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GALOIS ACTION ON THE FUNDAMENTAL GROUPS OF CURVES AND THE CYCLE $C-C^{-}$

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1. Introduction

This is an announcement of a joint work with Richard Hain at Duke University. This manuscript announces some results which closely relate the action of Galois groups on the fundamental groups of curves and the vanishing of the image of the algebraic cycles $C-C^{-}$ by so-called ℓ-adic Abel-Jacobi map. The main ingredients in the proof are Dennis Johnson's computation of the abelianization of the Torreli groups [8] and the theory of relative and weighted completions of classical and arithmetic mapping class groups.

Suppose that K is a field of characteristic zero and that X is a smooth projective variety defined over K. By standard constructions (see §4), a homologically trivial algebraic cycle Z of codimension r in X, defined over K, determines an extension of $\mathbb{Z}_\ell$ by an etale cohomology

$$0 \to H^{2r-1}(X\otimes_K \overline{K}, \mathbb{Z}_\ell(r)) \to E_Z \to \mathbb{Z}_\ell(0) \to 1$$

as $G_K := \mathrm{Gal}(\overline{K}/K)$-modules, and therefore a class

$$e_Z \in H^1(G_K, H^{2r-1}(X\otimes_K \overline{K}, \mathbb{Z}_\ell(r))).$$

This class depends only on the rational equivalence class of Z (over $\overline{K}$), and hence defines the ℓ-adic Abel Jacobi map

$$\mathrm{AJ} : CH^r_{hom}(X) \to H^1(G_K, H^{2r-1}(X\otimes_K \overline{K}, \mathbb{Z}_\ell(r)).$$

Suppose that C is a smooth projective curve of genus $g\geq 3$ over K and that x is a K-rational point of C. The morphism

$$\sigma_x : C \to \mathrm{Jac}\, C$$

that takes $z\in C$ to the divisor class of $z-x$ is an embedding and defines an algebraic 1-cycle $C_x := (\sigma_x)_* C$ in $\mathrm{Jac}\, C$. One also has the cycle $C_x^{-} := i_* C_x$, where i is the involution of the jacobian that takes each point to its inverse.

Two algebraic cycles are particularly relevant to Galois actions on fundamental groups:

(i) here X is the curve C and Z is the divisor $(2g-2)x - K_C$ in C, where K_C is any canonical divisor of C;
(ii) here X is the jacobian $\mathrm{Jac}\, C$ of C and $Z = C_x - C_x^{-}$.

Both are homologically trivial. The first defines a class

$$\kappa(C,x) \in H^1(G_K, H^1(C\otimes \overline{K}, \mathbb{Z}_\ell(1)))$$

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and the second a class

$$\mu(C,x) \in H^1(G_K, H^{2g-3}(\operatorname{Jac} C \otimes \overline{K}, \mathbf{Z}_\ell(g-1))).$$

Set $H_{\mathbf{Z}_\ell} = H^1(C \otimes \overline{K}, \mathbf{Z}_\ell(1))$ and $L_{\mathbf{Z}_\ell} = (\Lambda^3 H)(-1)$. Both are of weight -1. Denote their tensor products with $\mathbf{Q}_\ell$ by H and L, respectively. Wedging with the polarization $q \in \Lambda^2 H(-1)$ defines a G_K-invariant embedding $H \to L$. Let

$$\nu(C) = \text{ the image of } \mu(C,x) \text{ in } H^1(G_K, L/H).$$

This is independent of the choice of x.

Suppose now that K is a subfield of $\mathbf{C}$. Denote the Lie algebra of the $\mathbf{Q}_\ell$-form of the unipotent (aka, Malcev) completion of $\pi_1(C(\mathbf{C}),x)$ by $\mathfrak{p}(X,x)$. This is a pronilpotent Lie algebra. Denote the mapping class group of a closed, pointed, orientable genus g surface by Γ_g^1. That is

$$\Gamma_g^1 = \pi_0 \operatorname{Diff}^+(S,o)$$

where S is a compact oriented surface of genus g and $o \in S$, where $^+$ means orientation preserving. The mapping class group Γ_g^1 acts on $\pi_1(C(\mathbf{C}),x)$, therefore there is a homomorphism

$$\theta^{1,\mathrm{geom}} : \Gamma_g^1 \to \operatorname{Aut} \mathfrak{p}(C,x).$$

On the other hand, the theory of algebraic fundamental groups gives an outer Galois representation

$$\theta^{1,\mathrm{arith}} : G_K \to \operatorname{Aut} \mathfrak{p}(C,x) \otimes \mathbf{Q}_\ell.$$

We have m-th truncated representation, namely

$$\theta_m^{1,\mathrm{arith}} : G_K \to \operatorname{Aut}(\mathfrak{p}(C,x)/L^{m+1}) \otimes \mathbf{Q}_\ell,$$

where L^m denotes the mth term of the lower central series of $\mathfrak{p}(C,x)$. If m is 1, then we have the Galois action on the abelianization of $\mathfrak{p}(C,x)$, which is isomorphic to H. It is well known that this action preserves the cup product, hence if $m = 1$ we have

$$\theta_1^{1,\mathrm{arith}} : G_K \to GSp(H).$$

The group $\operatorname{Aut} \mathfrak{p}(C,x)$ is an affine proalgebraic group, being the inverse limit of the automorphism groups of the $\mathfrak{p}(C,x)/L^m$. Similarly we define θ_m^{geom}, etc.

Theorem 1. *The Zariski closure of the image of $\theta^{1,\mathrm{arith}}$ contains the image of $\theta^{1,\mathrm{geom}}$ if and only if the following 3 conditions are satisfied:*

(i) *the homomorphism $\theta_1^{1,\mathrm{arith}} : G_K \to GSp(H)$ is Zariski dense;*
(ii) *the class $\kappa(C,x)$ is non-zero in $H^1(G_K,H)$;*
(iii) *the class $\nu(C)$ is non-zero in $H^1(G_K, L/H)$.*

These conditions are equivalent to that the Zariski closure of the image of $\theta 1,^{\mathrm{arith}}{}_2$ contains the image of $\theta_2^{1,\mathrm{geom}}$, too.

The Zariski closure of the image of $\theta^{1,\mathrm{arith}}$ contains the image of the conjugation mapping $\pi_1(C(\mathbf{C}),x) \to \operatorname{Aut} \mathfrak{p}(C,x)$ if and only if $\kappa(C,x)$ is not zero in $H^1(G_K,H)$.

The mapping class group Γ_g is defined by

$$\Gamma_g = \pi_0 \operatorname{Diff}^+ S.$$

It is the quotient of Γ_g^1 by the canonical copy of $\pi_1(S,o)$.

The group $\pi_1(C(\mathbb{C}), x)$ acts on $\mathfrak{p}(C, x)$ by conjugation. Denote the Zariski closure of its image by $\operatorname{Inn}\mathfrak{p}(C, x)$. Set

$$\operatorname{Out}\mathfrak{p}(C, x) = \operatorname{Aut}\mathfrak{p}(C, x)/\operatorname{Inn}\mathfrak{p}(C, x).$$

This is a proalgebraic group. The homomorphisms $\theta^{1,\mathrm{arith}}$ and $\theta^{1,\mathrm{geom}}$ induce homomorphisms

$$\theta^{\mathrm{arith}} : G_K \to \operatorname{Out}\mathfrak{p}(C, x) \text{ and } \theta^{\mathrm{geom}} : \Gamma_g \to \operatorname{Out}\mathfrak{p}(C, x).$$

These are independent of the choice of x (actually these can be defined even if C has no K-rational point).

Theorem 2. *The Zariski closure of the image of θ^{arith} contains the image of θ^{geom} if and only if the following 2 conditions are satisfied:*

(i) *the homomorphism $\theta_1^{\mathrm{arith}} : G_K \to GSp(H)$ is Zariski dense;*
(ii) *the class $\nu(C)$ is non-zero in $H^1(G_K, L/H)$.*

Let $J := J(C, x)$ be the augumentation ideal of completed group ring $\mathbb{Z}_\ell[[\pi_1(C \otimes \overline{K})]]$. The extension

$$0 \to (H_{\mathbb{Z}_\ell} \otimes H)/q \to J/J^3 \to H_{\mathbb{Z}_\ell} \to 0$$

given by the Galois action on J determines an element of

$$e(C, x) \in H^1(G_K, H_{\mathbb{Z}_\ell} \otimes (H_{\mathbb{Z}_\ell} \otimes H)/q(-1)).$$

There is an inclusion

$$L_{\mathbb{Z}_\ell} \to H_{\mathbb{Z}_\ell} \otimes (H_{\mathbb{Z}_\ell} \otimes H)/q(-1)$$

defined by

$$a \wedge b \wedge c \mapsto a \otimes (b \wedge c) - b \otimes (a \wedge c) + c \otimes (a \wedge b)$$

It induces a mapping of Galois cohomology groups.

Theorem 3. *The class e lies in the image of*

$$p : H^1(G_K, L_{\mathbb{Z}_\ell}) \to H^1(G_K, H_{\mathbb{Z}_\ell} \otimes (H_{\mathbb{Z}_\ell} \otimes H)/q(-1))$$

and $p\mu(C, x) = 2e(C, x)$.

In the rest, we shall explain briefly some concepts around the proof of the theorems, and sketch the proof of Theorem 2 only.

2. Monodromy representation on fundamental groups

Let K be a field. In this article, a variety over K means a smooth geometrically connected scheme of finite type over K. Let $\mathcal{C} \to \mathcal{B}$ be a family of smooth (g, n)-curves, where g denotes the genus and n denotes the number of punctures. In our terminology, this means that there is a proper smooth morphism $\bar{\mathcal{C}} \to \mathcal{B}$ of schemes with each geometric fiber being a proper smooth curve of genus g, and $\mathcal{C}$ is the compliment in $\bar{\mathcal{C}}$ of a relatively normal crossing divisor $\mathcal{D} \to \mathcal{B}$ of relative degree n.

Let x be a geometric point on $\mathcal{B}$, and $\mathcal{C}_x$ the fiber over x, which is a (g, n)-curve. Suppose that the hyperbolicity condition $2g - 2 + n > 0$ is satisfied. Then we have the injectivity at the left in the fiber homotopy exact sequence, namely

$$1 \to \pi_1(\mathcal{C}_x) \to \pi_1(\mathcal{C}) \to \pi_1(\mathcal{B}, x) \to 1$$

is exact, where $\pi_1()$ denotes the algebraic fundamental group in the sense of SGA1[2], and we omit the base point if it does not cause a confusion.

Since the left group is normal in the middle, by conjugation we have a representation

$$\pi_1(\mathcal{C}) \to \operatorname{Aut} \pi_1(\mathcal{C}_x).$$

The subgroup $\pi_1(\mathcal{C}_x)$ in the left is mapped to the inner automorphisms, so by taking quotient we have

$$\pi_1(\mathcal{B}, x) \to \operatorname{Out} \pi_1(\mathcal{C}_x) := \operatorname{Aut} \pi_1(\mathcal{C}_x)/\operatorname{Inn} \pi_1(\mathcal{C}_x),$$

which is called *the monodromy representation on the fundamental group*, since this construction in the classical topological category coincides with the classical monodromy representation.

For simplicity, we assume that K has characteristic zero from now on. Then by GAGA in SGA1, as an abstract topological group,

$$\pi_1(\mathcal{C}_x) \cong \Pi_{g,n}^{\wedge}$$

holds, where $\Pi_{g,n}$ is the fundamental group of an oriented (real) surface with genus g and n punctures, and $^\wedge$ denotes its profinite completion. Thus, we have the monodromy representation

$$\rho_{\mathcal{C}}^{\wedge} : \pi_1(\mathcal{B}, x) \to \operatorname{Out} \Pi_{g,n}^{\wedge}. \tag{1}$$

It is often interesting to treat other completion than the profinite completion. One of such variants is the pro-ℓ completion, which we denote by $\Pi_{g,n}^{(\ell)}$. Another is the Malcev completion $\Pi_{g,n}^{\mathrm{unip}}$. Let us define the Malcev completion briefly. Let F be a topological field of characteristic zero. (If F is not given a topology, we may regard it with discrete topology.) We consider the category of pro-algebraic groups over F, which is defined as the pro-category of the category of affine algebraic groups over F. The category of pro-unipotent groups over F is the full subcategory of the pro-objects of unipotent groups over F.

Consider the functor from the category of pro-unipotent groups over F to the category of topological groups, given by taking the F-rational points with the topology induced from F. Then, one can prove that there is a left adjoint functor to this functor, which is the *continuous Malcev completion* over F. It is a functor from the category of topological groups to the category of pro-algebraic groups over F. The Malcev completion of a topological group Γ over F is denoted by $\Gamma_{/F}^{\mathrm{unip}}$.

One can show that if Γ is a finitely generated discrete group, then

$$\Gamma_{/\mathbb{Q}}^{\mathrm{unip}} \otimes \mathbb{Q}_\ell \cong \Gamma_{/\mathbb{Q}_\ell}^{\mathrm{unip}} \cong (\Gamma^{(\ell)})_{/\mathbb{Q}_\ell}^{\mathrm{unip}} \cong (\Gamma^{\wedge})_{/\mathbb{Q}_\ell}^{\mathrm{unip}}.$$

Thus, the profinite monodromy representation (1) yields by functority the pro-unipotent monodromy representation

$$\rho_{\mathcal{C}}^{\mathrm{unip}} : \pi_1(\mathcal{B}, x) \to \operatorname{Out} \Pi_{g,n}^{\mathrm{unip}}, \tag{2}$$

which factors through

$$\rho_{\mathcal{C}}^{(\ell)} : \pi_1(\mathcal{B}, x) \to \operatorname{Out} \Pi_{g,n}^{(\ell)}. \tag{3}$$

Our basic interest is to know the image of

$$\rho_{\mathcal{C}}^{*} : \pi_1(\mathcal{B}, x) \to \operatorname{Out} \Pi_{g,n}^{*},$$

for $* = \wedge, (\ell)$, or unip.

Assume that $\mathcal{C} \to \mathcal{B}$ is a curve $C \to \operatorname{Spec} K$. Our main result says that the Zariski closure of the image of ρ_{C}^{unip} ($= \theta^{\mathrm{arith}}$ in the previous section) is well controlled by the vanishing of the image of the algebraic cycle $C - C^-$ by the ℓ-adic Abel Jacobi map.

3. Universal monodromy

Assume $2g-2+n>0$. Let $\mathcal{M}_{g,n,\mathbb{Z}}$ be the moduli stack of (g,n)-curves over $\mathbb{Z}$ (see [1]). Then $\mathcal{M}_{g,n+1,\mathbb{Z}} \to \mathcal{M}_{g,n,\mathbb{Z}}$ is the universal family of (g,n)-curves, i.e., for any family of (g,n)-curves $\mathcal{C} \to \mathcal{B}$, there exists a unique morphism $[\mathcal{C}] : \mathcal{B} \to \mathcal{M}_{g,n}$ called the *classifying morphism* such that $\mathcal{C} \to \mathcal{B}$ is isomorphic to the pull back of $\mathcal{M}_{g,n+1,\mathbb{Z}} \to \mathcal{M}_{g,n,\mathbb{Z}}$ along $[\mathcal{C}]$.

Let us denote $\mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Q}$ simply by $\mathcal{M}_{g,n}$. Then $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ has the universality that any family of (g,n)-curves $\mathcal{C} \to \mathcal{B}$ over a field of characteristic zero is realized by pulling back along a unique classifying map, and thus

$$\begin{array}{rccc} \rho^*_{\mathcal{C}} : & \pi_1(\mathcal{B}) & \to & \operatorname{Out}\Pi^*_{g,n} \\ & {\scriptstyle \pi_1[\mathcal{C}]}\downarrow & & \| \\ \rho^*_{\mathcal{M}_{g,n+1}} : & \pi_1(\mathcal{M}_{g,n}) & \to & \operatorname{Out}\Pi^*_{g,n} \end{array}$$

commutes. Consequently, every monodromy representation $\rho^*_{\mathcal{C}}$ factors through $\rho^*_{\mathcal{M}_{g,n+1}}$, hence this is called the universal monodromy representation of (g,n)-curves.

It is known that

$$1 \to \pi_1(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}) \to \pi_1(\mathcal{M}_{g,n}) \to \pi_1(\mathbb{Q}) \to 1$$

is exact, the left group $\pi_1(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}})$ (often called the *geometric part* of $\pi_1(\mathcal{M}_{g,n})$) is isomorphic to the profinite completion of the mapping class group of (g,n)-surface, and the restriction of $\rho^*_{\mathcal{M}_{g,n+1}}$ to the geometric part is given as the completion of the natural action of the mapping class group on the fundamental group of (g,n)-real surface [11].

Let C be a (g,n)-curve over a field K of characteristic zero. Then, we may consider $\mathcal{C} = C$ and $\mathcal{B} = \operatorname{Spec} K$, and so $\pi_1(\mathcal{B})$ is isomorphic to the absolute Galois group G_K of K. In this case $\rho^*_{\mathcal{C}}$ is called the *pro-$*$ outer Galois representation on the fundamental group of C*,

$$\rho^*_C : G_K \to \operatorname{Out}\Pi^*_{g,n}.$$

For $* =$ unip, this is nothing but θ^{arith} in §1. By the universality, the following inclusion is obvious:

$$\operatorname{im}(\rho^*_C) \subset \operatorname{im}(\rho^*_{\mathcal{M}_{g,n+1}}) \; (\subset \operatorname{Out}\Pi^*_{g,n}).$$

When does this inclusion become equality?

The second author proved with A. Tamagawa that this can never be equality if $n \geq 1$ and $* = \wedge$, except for the trivial case $(g,n) = (0,3)$. (More strongly, no nontrivial element in the geometric part $\rho^*_{\mathcal{M}_{g,n+1} \otimes \overline{\mathbb{Q}}}$ lies in $\operatorname{im}(\rho^*_C)$.) On the contrary, for $*$ being pro-ℓ completion, the equality holds for infinitely many curves C ([10]).

Theorem 2 treats the case $*$ being prounipotent completion over $\mathbb{Q}_\ell$.

4. Abel-Jacobi map

Let S be a connected scheme, and let $f : \mathcal{X} \to S$ be a proper smooth morphism of locally finite type.

Let α be a relative algebraic cycle in $\mathcal{X}/S$ of codimension r. That is, α is a formal sum of codimension-r closed subvarieties of $\mathcal{X}$ equidimensional over S, with integer coefficients. Let $\mathrm{Cyc}^r(\mathcal{X}/S)$ be the set of such algebraic cycles.

Let

$$cl : \mathrm{Cyc}^r(\mathcal{X}/S) \to H^0(S, R^{2r}f_*(\mathbb{Z}_l)(r))$$

be the cycle map, where the right hand side is considered in the etale site, and (r) denotes the Tate twists. The kernel of this map is called homologically trivial cycles, denoted by $\mathrm{Cyc}^r_{hom}(\mathcal{X}/S)$. Then, one can construct (ℓ-adic) pre Abel Jacobi map

$$\text{(4)} \qquad \mathrm{preAJ} : \mathrm{Cyc}^r_{hom}(\mathcal{X}/S) \to H^1(S, R^{2r-1}f_*(\mathbb{Z}_l)(r))$$

as follows.

Let α be an element of $\mathrm{Cyc}^r_{hom}(\mathcal{X}/S)$, and $\mathcal{Z} \xhookrightarrow{\iota} \mathcal{X}$ be the support of α. Then, the Gysin sequence in the etale cohomology

$$0 \to R^{2r-1}f_*(\mathbb{Z}_\ell)(r) \to R^{2r-1}f|_{\mathcal{X}-\mathcal{Z}*}(\mathbb{Z}_\ell) \to (f\iota)_*(\mathbb{Z}_\ell) \to R^{2r}f_*(\mathbb{Z}_\ell)(r)$$

gives a short exact sequence

$$0 \to R^{2r-1}f_*(\mathbb{Z}_\ell)(r) \to \mathcal{E} \to \mathbb{Z}_\ell \cdot \alpha \to 0$$

by pulling back along the cycle map $\mathbb{Z}_\ell \cdot \alpha \to f\iota_*(\mathbb{Z}_\ell)$.

This is an extension of etale locally constant sheaf, giving an element of

$$\mathrm{Ext}^1(\mathbb{Z}_\ell, R^{2r-1}f_*(\mathbb{Z}_\ell)) = H^1(S, R^{2r-1}f_*(\mathbb{Z}_\ell)(r)),$$

which defines the pre Abel Jacobi map preAJ.

In the case that $S = \mathrm{Spec}\, K$ for a field K, AJ factors through the usual Chow group $CH^r_{hom}(\mathcal{X}/K) := \mathrm{Cyc}^r_{hom}(\mathcal{X}/K)/\{\text{rational equivalence}\}$, which coincides the usual ℓ-adic Abel-Jacobi map

$$\mathrm{AJ} : CH^r_{hom}(\mathcal{X}/K) \to H^1(K, R^{2r-1}f_*(\mathbb{Z}_\ell)(r)) = H^1(G_K, H^{2r-1}(X, \mathbb{Z}_\ell(r)))$$

(see [7, Lemma 9.4]). One may define the relative Chow group $CH^r(\mathcal{X}/S)$ and the relative Abel Jacobi map, but we don't need it here.

5. Class $C - C^-$

In the setting in the previous section, we consider the case where $\mathcal{C} \to S$ is a proper smooth relative curve of genus $g \geq 2$, and let $\mathcal{X} \to S$ be the relative Jacobian variety of $\mathcal{C} \to S$.

We shall define a homologically torsion relative algebraic cycle called $C - C^-$ in $\mathrm{Cyc}^{g-1}(\mathcal{X}/S)$.

Suppose for simplicity that there is a section $x : S \to \mathcal{C}$. Then, we have two embeddings of $\mathcal{C}$ in its Jacobian $\mathcal{X}$ given by $y \mapsto [y] - [x]$ and $y \mapsto [x] - [y]$. Let C_x denote the image by the former closed imersion, and C_x^- be the one by the latter. Let α be the algebraic cycle $\alpha := [C_x] - [C_x^-]$. The -1 in the endomorphism of the Jacobian acts on α by multiplication by -1, and acts trivially on $H^0(S, R^{2g-2}f_*(\mathbb{Z}_\ell)(g-1))$. Thus, $cl(\alpha)$ is a two-torsion, and vanishes in $H^0(S, R^{2g-2}f_*(\mathbb{Q}_\ell)(g-1))$. It follows that α lies in the rationally homologically trivial part of $\mathrm{Cyc}(\mathcal{C}/S)$, so we may define

$$\mathrm{preAJ}(\alpha) \in H^1(S, R^{2g-3}f_*(\mathbb{Q}_\ell)(g-1))$$

which is denoted by $\mu(C,x) := \mathrm{preAJ}(\alpha)$. Suppose that S is geometrically connected, and η is a geometric point of S. Then, the following etale cohomology is canonically isomorphic to the continuous Galois cohomology:

$$H^1(S, R^{2g-3}f_*(\mathbb{Q}_\ell)(g-1)) \cong H^1(\pi_1(S,\eta), H^{2g-3}(\mathcal{X}_\eta, \mathbb{Q}_\ell)(g-1)),$$

hence we consider $\mu(\mathcal{C},x)$ as an element of this Galois cohomology. This coincides $\mu(C,x)$ in §1 if $S = \operatorname{Spec} K$.

As in §1, let us denote $H := H^1(\mathcal{C}_\eta, \mathbb{Q}_\ell(1))$, which is of weight -1. Let q be the polarization $q \in \wedge^2 H(-1)$, which spans a one-dimensional vector space $\cong \mathbb{Q}_\ell(0)$. Using $H^m(\mathcal{X}_\eta, \mathbb{Q}_\ell) = \wedge^m H^1(\mathcal{C}_\eta, \mathbb{Q}_\ell) = \wedge^{2g-m} H^1(\mathcal{C}_\eta, \mathbb{Q}_\ell)(g-m)$, we have

$$\mu(\mathcal{C},x) \in H^1(\pi_1(S,\eta), (\wedge^3 H^1(\mathcal{X}_\eta, \mathbb{Q}_\ell))(2)) = H^1(\pi_1(S,\eta), (\wedge^3 H))(-1)).$$

Let $GSp(H)$ denote the subgroup of $GL(H)$ acting on q by a scalar multiple. Then $H \mapsto \wedge^3 H(-1)$ given by $x \mapsto x \wedge q$ is a morphism of $GSp(H)$-modules, and it splits. Thus, $\wedge^3 H(-1) = U \oplus H$ with $U := \wedge^3 H(-1)/H$. It is known that U, H are irreducible algebraic representations of $GSp(H)$ for $g \geq 3$.

Since $\pi_1(S,\eta)$ acts on H via $GSp(H)$, according to this decomposition we have

$$\mu(\mathcal{C},x) = \nu(\mathcal{C}) \oplus \kappa(\mathcal{C},x) \in H^1(\pi_1(S,\eta), U) \oplus H^1(\pi_1(S,\eta), H) = H^1(\pi_1(S,\eta), (\wedge^3 H))(-1)).$$

Here, one can show that $\nu(\mathcal{C})$ is independent of the choice of the section x. Actually, even if there is no section, one can define $\nu(\mathcal{C})$. One way to do this is to construct the corresponding element etale locally on S, and then patching together. One can show that $\nu(\mathcal{C}), \kappa(\mathcal{C},x)$ coincide with those in §1.

6. Sketch of Proof of Theorem 2

Here we sketch the proof of Theorem 2. The proof of Theorem 1 is a refinement of this, and is omitted here.

6.1. Universal monodromy and $C - C^-$. The image of pre Abel-Jacobi map $\nu(\mathcal{M}_{g,n+1})$ for the universal family is given by universal monodromy. Let J be the augumentation ideal of the group ring of fundamental group, as in §1. The extension

$$0 \to (H_{\mathbb{Z}_\ell} \otimes H)/q \to J/J^3 \to H_{\mathbb{Z}_\ell} \to 0$$

is an extension of $\pi_1(\mathcal{M}_{g,1})$-modules, hence giving an element in the continuous Galois cohomology

$$H^1(\pi_1(\mathcal{M}_{g,1}), H_{\mathbb{Z}_\ell} \otimes (H_{\mathbb{Z}_\ell} \otimes H)/q(-1)) \to H^1(\pi_1(\mathcal{M}_g), U).$$

Theorem 6.1. *Two times the above image coincides with $\nu(\mathcal{M}_{g,1})$.*

Let C/K be a $(g,0)$-curve. Then the classifying map $[C] : \operatorname{Spec} K \to \mathcal{M}_{g,0}$ yields $\pi_1(\operatorname{Spec} K) \to \pi_1(\mathcal{M}_g)$ and hence

$$H^1(\pi_1(\mathcal{M}_g), U) \to H^1(G_K, U).$$

Then the image of $\nu(\mathcal{M}_{g,1})$ is $\nu(C)$.

6.2. **Relative and weighted completion.** Let F be a topological field of characteristic zero, Γ a topological group, S a reductive algebraic group over F, $r : \Gamma \to S(F)$ be a continuous morphism with Zariski dense image. The relative Malcev completion of Γ with respect to r, denoted by $\Gamma^{rel-\mathrm{unip}}$, is defined to be the Tannakian fundamental group of the Tannakian category of finite dimensional Γ-modules over F with a Γ-stable filtration whose graded quotients are S-modules compatible with r.

Let $\omega : \mathbb{G}_m \to Z(S)$ be a morphism from $\mathbb{G}_m$ to the center of S, which is called a weight structure of S. Let $F(m)$ denotes a one-dimensional vector space on which $\mathbb{G}_m$ acts by m-th power multiplication. An S-module is of pure weight m if it is a sum of copies of $F(m)$ as a $\mathbb{G}_m$-module. An irreducible S-module is pure of weight m for some integer m.

A negatively weighted Γ-module is a Γ-stable filetered module, whose m-th graded quotient is an S-module of pure weight $m < 0$. The weighted completion of Γ with respect to r, ω is the Tannakian fundamental group of negatively weighted Γ-modules. It is denoted by Γ^{wt}.

Theorem 6.2 (Hain, M). *The kernel of $\Gamma^{rel-\mathrm{unip}} \to S$ is prounipotent, and its abelianization is isomorphic to*

$$\prod_{V_\alpha} H^1(\Gamma, V_\alpha)^* \otimes V_\alpha$$

as S-modules, where V_α spans a representative set of isomorphic classes of irreducible S-modules.

The kernel of $\Gamma^{wt} \to S$ is prounipotent, and its abelianization is isomorphic to

$$\prod_{V_\alpha} H^1(\Gamma, V_\alpha)^* \otimes V_\alpha$$

as S-modules, where V_α spans a representative set of isomorphic classes of irreducible S-modules of negative weights.

For proofs, see [4][6]. By the definition of relative completions, one can show that

$$\theta^{\mathrm{geom}} : \Gamma_g \to \mathrm{Out}\,\mathfrak{p}(C, x)$$

in §1 factors through $\Gamma_g \to \Gamma_g^{rel-\mathrm{unip}} \to \mathrm{Out}\,\mathfrak{p}(C,x)$, where the first map is Zariski dense. Thus, the Zariski closure of the image of θ^{geom} is the image of $\Gamma_g^{rel-\mathrm{unip}}$. Similarly, the universal monodromy ρ^{unip} factors as

$$\pi_1(\mathcal{M}_g) \to \pi_1(\mathcal{M}_g)^{wt} \to \mathrm{Out}\,\mathfrak{p}(C, x).$$

6.3. **Weighted completion of arithmetic mapping class groups.** The following theorem is essential.

Theorem 6.3 (D. Johnson). *Let $g \geq 3$. Let V be an irreducible Sp_g-module, considered as a Γ_g-module via $\Gamma_g \to \mathrm{Sp}_g$. Then, $H^1(\Gamma_g, V) = \mathbb{Q}$ if $V \cong U$, and $H^1(\Gamma_g, V) = \{0\}$ otherwise.*

Corollary 6.4 (R.Hain). *The relative Malcev completion of $\Gamma_g \to \mathrm{Sp}(g, \mathbb{Q})$ is*

$$1 \to \mathcal{T} \to \Gamma_g^{rel-\mathrm{unip}} \to \mathrm{Sp} \to 1$$

with $\mathcal{T}^{ab} \cong U$.

This corollary follows from Theorem 6.2.

Consider the diagram:

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(\mathcal{M}_{g,n}\otimes\overline{\mathbb{Q}}) & \to & \pi_1(\mathcal{M}_{g,n}) & \to & \pi_1(\mathbb{Q}) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & Sp(\mathbb{Q}_\ell) & \to & GSp(\mathbb{Q}_\ell) & \to & \mathbb{G}_m(\mathbb{Q}) & \to & 1 \end{array}$$

Here the left and middle vertical arrows come from the action on H, and the right vertical arrow comes from the action on q, namely the ℓ-adic cyclotomic character. We consider the relative Malcev completion of the left vertical arrow, and the weighted completions of the middle and the right vertical arrows, where the weight structure is given by the isomorphism $\mathbb{G}_m \to Z(GSp)$ by $\alpha \to \alpha^{-1}I$ and $\mathbb{G}_m \to \mathbb{G}_m$ by $\alpha \to \alpha^{-2}$. Then we have a sequence of pro-algebraic groups

$$\Gamma_g^{rel-\mathrm{unip}} \otimes \mathbb{Q}_\ell \to \pi_1(\mathcal{M}_g)^{wt} \to G_{\mathbb{Q}}^{wt} \to 1.$$

Proposition 6.5. *The above sequence is exact.*

Let us prove Theorem 2. The Zariski density of $G_K \to GSp(H)$ is a trivial necessary condition, so we may assume this.

Consider the weighted completion of $G_K \to GSp(H)$, and denote it by $G_K^{wt} \to GSp$. By the structure theorem (Theorem 6.2),

$$1 \to \mathcal{K} \to G_K^{wt} \to GSp \to 1$$

is exact, $\mathcal{K}$ is prounipotent and

$$\mathcal{K}^{ab} \cong \prod_{V_\alpha} H^1(G_K, V_\alpha)^* \otimes V_\alpha$$

where $*$ denotes the dual as $\mathbb{Q}_\ell$-vector space and V_α runs over the representatives of negative weight irreducible representations of GSp.

Assume that $\nu(C) \neq 0$. We want to show that $G_K^{wt} \to \pi_1(\mathcal{M}_g)^{wt}$ is surjective. Since $G_K^{wt} \to G_{\mathbb{Q}}^{wt}$ is surjective, it suffices to show that the image of G_K^{wt} contains the image of $\Gamma_g^{rel-\mathrm{unip}}$, and then since $G_K^{wt} \to GSp(H)$ is surjective, it suffices to show that the image of G_K^{wt} contains $\mathcal{T}$. To show this, it is sufficient to find an element of $\mathcal{K}$ mapping to a nontrivial element in $\mathcal{T}^{ab} = U$. This is because the intersection of the image of $\mathcal{K}$ and $\mathcal{T}^{ab}$ is a Sp-module, hence coincide with $\mathcal{T}^{ab}$ by its itrreducibility, then by prounipotency, the image of $\mathcal{K}$ contains $\mathcal{T}$. Now, $\mathcal{T}^{ab} = U$ is the unique U-component of the abelianization of the prounipotent radical of $\pi_1(\mathcal{M}_g)^{wt}$:

$$\prod_{V_\alpha} H^1(\pi_1(\mathcal{M}_g), V_\alpha)^* \otimes V_\alpha,$$

that is, $H^1(\pi_1(\mathcal{M}_g), U)^* \otimes U$. This follows from the Hochshild-Serre exact sequence

$$0 = H^1(G_K, H^0(\Gamma_g, U)) \to H^1(\pi_1(\mathcal{M}_g), U) \to H^0(G_K, H^1(\Gamma_g, U)),$$

and thus

$$H^1(\pi_1(\mathcal{M}_g), U) \hookrightarrow H^0(G_K, H^1(\Gamma_g, U)) \subset H^1(\Gamma_g, U) \cong \mathbb{Q}_\ell$$

and the element $\nu(\mathcal{M}_{g,1})$ in the left is mapped to the generator of $H^1(\Gamma_g, U)$, due to Johnson. Thus, $H^1(\pi_1(\mathcal{M}_g), U)$ is one-dimensional, and the generator $\nu(\mathcal{C})$ is mapped to $\nu(C)$ if restricted to $H^1(G_K, U)$ via the classifying map. By assumption, $\nu(C)$ is nontrivial, and hence

$$H^1(G_K, U)^* \to H^1(\pi_1(\mathcal{M}_g), U)^*$$

is surjective. Thus an element of $\mathcal{K}$ hits nontrivial element of $H^1(\pi_1(\mathcal{M}_g), U)^* \otimes U$, hence the conclusion.

If $\nu(C)$ is trivial, then

$$H^1(G_K, U)^* \to H^1(\pi_1(\mathcal{M}_g), U)^*$$

is trivial, which implies that G_K never hits $\mathcal{T}^{ab}$, hence the conclusion.

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