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Segal–Bargmann Transform of White Noise Operators and White Noise Differential Equations

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Abstract The Segal–Bargmann transform is applied to characterization for symbols of white noise operators. A general formulation of an initial value problem for white noise operators is given and unique existence of a solution is proved by means of symbols. Regularity properties of the solution is discussed by introducing Fock spaces interpolating the space of white noise distributions and the original Boson Fock space.

Keywords: Bargmann–Segal space, Segal–Bargmann transform, white noise theory, Gaussian analysis, operator symbol, white noise differential equation, weighted Fock space, Wick product

1 Introduction

An interesting framework for nonlinear stochastic analysis is offered by *white noise operator theory* or *quantum white noise calculus*, where singular noises such as higher powers of quantum white noises are discussed systematically. In particular, as an extension of quantum stochastic differential equations (quantum Itô theory) *white noise differential equations (WNDEs)* has become a central topic for substantial development of white noise theory [4], [5], [27], [28]. Among others, investigation of regularity properties of solutions is important but has not yet achieved satisfactorily. In this paper we show that the Segal–Bargmann transform, which has been extensively studied, see e.g., [9], [10], [22], [30], is naturally extended for white noise operators and can be a new clue to answer this question.

Consider the Boson Fock space $\Gamma(L^2(\mathbf{R}))$. In quantum physics, $\Gamma(L^2(\mathbf{R}))$ describes a quantum field theory on the 1-dimensional space $\mathbf{R}$; while, in quantum stochastic calculus [23], [29] this $\mathbf{R}$ is understood as a time axis. Then, field operators at each time point $t \in \mathbf{R}$ are considered as *noise generators*. In particular, the pair of annihilation and creation operators $\{a_t, a_t^*\}$ is called a *quantum white noise* and plays a fundamental role in quantum white noise calculus. One traditional way of giving a meaning of a_t, a_t^* is to smear the

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time, i.e., such field operators are formulated as (unbounded) operator-valued distributions in $t \in \mathbf{R}$. Then, the time parameter t disappears and observation of the time evolution is always indirect. Another is to introduce a Gelfand triple:

$$\mathcal{W} \subset \Gamma(L^2(\mathbf{R})) \subset \mathcal{W}^*,$$

where such field operators at a point are formulated as continuous operators from $\mathcal{W}$ into $\mathcal{W}^*$. From the Fock space viewpoint, these are not proper operators but something like distribution (or generalized operators). The *white noise theory* is based on the latter idea, see e.g., [12], [14], [21]. In this paper we adopt the recent framework proposed by Cochran, Kuo and Sengupta [6]. In general, a continuous operators from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator* and we denote by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ the space of such operators. A systematic study of white noise operators has been launched out in [24] and developed extensively along with the symbol calculus, see e.g., [3], [26].

It is our long-range project to develop a theory of differential equations for white noise operators. During the last years our main attention has been paid to a normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi|_{t=0} = \Xi_0, \tag{1.1}$$

where $t \mapsto L_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous map (also called a quantum stochastic process). Such a linear equation arises from a variety of mathematical models in theoretical physics. For example, a quantum stochastic differential equation introduced by Hudson and Parthasarathy [15] is equivalent to a normal-ordered white noise differential equation with $\{L_t\}$ involving only lower powers (i.e., linear terms) of quantum white noises. In the series of papers [4], [5], [27], [28], we proved unique existence of a solution in the space of white noise operators and established a method of examining its regularity properties in terms of weighted Fock spaces. Moreover, in the recent paper [18] we started an approach on the basis of infinite dimensional holomorphic functions.

The next step is to discuss a nonlinear equation beyond the normal-ordered white noise equations (1.1). In this paper we focus on an initial value problem for white noise operators:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \qquad 0 \le t \le T, \tag{1.2}$$

where $F : [0, T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function. The usual characterization theorem for operator symbols is powerful to solve (1.2), however, is not sufficient to claim regularity properties of the solution. To overcome this situation, in the recent paper Ji–Obata [17], a new aspect of operator symbols is introduced from the viewpoint of the Segal–Bargmann transform. In this paper, we show that the new idea helps to investigate a proper Fock space in which the solution acts as a usual (unbounded) operator rather than a generalized operator. We hope that the main result stated in Theorem 7.2, which needs more mature consideration, is a small step toward our goal.

2 Preliminaries

2.1 Boson Fock space and weighted Fock space

Let H be a real or complex Hilbert space with norm $|\cdot|$. For $n \ge 0$ let $H^{\widehat{\otimes} n}$ denote the n-fold symmetric tensor power of a Hilbert space H. Their norms are denoted by the common symbol $|\cdot|$ for simplicity. Given a positive sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ we put

$$\Gamma_\alpha(H) = \left\{ \phi = (f_n)_{n=0}^{\infty};\, f_n \in H^{\widehat{\otimes} n},\, \|\phi\|_+^2 \equiv \sum_{n=0}^{\infty} n!\,\alpha(n)\,|f_n|^2 < \infty \right\}.$$

Then $\Gamma_\alpha(H)$ becomes a Hilbert space and is called a *weighted Fock space* with weight sequence α. The Boson Fock space $\Gamma(H)$ is a special case of $\alpha(n) \equiv 1$.

Lemma 2.1 *Assume that a Hilbert space H_2 is densely imbedded in another Hilbert space H_1 and the inclusion map $H_2 \hookrightarrow H_1$ is a contraction. Let $\alpha = \{\alpha(n)\}$ be a positive sequence such that* $\inf \alpha(n) > 0$. *Then we have continuous inclusions with dense images:*

$$\Gamma_\alpha(H_2) \hookrightarrow \Gamma(H_2) \hookrightarrow \Gamma(H_1).$$

Moreover, the second inclusion is a contraction.

2.2 Rigged Hilbert space constructed from a selfadjoint operator

This is a standard construction, see e.g., [2], [8]. Let H be a complex Hilbert space and T a selfadjoint operator with dense domain $\mathrm{Dom}\,(T) \subset H$ such that $\inf \mathrm{Spec}\,(T) > 0$. We note that T^{-1} becomes a bounded operator on H and put

$$\rho_T = \|T^{-1}\|_{\mathrm{OP}} = (\inf \mathrm{Spec}\,(T))^{-1}.$$

Then, for each $p \geq 0$, the dense subspace $\mathcal{D}_p \equiv \mathrm{Dom}\,(T^p) \subset H$ becomes a Hilbert space equipped with the norm

$$|\xi|_{T,p} = |T^p \xi|_0, \qquad \xi \in \mathrm{Dom}\,(T^p),$$

where $|\cdot|_0$ is the norm of H. Furthermore, we define $\mathcal{D}_{-p}$ to be the completion of H with respect to the norm $|\xi|_{T,-p} = |T^{-p}\xi|_0$, $\xi \in H$. In view of a straightforward inequality:

$$|\xi|_{T,p} \leq \rho_T^{q-p}\,|\xi|_{T,q}, \qquad \xi \in \mathcal{D}_q, \quad -\infty < p \leq q < +\infty,$$

we come to a Hilbert riggings:

$$\cdots \subset \mathcal{D}_q \subset \cdots \subset \mathcal{D}_p \subset \cdots \subset \mathcal{D}_0 = H \subset \cdots \subset \mathcal{D}_{-p} \subset \cdots \subset \mathcal{D}_{-q} \subset \cdots, \qquad (2.1)$$

where each inclusion is continuous and has a dense image. Moreover, for any $p, q \in \mathbf{R}$ the operator T^{p-q} is naturally considered as an isometry from $\mathcal{D}_p$ onto $\mathcal{D}_q$. From (2.1) we obtain

$$\mathcal{D}_\infty = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{D}_p, \qquad \mathcal{D}_\infty^* = \operatorname*{ind\,lim}_{p\to\infty} \mathcal{D}_{-p}.$$

Obviously, $\mathcal{D}_\infty$ is a countable Hilbert space. It is nuclear if and only if there exists $p > 0$ such that T^{-p} is of Hilbert–Schmidt type.

2.3 Riggings of Fock spaces

Let $\alpha = \{\alpha(n)\}$ be a positive sequence such that $\inf \alpha(n) > 0$. Based on a rigged Hilbert space (2.1), we obtain a chain of weighted Fock spaces:

$$\cdots \subset \Gamma_\alpha(\mathcal{D}_q) \subset \cdots \subset \Gamma_\alpha(\mathcal{D}_p) \subset \cdots \subset \Gamma_\alpha(\mathcal{D}_0) = \Gamma_\alpha(H) \subset \Gamma(H), \qquad 0 \le p \le q,$$

where Lemma 2.1 is taken into account. By definition the norm of $\Gamma_\alpha(\mathcal{D}_p)$, $p \ge 0$, is given by

$$\| \phi \|_{p,+}^2 = \sum_{n=0}^{\infty} n!\, \alpha(n)\, | f_n |_p^2, \qquad \phi = (f_n) \in \Gamma_\alpha(\mathcal{D}_p).$$

Identifying $\Gamma(H)$ with its dual, we have

$$\Gamma_\alpha(\mathcal{D}_p)^* \cong \Gamma_{\alpha^{-1}}(\mathcal{D}_{-p}),$$

where the norm of $\Gamma_{\alpha^{-1}}(\mathcal{D}_{-p})$ is defined by

$$\| \Phi \|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)}\, | F_n |_{-p}^2, \qquad \Phi = (F_n).$$

The canonical complex bilinear form on $\Gamma_\alpha(\mathcal{D}_p)^* \times \Gamma_\alpha(\mathcal{D}_p)$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Then for $\Phi = (F_n) \in \Gamma_\alpha(\mathcal{D}_p)^*$ and $\phi = (f_n) \in \Gamma_\alpha(\mathcal{D}_p)$ it holds that

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\, \langle F_n, f_n \rangle .$$

With these notations we come to a rigging of the Fock space $\Gamma(H)$:

$$\cdots \subset \Gamma_\alpha(\mathcal{D}_q) \subset \cdots \subset \Gamma_\alpha(\mathcal{D}_p) \subset \cdots \subset \Gamma(H) \subset \cdots \subset \Gamma_{\alpha^{-1}}(\mathcal{D}_{-p}) \subset \cdots \subset \Gamma_{\alpha^{-1}}(\mathcal{D}_{-q}) \subset \cdots,$$

where $0 \le p \le q$. Furthermore, we obtain

$$\Gamma_\alpha(\mathcal{D}) = \operatorname*{proj\,lim}_{p \to \infty} \Gamma_\alpha(\mathcal{D}_p) \subset \Gamma(H) \subset \Gamma_\alpha(\mathcal{D})^* = \operatorname*{ind\,lim}_{p \to \infty} \Gamma_{\alpha^{-1}}(\mathcal{D}_{-p}),$$

where $\Gamma_\alpha(\mathcal{D})$ is a countable Hilbert space. The canonical bilinear form is denoted by the same symbol $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

3 Two Riggings of Fock Space

From now on we denote by $\mathcal{H}$ and $\mathcal{H}_{\mathbf{R}}$ the space of complex valued L^2-functions and that of real valued ones, respectively. We shall construct two riggings of $\Gamma(\mathcal{H})$:

$$\mathcal{W}_\alpha \equiv \operatorname*{proj\,lim}_{p \to \infty} \Gamma_\alpha(\mathcal{E}_p) \subset \Gamma(\mathcal{H}) \subset \operatorname*{ind\,lim}_{p \to \infty} \Gamma_{\alpha^{-1}}(\mathcal{E}_{-p}) = \mathcal{W}_\alpha^*,$$

$$\mathcal{G}_\infty \equiv \operatorname*{proj\,lim}_{p \to \infty} \Gamma(\mathcal{D}_p) \subset \Gamma(\mathcal{H}) \subset \operatorname*{ind\,lim}_{p \to \infty} \Gamma(\mathcal{D}_{-p}) = \mathcal{G}_\infty^*.$$

The former will be referred as a CKS-space and the latter as a Fock chain.

3.1 CKS-space

Consider the famous selfadjoint operator:

$$A = 1 + t^2 - \frac{d^2}{dt^2}.$$

As is well known, there exists an orthonormal basis $\{e_k\}_{k=0}^{\infty} \subset \mathcal{H}_{\mathbf{R}}$ of $\mathcal{H}$ such that $Ae_k = (2k+2)e_k$, $k \ge 0$. In particular, $\inf \mathrm{Spec}\,(A) = 2$ and

$$\rho \equiv \| A^{-1} \|_{\mathrm{OP}} = \frac{1}{2}, \qquad \| A^{-q} \|_{\mathrm{HS}}^2 = \sum_{k=0}^{\infty} \frac{1}{(2k+2)^{2q}} < \infty, \quad q > \frac{1}{2}.$$

By the standard method mentioned in §2.2 we obtain a Gelfand triple:

$$\mathcal{E} \equiv \operatorname*{proj\,lim}_{p\to\infty} \mathcal{E}_p \subset \mathcal{H} \subset \mathcal{E}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{E}_{-p}.$$

For simplicity, we write

$$|\,\xi\,|_p = |\,A^p\xi\,|_0\,, \qquad \xi \in \mathcal{E}_p.$$

The real part $\mathcal{E}_{\mathbf{R}}$ of $\mathcal{E}$ is also defined by a similar method for A is a real operator. We note the topological isomorphisms:

$$\mathcal{E}_{\mathbf{R}} \cong \mathcal{S}(\mathbf{R}), \qquad \mathcal{E}_{\mathbf{R}}^* \cong \mathcal{S}'(\mathbf{R}),$$

where $\mathcal{S}(\mathbf{R})$ is the space of rapidly decreasing functions and $\mathcal{S}'(\mathbf{R})$ the space of tempered distributions.

For our purpose we choose a weight sequence $\alpha = \{\alpha(n)\}$ satisfying the following four conditions:

(A1) $\alpha(0) = 1$ and there exists some $\sigma \ge 1$ such that $\inf_{n\ge0} \alpha(n)\sigma^n > 0$;

(A2) $\displaystyle\lim_{n\to\infty} \left\{\frac{\alpha(n)}{n!}\right\}^{1/n} = 0$;

(A3) α is equivalent to a positive sequence $\gamma = \{\gamma(n)\}$ such that $\{\gamma(n)/n!\}$ is log-concave;

(A4) there exists a constant $C_{1\alpha} > 0$ such that $\alpha(m)\alpha(n) \le C_{1\alpha}^{m+n}\alpha(m+n)$ for all m, n.

Given such a weight sequence α, we obtain

$$\mathcal{W}_\alpha \equiv \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(\mathcal{E}_p) \subset \Gamma(\mathcal{H}) \subset \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{\alpha^{-1}}(\mathcal{E}_{-p}) = \mathcal{W}_\alpha^*, \tag{3.1}$$

which is referred to as a *CKS-space*. Recall that $\mathcal{W}_\alpha$ is a nuclear space.

The generating function of $\{\alpha(n)\}$ is defined by

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n,$$

which is entire holomorphic by condition (A2). Moreover, we have

$$\begin{aligned} G_\alpha(0) &= 1, \\ G_\alpha(s) &\le G_\alpha(t), \qquad 0 \le s \le t, \\ \gamma[G_\alpha(t) - 1] &\le G_\alpha(\gamma t) - 1, \qquad \gamma \ge 1, \quad t \ge 0. \end{aligned} \tag{3.2}$$

It is known [1] that condition (A3) is necessary and sufficient for the power series

$$\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n$$

to have a positive radius of convergence $R_\alpha > 0$.

Concrete examples of $\{\alpha(n)\}$ satisfying conditions (A1)–(A4) are (i) $\alpha(n) \equiv 1$; (ii) $\alpha(n) = (n!)^\beta$ with $0 \le \beta < 1$; (iii) $\alpha(n) = \mathrm{Bell}_k(n)$, called the *Bell numbers* of order k, defined by

$$\frac{\overbrace{\exp(\exp \ldots (\exp}^{k \text{ times}} t) \ldots)}{\exp(\exp \ldots (\exp 0) \ldots)} = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n.$$

The corresponding CKS-spaces in the case of (i) and (ii) are called the *Hida–Kubo–Takenaka space* [20] and *Kondratiev–Streit space* [19], respectively. CKS-spaces are also constructed by means of infinite dimensional holomorphic functions, see Gannoun–Hachaichi–Ouerdiane–Rezgui [7].

3.2 Fock chain

Let K be a selfadjoint operator in $\mathcal{H}$ satisfying the following conditions:

(i) $\inf \mathrm{Spec}\,(K) \ge 1$;

(ii) $\mathcal{E}_{\mathbf{R}}$ is invariant under K;

(iii) $\mathcal{E}$ is densely and continuously imbedded in $\mathcal{D}_p$ for all $p \ge 0$.

Here $\mathcal{D}_p$ stands for the Hilbert space obtained from $\mathrm{Dom}\,(K^p)$ equipped with the norm $|\xi|_{K,p} = |K^p \xi|_0$. We set

$$\mathcal{G}_p = \Gamma(\mathcal{D}_p), \qquad p \in \mathbf{R}.$$

By definition, the norm of $\mathcal{G}_p$ is given by

$$\|\phi\|_{K,p}^2 = \sum_{n=0}^{\infty} n! \, |f_n|_{K,p}^2 \qquad \phi = (f_n), \quad f_n \in \mathcal{D}_p^{\widehat{\otimes} n}. \tag{3.3}$$

Then we come to

$$\mathcal{G}_\infty \equiv \operatorname*{proj\,lim}_{p\to\infty} \mathcal{G}_p \subset \Gamma(\mathcal{H}) \subset \operatorname*{ind\,lim}_{p\to\infty} \mathcal{G}_{-p} = \mathcal{G}_\infty^*, \tag{3.4}$$

where $\mathcal{G}_\infty$ becomes a countable Hilbert space equipped with the Hilbertian norms defined by (3.3). In general, $\mathcal{G}_\infty$ is not a nuclear space.

Lemma 3.1 *For any weight sequence α satisfying conditions (A1)–(A3) and $p \geq 0$ we have continuous inclusions with dense images:*

$$\mathcal{W}_\alpha \subset \mathcal{G}_p \subset \Gamma(\mathcal{H}) \subset \mathcal{G}_{-p} \subset \mathcal{W}_\alpha^*.$$

PROOF. Since $\mathcal{E} \hookrightarrow \mathcal{D}_p$ is continuous, there exist $C \geq 0$ and $p' \geq 0$ such that

$$|\xi|_{K,p} \leq C|\xi|_{p'} \leq C\rho^{q-p'}|\xi|_q, \qquad \xi \in \mathcal{E}, \quad q \geq p'.$$

Hence for a sufficiently large $q \geq 0$ we have $|\xi|_{K,p} \leq |\xi|_q$, $\xi \in \mathcal{E}$; in other words, $\mathcal{E}_q \hookrightarrow \mathcal{D}_p$ is a contraction. It then follows from Lemma 2.1 that $\Gamma_\alpha(\mathcal{E}_q) \hookrightarrow \Gamma(\mathcal{D}_p)$ is continuous. ∎

It is reasonable by Lemma 3.1 to denote the canonical complex bilinear form on $\mathcal{G}_\infty^* \times \mathcal{G}_\infty$ by $\langle\langle \cdot, \cdot \rangle\rangle$. Then, obviously,

$$|\langle\langle \Phi, \phi \rangle\rangle| \leq \|\Phi\|_{K,-p} \|\phi\|_{K,p}, \qquad \Phi \in \mathcal{G}_\infty^*, \quad \phi \in \mathcal{G}_\infty.$$

4 Gaussian Space and Bargmann–Segal Space

4.1 Gaussian space

Recall the Gelfand triple:

$$\mathcal{E}_{\mathbf{R}} = \mathcal{S}(\mathbf{R}) \subset \mathcal{H}_{\mathbf{R}} = L^2(\mathbf{R}, dt)_{\mathbf{R}} \subset \mathcal{E}_{\mathbf{R}}^* = \mathcal{S}'(\mathbf{R}). \tag{4.1}$$

By the Bochner–Minlos theorem, for each $\sigma > 0$ there exists a probability measure μ_{σ^2} on $\mathcal{E}_{\mathbf{R}}^*$ such that

$$\exp\left\{-\frac{\sigma^2}{2}\langle \xi, \xi \rangle\right\} = \int_{\mathcal{E}_{\mathbf{R}}^*} e^{i\langle x, \xi \rangle} \mu_{\sigma^2}(dx), \qquad \xi \in \mathcal{E}_{\mathbf{R}}.$$

We put $\mu = \mu_1$ for simplicity. Then the probability space $(\mathcal{E}_{\mathbf{R}}^*, \mu)$ is called the *(standard) Gaussian space.* Define a probability measure ν on $\mathcal{E}^* = \mathcal{E}_{\mathbf{R}}^* + i\mathcal{E}_{\mathbf{R}}^*$ in such a way that

$$\nu(dz) = \mu_{1/2}(dx) \times \mu_{1/2}(dy), \qquad z = x + iy, \quad x, y \in \mathcal{E}_{\mathbf{R}}^*.$$

Following Hida [13] the probability space $(\mathcal{E}^*, \nu)$ is called the *(standard) complex Gaussian space* associated with (4.1).

4.2 Wiener–Itô–Segal isomorphism

Theorem 4.1 (Wiener–Itô–Segal) *There exists a unitary isomorphism between $L^2(\mathcal{E}_{\mathbf{R}}^*, \mu)$ and $\Gamma(\mathcal{H})$, which is uniquely determined by the correspondence:*

$$e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle/2} \quad \longleftrightarrow \quad \phi_\xi \equiv \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right), \tag{4.2}$$

where ξ runs over $\mathcal{E}$.

The above ϕ_ξ is called an *exponential vector* or a *coherent vector.* We often use the same symbol for the left hand side of (4.2).

4.3 Bargmann–Segal space

The *Bargmann–Segal space*, denoted by $E^2(\nu)$, is by definition the space of entire functions $g : \mathcal{H} \to \mathbf{C}$ such that

$$\| g \|_{E^2(\nu)}^2 \equiv \sup_{P \in \mathcal{P}} \int_{\mathcal{E}^*} |g(Pz)|^2 \nu(dz) < \infty,$$

where $\mathcal{P}$ is the set of all finite rank projections on $\mathcal{H}_{\mathbf{R}}$ with range contained in $\mathcal{E}_{\mathbf{R}}$. Note that $P \in \mathcal{P}$ is naturally extended to a continuous operator from $\mathcal{E}^*$ into $\mathcal{H}$ (in fact into $\mathcal{E}$), which is denoted by the same symbol. The Bargmann–Segal space $E^2(\nu)$ is a Hilbert space with norm $\| \cdot \|_{E^2(\nu)}$. For $\phi = (f_n)_{n=0}^{\infty} \in \Gamma(\mathcal{H})$ define

$$J\phi(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle, \qquad \xi \in \mathcal{H}. \tag{4.3}$$

Since the right hand side converges uniformly on each bounded subset of $\mathcal{H}$, $J\phi$ becomes an entire function on $\mathcal{H}$. Moreover, it is known (e.g., [9], [10]) that J becomes a unitary isomorphism from $\Gamma(\mathcal{H})$ onto $E^2(\nu)$. In fact, for $\phi \in \Gamma(\mathcal{H})$ we have

$$\| J\phi \|_{E^2(\nu)}^2 = \sup_{P \in \mathcal{P}} \int_{\mathcal{E}^*} | \langle\!\langle \phi, \phi_{Pz} \rangle\!\rangle |^2 \nu(dz) = \sup_{P \in \mathcal{P}} \| \Gamma(P)\phi \|_0^2 = \| \phi \|_0^2 .$$

The map J defined in (4.3) is called the ***duality transform*** and is related with the S-transform (see (4.5) below) in an obvious manner:

$$J\phi|_{\mathcal{D}_\infty} = S\phi, \qquad \phi \in \Gamma(\mathcal{H}),$$

which follows from (4.5) and (4.3).

4.4 White noise functions

The riggings obtained from (3.1) and (3.4) through the Wiener–Itô–Segal isomorphism are denoted respectively by

$$\mathcal{W}_\alpha \subset \mathcal{W}_p \subset L^2(\mathcal{E}_{\mathbf{R}}^*, \mu) \subset \mathcal{W}_{-p} \subset \mathcal{W}_\alpha^*,$$
$$\mathcal{G}_\infty \subset \mathcal{G}_p \subset L^2(\mathcal{E}_{\mathbf{R}}^*, \mu) \subset \mathcal{G}_{-p} \subset \mathcal{G}_\infty^*,$$

where $p \geq 0$. In this context, elements of $\mathcal{W}_\alpha$ and of $\mathcal{W}_\alpha^*$ are called a ***white noise test function*** and a ***white noise distribution***, respectively. We note also

$$\mathcal{W}_\alpha \subset \mathcal{G}_\infty \subset \mathcal{G}_p \subset L^2(\mathcal{E}_{\mathbf{R}}^*, \mu) \subset \mathcal{G}_{-p} \subset \mathcal{G}_{-\infty} \subset \mathcal{W}_\alpha^*, \qquad p \geq 0, \tag{4.4}$$

which is proved in Lemma 3.1. When there is no danger of confusion, we write $\mathcal{W} = \mathcal{W}_\alpha$ for simplicity.

4.5 S-transforms

For $\Phi \in \mathcal{W}^*$, the *S-transform* is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in \mathcal{E}. \tag{4.5}$$

Since the exponential vectors $\{\phi_\xi ; \xi \in \mathcal{E}\}$ span a dense subspace of $\mathcal{W}$, each Φ is uniquely specified by the S-transform. Obviously, the S-transform $F = S\Phi$ possesses the following properties:

(F1) for each $\xi, \eta \in \mathcal{E}$, the function $z \mapsto F(z\xi + \eta)$ is entire holomorphic on $\mathbf{C}$;

(F2) there exist $C \geq 0$ and $p \geq 0$ such that

$$|F(\xi)|^2 \leq CG_\alpha(|\xi|_p^2), \qquad \xi \in \mathcal{E}.$$

It is emphasized in white noise theory that the converse assertion is also true. This famous characterization theorem for S-transform was first proved for the Hida–Kubo–Takenaka space by Potthoff and Streit.

Theorem 4.2 (Cochran–Kuo–Sengupta [6]) *Let F be a complex valued function on $\mathcal{E}$. Then F is the S-transform of some $\Phi \in \mathcal{W}^*$ if and only if F satisfies conditions (F1) and (F2). In that case, for any $q > 1/2$ with $\| A^{-q} \|_{\mathrm{HS}}^2 < R_\alpha$ we have*

$$\| \Phi \|_{-(p+q),-}^2 \leq C\widetilde{G}_\alpha(\|A^{-q}\|_{\mathrm{HS}}^2).$$

In the proof of Theorem 4.2, the nuclearity of the space $\mathcal{W}$ plays an essential role. While, in general the countable Hilbert space $\mathcal{G}_\infty$ is not nuclear and hence the method of those used in the proof of Theorem 4.2 is not applicable to characterize S-transforms of elements of $\mathcal{G}_\infty^*$. However, we have the following characterization theorem for S-transforms of elements of $\mathcal{G}_\infty^*$ by using Bargmann–Segal space, see [10], [17].

Theorem 4.3 *Let $p \in \mathbf{R}$. Then a complex valued function g on $\mathcal{D}_\infty$ is the S-transform of some $\Phi \in \mathcal{G}_p$ if and only if g can be extended to a continuous function on $\mathcal{D}_{-p}$ and $g \circ K^p \in E^2(\nu)$. In this case,*

$$\| \Phi \|_{K,p} = \| g \circ K^p \|_{E^2(\nu)} .$$

5 White Noise Operators

A continuous linear operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is called a *white noise operator*.[‡] Note that $\mathcal{L}(\mathcal{W}, \mathcal{W})$, $\mathcal{L}(\Gamma(\mathcal{H}), \Gamma(\mathcal{H}))$ and $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ are subspaces of $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, see (4.4). Moreover, $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ is isomorphic to $\mathcal{L}(\mathcal{W}, \mathcal{W})$ by duality. A general theory for white noise operators has been extensively developed in [3], [24], [26]. In this section we shall focus on regularity properties of a white noise operator in terms of Fock riggings.

[‡]In general, for two locally convex spaces $\mathfrak{X}, \mathfrak{Y}$, the space of all continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$ is denoted by $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$. We always assume that $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is equipped with the topology of uniform convergence on every bounded subset.

5.1 Integral kernel operators

Let a_t and a_t^* be the annihilation and creation operators at a point $t \in \mathbf{R}$. For $\phi \in \mathcal{W}$ we have

$$a_t\phi(x) = \lim_{\theta \to 0} \frac{\phi(x+\theta\delta_t) - \phi(x)}{\theta}, \qquad t \in \mathbf{R}, \quad x \in \mathcal{E}_{\mathbf{R}}^*,$$

where the limit always exists. It is known that $a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$. Moreover, the maps $t \mapsto a_t$ and $t \mapsto a_t^*$ are both infinitely many times differentiable. The pair $\{a_t, a_t^*\}$ is referred to as the *quantum white noise process.*

Let $l, m \geq 0$ be integers. Given $\kappa \in (\mathcal{E}^{\otimes(l+m)})^*$ we define an *integral kernel operator* by

$$\Xi_{l,m}(\kappa) = \int_{\mathbf{R}^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where the integral is understood in a formal sense. To be more precise, for $\phi = (f_n) \in \mathcal{W}$ we define $\Xi_{l,m}(\kappa)\phi = (g_n)$ by

$$g_n = 0, \quad 0 \leq n < l; \qquad g_{l+n} = \frac{(n+m)!}{n!} \kappa \otimes_m f_{n+m}, \quad n \geq 0,$$

where $\otimes_m$ is the *right m-contraction*, see [5]. An integral kernel operator is always a white noise operator, that is, $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ for an arbitrary kernel $\kappa \in (\mathcal{E}^{\otimes(l+m)})^*$. Moreover, if the weight sequence $\{\alpha(n)\}$ fulfills conditions (A1)–(A4), then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ if and only if $\kappa \in \mathcal{E}^{\otimes l} \otimes (\mathcal{E}^{\otimes m})^*$.

It is an interesting question to characterize the integral kernel operators belonging to $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ for some $p \in \mathbf{R}$. We here only mention the following

Theorem 5.1 (Chung–Ji–Obata [4]) *Let $\alpha = \{\alpha(n)\}$ be a weight sequence satisfying conditions (A1)–(A4) and $p \in \mathbf{R}$. Then for $\kappa \in (\mathcal{E}^{\otimes(l+m)})^*$, $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ if and only if $\kappa \in \mathcal{D}_p^{\otimes l} \otimes (\mathcal{E}^{\otimes m})^*$ if and only if $\kappa \otimes_m \in \mathcal{L}(\mathcal{E}^{\otimes m}, \mathcal{D}_p^{\otimes l})$.*

5.2 Operator symbols from the viewpoint of Segal–Bargmann transform

The *symbol*, which is an operator version of the famous Segal–Bargmann transform, of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a complex valued function on $\mathcal{E} \times \mathcal{E}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in \mathcal{E}.$$

Every white noise operator is uniquely determined by its symbol. By definition the symbol and the S-transforms are related as

$$\widehat{\Xi}(\xi, \eta) = S(\Xi\phi_\xi)(\eta) = S(\Xi^*\phi_\eta)(\xi), \qquad \xi, \eta \in \mathcal{E}.$$

It is straightforward to see that the symbol $\Theta = \widehat{\Xi}$ of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ possesses the following properties:

(O1) for any $\xi, \xi_1, \eta, \eta_1 \in \mathcal{E}$ the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbf{C} \times \mathbf{C}$;

(O2) there exist constant numbers $C \geq 0$ and $p \geq 0$ such that

$$|\Theta(\xi,\eta)|^2 \leq C G_\alpha(|\,\xi\,|_p^2) G_\alpha(|\,\eta\,|_p^2), \qquad \xi,\eta \in \mathcal{E}.$$

As in the case of S-transform, the characterization theorem for symbols, which was first proved by Obata for the Hida–Kubo–Takenaka space, is a significant consequence of white noise theory. The characterization in the case of CKS–space was proved by Chung–Ji–Obata [3].

We now prove the characterization theorem for symbols of operators on Fock spaces, in this connection see also [17]. Let $p \in \mathbf{R}$. Then it is easily shown that the symbol of $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is extended to an entire function on $\mathcal{E} \times \mathcal{D}_{-p}$.

Theorem 5.2 *Let $p \in \mathbf{R}$ and let Θ a complex valued function defined on $\mathcal{E} \times \mathcal{E}$. Then there exists $\Xi \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ such that $\Theta = \widehat{\Xi}$ if and only if*

(i) *Θ can be extended to an entire function on $\mathcal{E} \times \mathcal{D}_{-p}$;*

(ii) *there exist $q \geq 0$ and $C \geq 0$ such that*

$$\|\Theta(\xi, K^p\cdot)\|_{E^2(\nu)}^2 \leq C G_\alpha(|\xi|_q^2), \qquad \xi \in \mathcal{E}.$$

PROOF. Suppose that there exists $\Xi \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ such that $\Theta = \widehat{\Xi}$. Then condition (i) is obvious and there exists $q \geq 0$ such that $\Xi \in \mathcal{L}(\mathcal{W}_q, \mathcal{G}_p)$. Hence there exists $C \geq 0$ such that

$$\|\Xi\phi\|_{K,p} \leq C\|\phi\|_{q,+}, \qquad \phi \in \mathcal{W}_q.$$

Therefore, we have

$$\|\Theta(\xi, K^p\cdot)\|_{E^2(\nu)}^2 = \|\Xi\phi_\xi\|_{K,p}^2 \leq C^2\|\phi_\xi\|_{q,+}^2 = C^2 G_\alpha(|\xi|_q^2).$$

Conversely, suppose that conditions (i) and (ii) are satisfied. Let $\xi \in \mathcal{E}$ be fixed and define a function $F_\xi : \mathcal{D}_{-p} \to \mathbf{C}$ by $F_\xi(\eta) = \Theta(\xi,\eta)$, $\eta \in \mathcal{D}_{-p}$. Then by (ii), $F_\xi(K^p\cdot) \in E^2(\nu)$. Hence by Theorem 4.3, there exists $\Phi_\xi \in \mathcal{G}_p$ such that $S\Phi_\xi = F_\xi$ and

$$\|\Phi_\xi\|_{K,p}^2 = \|F_\xi \circ K^p\|_{E^2(\nu)}^2 = \|\Theta(\xi, K^p\cdot)\|_{E^2(\nu)}^2 \leq C G_\alpha(|\xi|_q^2).$$

Now, fix $\phi \in \mathcal{G}_{-p}$ and define a function $G_\phi : \mathcal{E} \to \mathbf{C}$ by

$$G_\phi(\xi) = \langle\langle \phi,\, \Phi_\xi \rangle\rangle\,, \qquad \xi \in \mathcal{E}.$$

Then we can easily show that G_ϕ satisfies conditions (F1) and (F2). In fact,

$$|G_\phi(\xi)|^2 \leq \|\phi\|_{K,-p}^2 \|\Phi_\xi\|_{K,p}^2 \leq C\|\phi\|_{K,-p}^2 G_\alpha(|\xi|_q^2).$$

Therefore, by Theorem 4.2, there exists $\Psi_\phi \in \mathcal{W}_\alpha^*$ such that

$$S(\Psi_\phi)(\xi) = G_\phi(\xi) = \langle\langle \phi,\, \Phi_\xi \rangle\rangle\,, \qquad \xi \in \mathcal{E}.$$

Moreover, we have

$$\|\,\Psi_\phi\,\|_{-(q+q'),-}^2 \leq C\widetilde{G}_\alpha(\|\,A^{-q'}\,\|_{\mathrm{HS}}^2)\|\,\phi\,\|_{K,-p}^2 \tag{5.1}$$

for some $q' > 1/2$ with $\|\,A^{-q'}\,\|_{\mathrm{HS}}^2 < R_\alpha$. Define a linear operator $\Xi^* : \mathcal{G}_{-p} \to \mathcal{W}_\alpha^*$ by $\Xi^*\phi = \Psi_\phi$, $\phi \in \mathcal{G}_{-p}$. Then $\Xi^* \in \mathcal{L}(\mathcal{G}_{-p}, \mathcal{W}_\alpha^*)$ by (5.1) and hence Θ is the symbol of $\Xi \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ (Ξ is the adjoint of Ξ^*). ∎

During the above theorem we are convinced that the symbol is an operator-version of the Segal–Bargmann transform.

5.3 Wick products

We first recall the following

Lemma 5.3 (Chung–Ji–Obata [5]) *If the weight sequence $\alpha = \{\alpha(n)\}$ satisfies conditions (A1)–(A4), then for two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ there exists a unique operator $\Xi \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ such that*

$$\widehat{\Xi}(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)e^{-\langle\xi,\eta\rangle}, \qquad \xi,\eta \in \mathcal{E}. \tag{5.2}$$

The operator Ξ defined in (5.2) is called the *Wick product* of Ξ_1 and Ξ_2, and is denoted by $\Xi = \Xi_1 \diamond \Xi_2$. Note that $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ equipped with the Wick product becomes a commutative $*$-algebra. As for the annihilation and creation operators we have

$$a_s \diamond a_t = a_s a_t, \qquad a_s^* \diamond a_t = a_s^* a_t, \qquad a_s \diamond a_t^* = a_t^* a_s, \qquad a_s^* \diamond a_t^* = a_s^* a_t^*. \tag{5.3}$$

More generally, it holds that

$$a_{s_1}^* \cdots a_{s_l}^* \Xi a_{t_1} \cdots a_{t_m} = (a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}) \diamond \Xi, \qquad \Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*).$$

In fact, the Wick product is a unique bilinear map from $\mathcal{L}(\mathcal{W}, \mathcal{W}^*) \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ into $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ which is (i) separately continuous; (ii) associative; and (iii) satisfying (5.3).

Proposition 5.4 *Let $p \in \mathbf{R}$ and $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$. Then $\Xi_1 \diamond \Xi_2 \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ if and only if there exist $q \geq 0$ and $C \geq 0$ such that*

$$\left\| \widehat{\Xi}_1(\xi, K^p\cdot)\widehat{\Xi}_2(\xi, K^p\cdot)e^{-\langle\xi, K^p\cdot\rangle} \right\|_{E^2(\nu)}^2 \leq C G_\alpha(|\xi|_q^2), \qquad \xi \in \mathcal{E}.$$

PROOF. An immediate consequence from Theorem 5.2. ∎

6 Quantum Stochastic Processes

A continuous map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ defined on an interval is called a *quantum stochastic process* (in the sense of white noise theory), see [25]. The quantum white noise process $\{a_t, a_t^*\}$ is a pair of quantum stochastic processes in this sense, see §5.1. In this section we discuss some detailed properties of quantum stochastic processes in $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$.

6.1 Continuity criterion

We first mention a criterion for the continuity of $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ in terms of operator symbols.

Theorem 6.1 *Let T be a locally compact space and $p \in \mathbf{R}$. Then for a map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$, $t \in T$, the following conditions are equivalent:*

(i) *$t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is continuous;*

(ii) *for any* $t_0 \in T$ *there exist* $q \geq 0$ *and an open neighborhood* U *of* t_0 *such that*

$$\{\Xi_t; t \in U\} \subset \mathcal{L}(\mathcal{W}_q, \mathcal{G}_p) \quad \text{and} \quad \lim_{t \to t_0} \|\Xi_t - \Xi_{t_0}\|_{\mathcal{L}(\mathcal{W}_q, \mathcal{G}_p)} = 0;$$

(iii) *for any* $t_0 \in T$ *there exist an open neighborhood* U *of* t_0*, a set of positive numbers* $\{\epsilon_t; t \in U\}$ *converging to* 0 *as* $t \to t_0$*, constant number* $q \geq 0$ *such that*

$$\|\widehat{\Xi}_t(\xi, K^p\cdot) - \widehat{\Xi}_{t_0}(\xi, K^p\cdot)\|^2_{E^2(\nu)} \leq \epsilon_t G_\alpha(|\,\xi\,|_q^2), \qquad \xi \in \mathcal{E}, \quad t \in U;$$

(iv) *for any* $t_0 \in T$ *there exist* $M \geq 0$*,* $q \geq 0$ *and an open neighborhood* U *of* t_0 *such that*

$$\|\widehat{\Xi}_t(\xi, K^p\cdot)\|^2_{E^2(\nu)} \leq M G_\alpha(|\,\xi\,|_q^2), \qquad \xi \in \mathcal{E}, \quad t \in U,$$

and for each $\xi \in \mathcal{E}$*,* $\widehat{\Xi}_t(\xi, K^p\cdot)$ *converges to* $\widehat{\Xi}_{t_0}(\xi, K^p\cdot)$ *in* $E^2(\nu)$*.*

The proof is a simple modification of that of [28, Theorem 1.8] and is omitted.

Proposition 6.2 (1) *The map* $t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ *is continuous for all* $p \geq 0$*.*

(2) *If there exists* $p \geq 0$ *such that* $t \mapsto \delta_t \in \mathcal{D}_{-p}$ *is continuous, so is* $t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$*.*

PROOF. (1) Note that there exists $q \geq 0$ such that the map $t \mapsto \delta_t \in \mathcal{E}_{-q}$ is continuous. Moreover, for all $p \geq 0$ there exists $p' \geq p$ such that

$$\|\widehat{a}_t(\xi, K^p\cdot)\|^2_{E^2(\nu)} = |\,\langle \delta_t, \xi\rangle\,|^2 \|\phi_\xi\|^2_{K,p} \leq |\delta_t|^2_{-q} |\xi|_q^2 \|\phi_\xi\|^2_{p',+}, \qquad \xi \in \mathcal{E}.$$

It then follows from Theorem 6.1 that $t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is continuous.

(2) By assumption we have

$$\|\widehat{a_t^*}(\xi, K^{-p}\cdot)\|^2_{E^2(\nu)} = \|\,\langle \delta_t, K^{-p}\cdot\rangle\, e^{\langle \xi, K^{-p}\cdot\rangle}\|^2_{E^2(\nu)} = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!^2} |\delta_t|^2_{K,-p} |\xi|^{2n}_{K,-p}, \qquad \xi \in \mathcal{E}.$$

Since the embedding $\mathcal{E} \hookrightarrow \mathcal{D}_{-p}$ is continuous, there exist $C \geq 0$ and $q \geq 0$ such that

$$\|\widehat{a_t^*}(\xi, K^{-p}\cdot)\|^2_{E^2(\nu)} \leq C |\delta_t|^2_{K,-p}\, e^{|\xi|_q^2}, \qquad \xi \in \mathcal{E}.$$

Then by Theorem 6.1 the map $t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is continuous. ∎

By Theorem 6.1, the following result is immediate.

Theorem 6.3 *Let* $p \in \mathbf{R}$ *and let* $\{\Xi_n\}_{n=0}^{\infty}$ *be a sequence in* $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ *and* $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$*. Then* Ξ_n *converges to* Ξ *in* $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ *if and only if there exist* $M \geq 0$ *and* $q \geq 0$ *such that*

$$\|\widehat{\Xi}_n(\xi, K^p\cdot)\|^2_{E^2(\nu)} \leq M G_\alpha(|\,\xi\,|_q^2), \qquad \xi \in \mathcal{E}, \quad n = 1, 2, \cdots,$$

and for each $\xi \in \mathcal{E}$*,* $\widehat{\Xi}_n(\xi, K^p\cdot)$ *converges to* $\widehat{\Xi}(\xi, K^p\cdot)$ *in* $E^2(\nu)$*.*

6.2 Quantum stochastic integrals

Recall that the topology of $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by the seminorms:

$$\| \Xi \|_{B,B'} = \sup \{ | \langle\langle \Xi\phi, \psi \rangle\rangle | \,;\, \phi \in B,\, \psi \in B' \}, \qquad B, B' \in \mathcal{B},$$

where $\mathcal{B}$ is the class of all bounded subsets of $\mathcal{W}$. Similarly, for fixed $p \in \mathbf{R}$, the topology of $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is defined by

$$\| \Xi \|_{B,p} = \sup \{ \| \Xi\phi \|_{K,p} \,;\, \phi \in B \}, \qquad B \in \mathcal{B}.$$

Lemma 6.4 *Let $\{L_t\}$ be a quantum stochastic process in $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$. Then for any $a, t \in \mathbf{R}$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ there exists a unique operator $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ such that*

$$\langle\langle \Xi_{a,t}(f)\phi, \psi \rangle\rangle = \int_a^t f(s) \langle\langle L_s\phi, \psi \rangle\rangle \, ds, \qquad \phi \in \mathcal{W}, \quad \psi \in \mathcal{G}_{-p}. \tag{6.1}$$

Moreover, $t \mapsto \Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is continuous.

PROOF. Since $s \mapsto L_s$ is continuous, the closed interval $[a, t]$ is mapped to a compact subset of $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$. Hence by applying (ii) in Theorem 6.1 there exists some $q \geq 0$ such that

$$C \equiv \sup_{a \leq s \leq t} \| L_s \|_{\mathcal{L}(\mathcal{W}_q, \mathcal{G}_p)} < \infty.$$

Then for any $s \in [a, t]$ we have

$$|\langle\langle L_s\phi, \psi \rangle\rangle| \leq \| L_s \|_{\mathcal{L}(\mathcal{W}_q, \mathcal{G}_p)} \| \phi \|_{q,+} \| \psi \|_{K,-p} \leq C \| \phi \|_{q,+} \| \psi \|_{K,-p},$$

and

$$\left| \int_a^t f(s) \langle\langle L_s\phi, \psi \rangle\rangle \, ds \right| \leq C \| \phi \|_{q,+} \| \psi \|_{K,-p} \int_a^t |f(s)| ds, \qquad \phi \in \mathcal{W}, \quad \psi \in \mathcal{G}_{-p}. \tag{6.2}$$

Therefore, for each fixed $\phi \in \mathcal{W}$ the right hand side of (6.1) is a continuous linear functional on $\mathcal{G}_{-p}$. Hence, by the Riesz representation theorem there exists a unique $\phi' \in \mathcal{G}_p$ such that

$$\langle\langle \phi', \psi \rangle\rangle = \int_a^t f(s) \langle\langle L_s\phi, \psi \rangle\rangle \, ds.$$

Define a linear operator $\Xi_{a,t}(f)$ from $\mathcal{W}$ into $\mathcal{G}_p$ by $\Xi_{a,t}(f)\phi = \phi'$. Then by (6.2), $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ and (6.1) holds.

It is sufficient to prove the continuity on any finite interval (a_1, b_1). Taking constant numbers $q \geq 0$, $C \geq 0$ as above (considering the closed interval $[a_1, b_1]$), we obtain that for any $a_1 < u < t < b_1$ and $\phi \in \mathcal{W}$, $\psi \in \mathcal{G}_{-p}$

$$| \langle\langle (\Xi_{a,t}(f) - \Xi_{a,u}(f))\phi, \psi \rangle\rangle | \leq C \| \phi \|_{q,+} \| \psi \|_{K,-p} \int_u^t |f(s)| \, ds.$$

Then for bounded subset $B \subset \mathcal{W}$ we have

$$\| \Xi_{a,t}(f) - \Xi_{a,u}(f) \|_{B,p} \leq C \| B \|_{q,+} \int_u^t |f(s)| \, ds, \qquad a_1 < u < t < b_1,$$

where $\| B \|_{q,+} = \sup\{\| \phi \|_{q,+} \; ; \; \phi \in B\} < \infty$. It follows the continuity immediately. ∎

The white noise operator $\Xi_{a,t}(f)$ defined in (6.1) is denoted by

$$\Xi_{a,t}(f) = \int_a^t f(s) L_s \, ds.$$

Theorem 6.5 *Let two quantum stochastic processes $\{L_t\}$ and $\{\Xi_t\}$ in $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ be related as*

$$\Xi_t = \int_a^t L_s \, ds, \qquad t \in \mathbf{R}.$$

Then the map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{G}_p)$ is differentiable and

$$\frac{d}{dt} \Xi_t = L_t$$

holds in $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$.

The proof is straightforward by modifying the argument in [25].

7 White Noise Differential Equations

In this section, we study the following white noise differential equation:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \quad 0 \le t \le T, \tag{7.1}$$

where $F : [0, T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function and Ξ_0 is a white noise operator. A solution of (7.1) must be a C^1-map defined on $[0, T]$ with values in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Obviously, the solution depends on the "regularity property" of the initial value Ξ_0.

7.1 Unique existence

We now consider two weight sequences $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ satisfying conditions (A1)–(A4), the generating functions of which are related in such a way that

$$G_\alpha(t) = \exp \gamma \{G_\omega(t) - 1\}, \tag{7.2}$$

where $\gamma > 0$ is a certain constant. In that case, we have continuous inclusions:

$$\mathcal{W}_\alpha \subset \mathcal{W}_\omega \subset L^2(\mathcal{E}^*_{\mathbf{R}}, \mu) \subset \mathcal{W}^*_\omega \subset \mathcal{W}^*_\alpha$$

and

$$\mathcal{L}(\mathcal{W}_\omega, \mathcal{W}^*_\omega) \subset \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}^*_\alpha).$$

The relation given as in (7.2) is abstracted from the case of Bell numbers, see [5], [6].

Theorem 7.1 (Ji–Obata [16]) *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences satisfying conditions (A1)–(A4), and assume that their generating functions are related as in (7.2). Let $F : [0,T] \times \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*) \to \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ be a continuous function and assume there exist $p \geq 0$ and a nonnegative function $M \in L^1[0,T]$ such that*

(i) *for all $\xi, \eta \in \mathcal{E}$, $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$, and $s \in [0,T]$*

$$|\widehat{F}(s,\Xi_1)(\xi,\eta) - \widehat{F}(s,\Xi_2)(\xi,\eta)|^2 \leq M(s)G_\omega(|\,\xi\,|_p^2)G_\omega(|\,\eta\,|_p^2)|\widehat{\Xi}_1(\xi,\eta) - \widehat{\Xi}_2(\xi,\eta)|^2;$$

(ii) *for all $\xi, \eta \in \mathcal{E}$, $\Xi \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$, and $s \in [0,T]$*

$$|\widehat{F}(s,\Xi)(\xi,\eta)|^2 \leq M(s)G_\omega(|\,\xi\,|_p^2)G_\omega(|\,\eta\,|_p^2)(1 + |\widehat{\Xi}(\xi,\eta)|^2).$$

Then, for any $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ the initial value problem (7.1) has a unique solution $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$, $t \in [0,T]$.

Example Let $\{L_t\}, \{M_t\} \subset \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ be two quantum stochastic processes, where t runs over $[0,T]$. Then the initial value problem

$$\frac{d}{dt}\Xi_t = L_t \diamond \Xi_t + M_t, \qquad \Xi|_{t=0} = \Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*), \tag{7.3}$$

has a unique solution in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$. Note that equation (7.3) is already beyond a traditional quantum stochastic differential equation.

7.2 Regularity of solutions

In this section we study regularity properties of solution Ξ_t of (7.1) as usual operators in $\mathcal{L}(\mathcal{W}, \mathcal{G}_p)$. Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences satisfying conditions (A1)–(A4), and assume that their generating functions are related as in (7.2).

Theorem 7.2 *Let $F : [0,T] \times \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*) \to \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ be a continuous function and assume that there exist $q \geq 0$ and a nonnegative function $M \in L^1[0,T]$, and a nonnegative, locally bounded function g on $\mathcal{E} \times \mathcal{D}_{-p}$ satisfying*

$$\|g(\xi, K^p\cdot)^n\|_{E^2(\nu)}^2 \leq n!\left(RG_\omega(|\xi|_q^2)\right)^n, \qquad n = 1, 2, \cdots \tag{7.4}$$

for some $R \geq 0$ such that

(i) *for all $\xi, \eta \in \mathcal{E}$, $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$, and $s \in [0,T]$*

$$\left|\widehat{F}(s,\Xi_1)(\xi,\eta) - \widehat{F}(s,\Xi_2)(\xi,\eta)\right|^2 \leq M(s)g(\xi,\eta)^2\left|\widehat{\Xi}_1(\xi,\eta) - \widehat{\Xi}_2(\xi,\eta)\right|^2;$$

(ii) *for all $\xi, \eta \in \mathcal{E}$, $\Xi \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$, and $s \in [0,T]$*

$$\left|\widehat{F}(s,\Xi)(\xi,\eta)\right|^2 \leq M(s)g(\xi,\eta)^2\left(1 + \left|\widehat{\Xi}(\xi,\eta)\right|^2\right).$$

Then, for any $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ *satisfying*

$$\left\| g(\xi, K^p\cdot)^n \widehat{\Xi}_0(\xi, K^p\cdot) \right\|_{E^2(\nu)}^2 \le n! \left(R' G_\omega(|\xi|_{q'}^2) \right)^n, \qquad n = 1, 2, \cdots, \tag{7.5}$$

for some $R' \ge 0$ *and* $q' \ge 0$, *the initial value problem* (7.1) *has a unique solution* $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$, $t \in [0, T]$.

PROOF. In principle, the proof is based on the standard Picard–Lindelöf method of successive approximations (see e.g., [11]) applied to the operator symbols. We define

$$\Xi_t^{(0)} = \Xi_0,$$
$$\Xi_t^{(n)} = \Xi_0 + \int_0^t F(s, \Xi_s^{(n-1)})\, ds, \qquad n \ge 1.$$

Then by applying Theorem 5.2 we see that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ for all n and $0 \le t \le T$. In fact, by (ii)

$$\begin{aligned} \left| \int_0^t \widehat{F}(s, \Xi_0)(\xi, \eta) ds \right|^2 &\le T \int_0^t |\widehat{F}(s, \Xi_0)(\xi, \eta)|^2 ds \\ &\le T \left(\int_0^t M(s) ds \right) g(\xi, \eta)^2 \left(1 + |\widehat{\Xi}_0(\xi, \eta)|^2 \right) \end{aligned}$$

and hence by assumption we have

$$\left\| \int_0^t \widehat{F}(s, \Xi_0)(\xi, K^p\cdot) ds \right\|_{E^2(\nu)}^2 \le T \left(\int_0^t M(s) ds \right) \left\{ R_1 G_\omega(|\xi|_{q_1}^2) + R_2 G_\omega(|\xi|_{q_2}^2) \right\} \tag{7.6}$$

for some $R_1, R_2 \ge 0$ and $q_1, q_2 \ge 0$. Moreover, since $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$, we see from Theorem 5.2 that

$$\|\widehat{\Xi}_0(\xi, K^p\cdot)\|_{E^2(\nu)}^2 \le R_3 G_\omega(|\xi|_{q_3}^2) \tag{7.7}$$

for some $R_3 \ge 0$ and $q_3 \ge 0$. Put

$$R = \max\left\{ 2T \left(\int_0^t M(s) ds \right) R_1, 2T \left(\int_0^t M(s) ds \right) R_2, 2R_3 \right\}$$

and $q = \max\{q_1, q_2, q_3\}$. Then by (7.6), (7.7) we have

$$\begin{aligned} \left\| \widehat{\Xi_t^{(1)}}(\xi, K^p\cdot) \right\|_{E^2(\nu)}^2 &\le 2 \left(\left\| \int_0^t \widehat{F}(s, \Xi_0)(\xi, K^p\cdot) ds \right\|_{E^2(\nu)}^2 + \|\widehat{\Xi}_0(\xi, K^p\cdot)\|_{E^2(\nu)}^2 \right) \\ &\le 3RG_\omega(|\xi|_q^2). \end{aligned}$$

Hence by Theorem 5.2 we see that $\Xi_t^{(1)} \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ for all $0 \le t \le T$. The above argument can be repeated to conclude that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ for all n and $0 \le t \le T$. Moreover, from Theorem 6.1 we see that $\{\Xi_t^{(n)}\} \subset \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ is a quantum stochastic process.

We shall prove step by step that $\lim_{n\to\infty} \Xi_t^{(n)}$ is the desired solution to our equation. For simplicity we put

$$\Theta_n(t;\xi,\eta) = \widehat{\Xi_t^{(n)}}(\xi,\eta) = \langle\!\langle \Xi_t^{(n)}\phi_\xi,\, \phi_\eta \rangle\!\rangle, \qquad \xi,\eta\in\mathcal{E}, \quad 0\le t\le T.$$

By (i) we see that

$$\begin{aligned}|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)| &= \left| \int_0^t \left\{ \widehat{F}(s,\Xi_s^{(n-1)})(\xi,\eta) - \widehat{F}(s,\Xi_s^{(n-2)})(\xi,\eta) \right\} ds \right| \\ &\le g(\xi,\eta) \int_0^t \sqrt{M(s)}\, |\Theta_{n-1}(s;\xi,\eta) - \Theta_{n-2}(s;\xi,\eta)|\, ds, \end{aligned} \tag{7.8}$$

for $\xi,\eta\in\mathcal{E}$ and $0\le t\le T$. Then, repeating this argument we come to

$$\begin{aligned}&|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)| \\ &\quad\le \{g(\xi,\eta)\}^{n-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \\ &\qquad\qquad \times \sqrt{M(t_1)}\sqrt{M(t_2)}\cdots\sqrt{M(t_{n-1})}|\Theta_1(t_{n-1};\xi,\eta) - \Theta_0(t_{n-1};\xi,\eta)|. \end{aligned} \tag{7.9}$$

As for the last quantity it follows from (ii) that

$$|\Theta_1(t;\xi,\eta) - \Theta_0(t;\xi,\eta)| = \left| \int_0^t \widehat{F}(s,\Xi_0)(\xi,\eta)\, ds \right| \le \int_0^t |\widehat{F}(s,\Xi_0)(\xi,\eta)| ds. \tag{7.10}$$

For simplicity we put

$$H(\xi,\eta) = \overline{M} g(\xi,\eta)\sqrt{1 + |\widehat{\Xi}_0(\xi,\eta)|^2}, \qquad \overline{M} = \int_0^T \sqrt{M(s)}\, ds.$$

Then (7.10) becomes

$$|\Theta_1(t;\xi,\eta) - \Theta_0(t;\xi,\eta)| \le H(\xi,\eta).$$

Similarly, (7.9) becomes

$$\begin{aligned}&|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)| \le \{g(\xi,\eta)\}^{n-1} H(\xi,\eta) \times \\ &\qquad \times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \sqrt{M(t_1)}\sqrt{M(t_2)}\cdots\sqrt{M(t_{n-1})} \\ &\quad \le \frac{1}{(n-1)!} \left\{\overline{M} g(\xi,\eta)\right\}^{n-1} H(\xi,\eta). \end{aligned} \tag{7.11}$$

It then follows that for each $\xi,\eta\in\mathcal{E}$, the series

$$\sum_{n=1}^{\infty} \{\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)\}$$

converges absolutely and uniformly in $t \in [0, T]$. In fact,

$$\begin{aligned}\sum_{n=1}^{\infty} |\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)| &\leq H(\xi,\eta)\exp\left\{\overline{M}\, g(\xi,\eta)\right\} \\ &\leq \exp\left\{2\overline{M}\, g(\xi,\eta)\right\}\sqrt{1+|\widehat{\Xi}_0(\xi,\eta)|^2}. \end{aligned} \tag{7.12}$$

We now put

$$\Theta_t(\xi,\eta) = \lim_{n\to\infty} \Theta_n(t;\xi,\eta) = \widehat{\Xi}_0(\xi,\eta) + \sum_{n=1}^{\infty}\Big\{\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)\Big\}.$$

Since g is bounded on every bounded subset of $\mathcal{E} \times \mathcal{D}_{-p}$, by (7.12) we can easily see that for any $\xi, \xi' \in \mathcal{E}$ and $\eta, \eta' \in \mathcal{D}_{-p}$, the series

$$\sum_{n=1}^{\infty}\Big\{\Theta_n(t;\lambda\xi+\xi',\gamma\eta+\eta') - \Theta_{n-1}(t;\lambda\xi+\xi',\gamma\eta+\eta')\Big\}$$

converges uniformly on every compact subset of $\mathbf{C}\times\mathbf{C}$. Therefore for each $0 \leq t \leq T$ the map $(\lambda,\gamma) \mapsto \Theta_t(\lambda\xi+\xi',\gamma\eta+\eta')$ is holomorphic on $\mathbf{C}\times\mathbf{C}$. Also, by (7.12)

$$\begin{aligned}&\left\|\sum_{n=1}^{\infty}\{\Theta_n(t;\xi,K^p\cdot) - \Theta_{n-1}(t;\xi,K^p\cdot)\}\right\|_{E^2(\nu)}^2 \\ &\qquad\leq \left\|\exp\left\{2\overline{M}\, g(\xi,K^p\cdot)\right\}\sqrt{1+|\widehat{\Xi}_0(\xi,K^p\cdot)|^2}\right\|_{E^2(\nu)}^2. \end{aligned} \tag{7.13}$$

On the other hand, by the Schwartz inequality, for any $0 < r < 1$

$$\exp\left\{2\overline{M}\, g(\xi,\eta)\right\} \leq \frac{1}{\sqrt{1-r}}\left(\sum_{n=0}^{\infty}\frac{(2\overline{M})^{2n}}{r^n n!^2} g(\xi,\eta)^{2n}\right)^{1/2}.$$

Therefore, by (7.4) and (7.5) there exist $R_1, R_2 \geq 0$ and $q_1, q_2 \geq 0$ such that

$$\begin{aligned}&\left\|\exp\left\{2\overline{M}\, g(\xi,K^p\cdot)\right\}\sqrt{1+|\widehat{\Xi}_0(\xi,K^p\cdot)|^2}\right\|_{E^2(\nu)}^2 \\ &\leq \frac{1}{1-r}\sum_{n=0}^{\infty}\frac{(2\overline{M})^{2n}}{r^n n!^2}\left\{n!\left(R_1 G_\omega(|\xi|_{q_1}^2)\right)^n + n!\left(R_2 G_\omega(|\xi|_{q_2}^2)\right)^n\right\} \\ &\leq \frac{2}{1-r}\sum_{n=0}^{\infty}\frac{(2\overline{M})^{2n}}{r^n n!^2} n!\left(R G_\omega(|\xi|_q^2)\right)^n \\ &= \frac{2}{1-r}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{4\overline{M}^2 R}{r} G_\omega(|\xi|_q^2)\right)^n, \end{aligned} \tag{7.14}$$

where $R = R_1 \vee R_2$ and $q = q_1 \vee q_2$. We fix $0 < r < 1$ in such a way that

$$M_0(\gamma) \equiv \frac{4\overline{M}^2 R}{r\gamma} \geq 1,$$

where γ is the constant defined in (7.2). Then by (3.2) we have

$$\begin{aligned}
\frac{4\overline{M}^2R}{r}G_\omega(|\xi|_q^2) &= \gamma M_0(\gamma)\,G_\omega\left(|\xi|_q^2\right) \\
&= \gamma\left\{M_0(\gamma)\left[G_\omega\left(|\xi|_q^2\right)-1\right]\right\}+\gamma M_0(\gamma) \\
&\le \gamma\left\{G_\omega\left(M_0(\gamma)\,|\xi|_q^2\right)-1\right\}+\gamma M_0(\gamma).
\end{aligned}$$

Choosing $r_0 \geq 0$ such that

$$M_0(\gamma)\rho^{2r_0} \leq 1,$$

we obtain

$$\frac{4\overline{M}^2R}{r}G_\omega(|\xi|_q^2) \leq \gamma\left\{G_\omega(|\xi|_{q+r_0}^2)-1\right\}+\gamma M_0(\gamma). \tag{7.15}$$

Hence, by (7.2), (7.13), (7.14) and (7.15) we have

$$\left\|\sum_{n=1}^{\infty}\{\Theta_n(t;\xi,K^p\cdot)-\Theta_{n-1}(t;\xi,K^p\cdot)\}\right\|_{E^2(\nu)}^2 \leq \frac{2}{1-r}e^{(4\overline{M}^2R)/r}G_\alpha(|\xi|_{q+r_0}^2).$$

Therefore, for the function Θ_t condition (ii) in Theorem 5.2 is satisfied since $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$. Hence by Theorem 5.2 there exists a unique operator $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ such that

$$\Theta_t(\xi,\eta) = \widehat{\Xi}_t(\xi,\eta), \qquad \xi,\eta\in\mathcal{E}, \quad t\in[0,T].$$

By applying Theorem 6.3 with (7.11) we see that

$$\Xi_t = \lim_{n\to\infty}\Xi_t^{(n)} \qquad \text{uniformly in } t,$$

and hence the map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$ is continuous.

We now prove that $\{\Xi_t\}$ is a solution of (7.1). We first note that, by Lemma 6.4, the integral

$$\int_0^t F(s,\Xi_s)ds$$

is well-defined as an operator in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$. On the other hand, by assumption (i) we have

$$\begin{aligned}
&\left\|\int_0^t\left(\widehat{F}(s,\Xi_s)(\xi,K^p\cdot)-\widehat{F}(s,\Xi_s^{(n)})(\xi,K^p\cdot)\right)ds\right\|_{E^2(\nu)}^2 \\
&\le \left\|\int_0^t\sqrt{M(s)}g(\xi,K^p\cdot)\left|\widehat{\Xi}_s(\xi,K^p\cdot)-\widehat{\Xi}_s^{(n)}(\xi,K^p\cdot)\right|ds\right\|_{E^2(\nu)}^2 \\
&\le \left\|\int_0^t\sqrt{M(s)}g(\xi,K^p\cdot)\sum_{k=n+1}^{\infty}|\Theta_k(s;\xi,K^p\cdot)-\Theta_{k-1}(s;\xi,K^p\cdot)|\,ds\right\|_{E^2(\nu)}^2.
\end{aligned}$$

Therefore, by (7.11), for any $0 < r < 1$ we have

$$\left\| \int_0^t \left(\widehat{F}(s, \Xi_s)(\xi, K^p\cdot) - \widehat{F}(s, \Xi_s^{(n)})(\xi, K^p\cdot) \right) ds \right\|_{E^2(\nu)}^2$$
$$\leq \left\| \overline{M} g(\xi, K^p\cdot) \sum_{k=n}^{\infty} \frac{1}{k!} \left\{ \overline{M} g(\xi, K^p\cdot) \right\}^k H(\xi, K^p\cdot) \right\|_{E^2(\nu)}^2$$
$$\leq \frac{r^n}{1-r} \sum_{k=n}^{\infty} \frac{\overline{M}^{2(k+2)}}{r^k k!^2} \left\| g(\xi, K^p\cdot)^{k+2} \sqrt{1 + |\widehat{\Xi}_0(\xi, K^p\cdot)|^2} \right\|_{E^2(\nu)}^2$$
$$\leq \frac{2r^n}{1-r} \sum_{k=n}^{\infty} \frac{\overline{M}^{2(k+2)}}{r^k k!^2} (k+2)! \left(R G_\omega(|\xi|_q^2) \right)^{k+2},$$

where $R \geq 0$ and $p \geq 0$ are pointed out in (7.14). It follows from Theorem 6.3 that

$$\lim_{n\to\infty} \int_0^t F(s, \Xi_s^{(n)}) ds = \int_0^t F(s, \Xi_s) ds.$$

Hence we see that

$$\Xi_t = \lim_{n\to\infty} \Xi_t^{(n)} = \Xi_0 + \int_0^t F(s, \Xi_s) ds$$

which shows from Theorem 6.5 that $\{\Xi_t\}$ is a solution.

Finally we prove the uniqueness. Assume that we have two solutions $\{\Xi_t\}$ and $\{X_t\}$, which satisfies the same integral equation. Modeled after the derivation of (7.8), we come to

$$\left| \widehat{\Xi}_t(\xi, \eta) - \widehat{X}_t(\xi, \eta) \right| \leq g(\xi, \eta) \int_0^t \sqrt{M(s)} \left| \widehat{\Xi}_s(\xi, \eta) - \widehat{X}_s(\xi, \eta) \right| ds.$$

Then $\widehat{\Xi}_t = \widehat{X}_t$ follows by a standard argument with the Gronwall inequality. ∎

The following result for regular solution of normal-ordered white noise differential equation is immediate from Theorem 7.2. For more relevant study of regularity of solutions of normal-ordered white noise differential equations, we refer to [5].

Corollary 7.3 *Let $p \in \mathbf{R}$ and let $\{L_t\}, \{M_t\} \subset \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ be two quantum stochastic processes, where t runs over $[0, T]$, satisfying the conditions: there exist $q \geq 0$ and a nonnegative function $H \in L^1[0, T]$, and a nonnegative, locally bounded function g on $\mathcal{E} \times \mathcal{D}_{-p}$ satisfying*

$$\|g(\xi, K^p\cdot)^n\|_{E^2(\nu)}^2 \leq n! \left(R G_\omega(|\xi|_q^2) \right)^n, \qquad n = 1, 2, \cdots$$

for some $R \geq 0$ such that for all $\xi, \eta \in \mathcal{E}$ and $s \in [0, T]$

$$\max\left\{ \left| \widehat{L}_s(\xi, \eta) e^{-\langle \xi, \eta \rangle} \right|^2, \left| \widehat{M}_s(\xi, \eta) \right|^2 \right\} \leq H(s) g(\xi, \eta)^2.$$

Then for any $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{G}_p)$ satisfying

$$\left\| g(\xi, K^p\cdot)^n \widehat{\Xi}_0(\xi, K^p\cdot) \right\|_{E^2(\nu)}^2 \leq n! \left(R G_\omega(|\xi|_q^2) \right)^n, \qquad n = 1, 2, \cdots,$$

for some $R \geq 0$ and $q \geq 0$, the (normal-ordered) white noise differential equation (7.3) has a unique solution Ξ_t in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{G}_p)$, $t \in [0, T]$.

References

[1] N. Asai, I. Kubo and H.-H. Kuo: *General characterization theorems and intrinsic topologies in white noise analysis*, Hiroshima Math. J. **31** (2001), 299–330.

[2] Yu. M. Berezansky and Yu. G. Kondratiev: "Spectral Methods in Infinite-Dimensional Analysis," Kluwer Academic, 1995.

[3] D. M. Chung, U. C. Ji and N. Obata: *Higher powers of quantum white noises in terms of integral kernel operators*, Infinite Dimen. Anal. Quantum Probab. **1** (1998), 533–559.

[4] D. M. Chung, U. C. Ji and N. Obata: *Normal-ordered white noise differential equations II: Regularity properties of solutions*, in "Probability Theory and Mathematical Statistics (B. Grigelionis et al. Eds.)," pp. 157–174, VSP/TEV, 1999.

[5] D. M. Chung, U. C. Ji and N. Obata: *Quantum stochastic analysis via white noise operators in weighted Fock space*, to appear in Rev. Math. Phys. 2002.

[6] W. G. Cochran, H.-H. Kuo and A. Sengupta: *A new class of white noise generalized functions*, Infinite Dimen. Anal. Quantum Probab. **1** (1998), 43–67.

[7] R. Gannoun, R. Hachaichi, H. Ouerdiane and A. Rezgui: *Un théorème de dualité entre espace de fonctions holomorphes à croissance exponentielle*, J. Funct. Anal. **171** (2000), 1–14.

[8] I. M. Gelfand and N. Ya. Vilenkin: "Generalized Functions, Vol.4," Academic Press, 1964.

[9] L. Gross and P. Malliavin: *Hall's transform and the Segal–Bargmann map*, in "Itô's Stochastic Calculus and Probability Theory (N. Ikeda, S. Watanabe, M. Fukushima and H. Kunita (Eds.)," pp. 73–116, Springer-Verlag, 1996.

[10] M. Grothaus, Yu. G. Kondratiev and L. Streit: *Complex Gaussian analysis and the Bargmann–Segal space*, Methods of Funct. Anal. and Topology **3** (1997), 46–64.

[11] P. Hartman: "Ordinary Differential Equations (second edition)," Birkhäuser, 1982.

[12] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes, no. 13, Carleton University, Ottawa, 1975.

[13] T. Hida: "Brownian Motion," Springer–Verlag, 1980.

[14] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White Noise: An Infinite Dimensional Calculus," Kluwer Academic Publishers, 1993.

[15] R. L. Hudson and K. R. Parthasarathy: *Quantum Itô's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[16] U. C. Ji and N. Obata: *Initial value problem for white noise operators and quantum stochastic processes*, in "Infinite Dimensional Harmonic Analysis (H. Heyer, T. Hirai and N. Obata, Eds.)," pp.203–216, D.+M. Gräbner, Tübingen, 2000.

[17] U. C. Ji and N. Obata: *A role of Bargmann-Segal spaces in characterization and expansion of operators on Fock space*, preprint, 2000.

[18] U. C. Ji, N. Obata and H. Ouerdiane: *Analytic characterization of generalized Fock space operators as two-variable entire functions with growth condition*, to appear in Infinite Dimen. Anal. Quantum Probab.

[19] Yu. G. Kondratiev and L. Streit: *Spaces of white noise distributions: Constructions, descriptions, applications I*, Rep. Math. Phys. **33** (1993), 341–366.

[20] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I*, Proc. Japan Acad. **56A** (1980), 376–380.

[21] H.-H. Kuo: "White Noise Distribution Theory," CRC Press, 1996.

[22] Y.-J. Lee and H.-H. Shih: *The Segal-Bargmann transform for Lévy functionals*, J. Funct. Anal. **168** (1999), 46–83.

[23] P.-A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer-Verlag, 1993.

[24] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.

[25] N. Obata: *Generalized quantum stochastic processes on Fock space*, Publ. RIMS **31** (1995), 667–702.

[26] N. Obata: *Integral kernel operators on Fock space – Generalizations and applications to quantum dynamics*, Acta Appl. Math. **47** (1997), 49–77.

[27] N. Obata: *Quantum stochastic differential equations in terms of quantum white noise*, Nonlinear Analysis, Theory, Methods and Applications **30** (1997), 279–290.

[28] N. Obata: *Wick product of white noise operators and quantum stochastic differential equations*, J. Math. Soc. Japan, **51** (1999), 613–641.

[29] K. R. Parthasarathy: "An introduction to quantum stochastic calculus," Birkhäuser, 1992.

[30] Y. Yokoi: *Simple setting for white noise calculus using Bargmann space and Gauss transform*, Hiroshima Math. J. **25** (1995), 97–121.