x,y) = \rho^{-k}F(\theta)$ will be our desired fundamental solution.

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