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Surgery construction of renormalizable polynomials

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Abstract

Renormalization can be considered as an operator extracting from a given polynoomial a skew map on $\mathbf{Z}_N \times \mathbf{C}$ over $k \to (k+1)$ on $\mathbf{Z}_N$ whose restriction on each fiber is a polynomial. By using a quasiconformal surgery, we construct the inverse of this renormalization operator in some case, that is, from a given N-polynomial with fiberwise connected Julia sets, gluing N-sheets of the complex plane together and construct a polynomial having a renormalization of period N which is hybrid equivalent to it and whose small filled Julia sets have a repelling fixed point of the constructed polynomial.

1 N-polynomial maps

We first give a notion of N-polynomial maps. An N-polynomial map is simply a skew map from a union of N sheets of the complex plane $\mathbf{Z}_N \times \mathbb{C}$ to itself, whose restriction of each sheet is a polynomial mapped to the next sheet. We can easily generalize the theory on dynamics of usual polynomials to N-polynomial maps. In this section, we give an overview of its dynamical properties. Furthermore, we consider a renormalization of a given polynomial as an N-polynomial-like restriction. So we can also consider it as the operator extracting an N-polynomial map from a given polynomial.

Definition. Let $N > 0$. An *N-polynomial map* is an N-tuple of polynomials. An N-polynomial map $F = (F_0, \dots, F_{N-1})$ is considered as a map on $\mathbf{Z}_N \times \mathbb{C}$ to itself as follows:

$$F(k, z) = (k+1, F_k(z)).$$

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The *filled Julia set* $K(F)$ is the set of all points whose forward orbits by F are bounded. The *Julia set* $J(F)$ is the boundary of $K(F)$. The *k-th small filled Julia set* is defined by $K_k(F) = \{z \mid (k, z) \in K(F)\}$ and *k-th small Julia set* $J_k(F) = \partial K_k(F)$.

Definition. An *N-polynomial-like map* is an N-tuple of holomorphic proper maps $F = (F_k : U_k \to V_{k+1})_{k \in \mathbf{Z}_N}$ such that:

- U_k and V_k are topological disks in $\mathbb{C}$.
- U_k is a relatively compact subset of V_k.

We also consider an N-polynomial-like map F as a map between disjoint union of disks:

$$F : \bigsqcup_{k \in \mathbf{Z}_N} U_k \to \bigsqcup_{k \in \mathbf{Z}_N} V_k \qquad F|_{U_k} = F_k.$$

The *k-th small filled Julia set* $K_k(F)$ is defined by

$$K_k(F) = \left\{ z \in U_k \;\middle|\; F^n(z) \in U_{n+k} \right\}$$

and the *k-th small Julia set* $J_k(F)$ is defined by the boundary of $K_k(F)$. The *(resp. filled) Julia set* is defined by the disjoint union of the k-th small (*resp.* filled) Julia sets. We say the (filled) Julia set is *fiberwise connected* if k-th small (filled) Julia set is connected for any k.

For an N-polynomial or an N-polynomial-like map $F = (F_k)$, we write

$$F_k^n = F_{k+n-1} \circ \cdots \circ F_{k+1} \circ F_k,$$

so that $F^n(k, z) = (k + n, F_k^n(z))$.

Although the degree of an N-polynomial map (or an N-polynomial-like map) F is not well-defined ($\deg(F_k)$ may be different), the degree of F^N is well-defined (it is equal to $\prod \deg(F_k)$). In this paper, we always assume $\deg F^n > 1$.

Definition. Let $F = (F_k : U_k \to V_{k+1})$ and $G = (G_k : U'_k \to V'_{k+1})$ be N-polynomial-like maps. We say F and G are *hybrid equivalent* if there exist quasiconformal homeomorphisms ϕ_k ($k \in \mathbf{Z}_N$) between some neighborhoods of $K_k(F)$ and $K_k(G)$ such that $G_k \circ \phi_k = \phi_{k+1} \circ F_k$ and $\bar{\partial}\phi_k \equiv 0$ on $K(F, k)$.

Theorem 1.1 (Straightening theorem for N-polynomial-like maps). *For any N-polynomial-like map F, there exist an N-polynomial map G of the same degree as F (that is, $\deg(F_k) = \deg(G_k)$ for all k) hybrid equivalent to F.*

Furthermore, if F has fiberwise connected Julia set, then G is unique up to affine conjugacy.

Usually, we consider a renormalization as a polynomial-like map with connected Julia set which is a restriction of some iterate of a polynomial. But here, we consider it as an N-polynomial-like map;

Definition. A polynomial f is *renormalizable* if there exist disks U_k and V_k ($k \in \mathbf{Z}_N$) such that:

- $G = (f : U_k \to V_{k+1})_{k \in \mathbf{Z}_N}$ is an N-polynomial-like map with fiberwise connected Julia set.
- $U_k \cap U_{k'}$ contains no critical point of f if $k \neq k'$.
- When $N = 1$, U_0 does not contain all the critical points of f.

We call G a *renormalization* of period N.

The small filled Julia sets of a renormalization are "almost disjoint" (they intersects only at a repelling periodic orbit [Mc], [In]). So we define the *(resp. filled) Julia set* of a renormalization by the union (not the *disjoint* union) of the small (*resp.* filled) Julia sets.

We may assume an N-polynomial map F is monic (that is, each F_k is monic). Let $\Delta = \{|z| < 1\}$. Easy calculation shows:

Proposition 1.2 (The existence of the Böttcher coordinates). *For a given monic N-polynomial map F, there exist conformal maps $\varphi_k : (\mathbb{C} \setminus \overline{\Delta}) \to (\mathbb{C} \setminus K(F, k))$ such that $\varphi_{k+1}(z^{\deg F_k}) = F_k \circ \varphi_k(z)$.*

In fact, we only take φ_k the Böttcher coordinate for the monic polynimoal $F_{k-1} \circ \cdots \circ F_{k+1} \circ F_k$.

So, we can define external rays for F just as the usual polynomial case.

Definition. Let F, φ_k as above. The *k-th external ray* $\mathcal{R}_k(F; \theta)$ of angle θ for an N-polynomial map F is defined by:

$$\mathcal{R}_k(F; \theta) = \left\{ \varphi(r \exp(2\pi i \theta)) \;\middle|\; 1 < r < \infty \right\}.$$

If the limit

$$\lim_{r \to 1} \varphi(r \exp(2\pi i \theta))$$

exists, say x, then we say $\mathcal{R}_k(F; \theta)$ lands at x and θ is the landing angle for (k, x).

Let $R > 1$. We also define

$$\mathcal{R}_k(F;\theta,R) = \left\{\varphi(r\exp(2\pi i\theta)) \,\middle|\, 1 < r < \infty\right\},$$
$$\mathcal{R}_k(F;\theta,R,\epsilon) = \left\{\varphi(r\exp(2\pi i\eta)) \,\middle|\, 1 < r < \infty,\ \eta = \theta + \epsilon\log r\right\}.$$

If $\mathcal{R}(F;\theta)$ lands at x, then $\mathcal{R}(F;\theta,R,\epsilon)$ also converges to x. By the proposition above,

$$F(\mathcal{R}_k(F;\theta)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta),$$
$$F(\mathcal{R}_k(F;\theta,R)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta,\ R^{\deg(F_k)}),$$
$$F(\mathcal{R}_k(F;\theta,R,\epsilon)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta,\ R^{\deg(F_k)},\ \epsilon).$$

We say the ray is *periodic* if $F^n(\mathcal{R}_k(F;\theta)) = \mathcal{R}_k(F;\theta)$ for some $n > 0$. The least such n is called the period of this ray. Clearly, the period of every periodic point is divisible by N.

Let $x = (k,z)$ be a periodic point of F with period n. If x is repelling or parabolic, then there are finite number of rays landing at x and they have the same period. Let q be the number of rays landing at x and let $\theta_1,\ldots\theta_q$ be the angle of these rays ordered counterclockwise. Since F^n permutes the rays landing at x and it perserves the cyclic order of them, there exist p such that $F^n(\mathcal{R}_k(F;\theta_i)) = F^n(\mathcal{R}_k(F;\theta_{i+p}))$ for every $i \in \mathbf{Z}_q$. We say that the *(combinatorial) rotation number* of this point x is p/q.

We also consider external rays for N-polynomial-like maps. They are defined by the inverse images of external rays for N-polynomial maps by the hybrid conjugacy in Proposition 1.1.

2 Results

Let F be an N-polynomial map with fiberwise connected Julia set and $O = \{(k,x_k) \mid k \in \mathbf{Z}_N\}$ be a repelling periodic orbit of period N with rotation number p_0/q_0.

Definition. We say a polynomial (g,x) with marked fixed point x is a *p-rotatory intertwining* of (F,O) if:

- g has a renormalization of period N hybrid equivalent to F.
- x corresponds to O by the hybrid conjugacy.
- x has a rotation number $p/(Nq_0)$.

- $\deg(g) = \sum(\deg(F_k) - 1) + 1$. (Equivalently, all critical points of g lie in the filled Julia set of the renormalization above.)

Note that the filled Julia set of such a polynomial is connected.

To construct a p-rotatory intertwining of (F, O), we need some combinatorial property of the dynamics near the fixed point x.

Definition. A 4-tuple of integers (N, p_0, q_0, p) is admissible if $p \equiv p_0 \mod q_0$ and p and N are relatively prime.

Note that the above definition also makes sense when N and q_0 are integers, $p_0 \in \mathbf{Z}_{q_0}$ and $p \in \mathbf{Z}_{Nq_0}$.

Proposition 2.1. *If a p-rotatary intertwining of (F, O) exists, then (N, p_0, q_0, p) is admissible.*

Theorem 2.2. *Let F be an N-polynomial map with fiberwise connected Julia set and $O = \{(k, x_k)\}$ is a repelling periodic orbit of period N with rotation number p_0/q_0.*

When an integer p satisfies that (N, p_0, q_0, p) is admissible, then there exists a p-rotatory intertwining (g, x) of (F, O) and it is unique up to affine conjugacy.

The following two sections are devoted to prove this theorem.

3 Construction

In this section, we prove the existence part of Theorem 2.2. We use the idea of the intertwining surgery [EY].

Let (F, O) be an N-polynomial map with marked periodic point satisfying the assumption of Theorem 2.2. Fix $R > 0$ and let

$$V_k = \left\{ (k, z) \;\middle|\; |\varphi_k(z)| < R \right\} \cup K_k(F)$$

and $U_k = F_k^{-1}(V_{k+1})$. Let $\mathcal{V} = \bigsqcup V_k$ and $\mathcal{U} = \bigsqcup U_k$. Then $(F_k : U_k \to V_{k+1})$ is an N-polynomial-like map (we also use the word F for it and write $F : \mathcal{U} \to \mathcal{V}$).

Let $\theta_0, \ldots, \theta_{q_0 - 1}$ be all the external angles for $(0, x_0)$ ordered counterclockwise.

Let $\epsilon > 0$ and $0 < \delta < \epsilon/2$. For $0 \le k < N$ and $l \in \mathbf{Z}_{q_0}$, consider arcs

$$\gamma_0(k + Nl) = \mathcal{R}_0\left(F; \theta_k, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon\right),$$

$$\gamma_0^{\pm}(k + Nl) = \mathcal{R}_0\left(F; \theta_k, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon \pm \delta\right).$$

When ϵ is sufficiently small, these arcs are mutually disjoint. For $j \in \mathbb{Z}_{Nq_0}$, let

$$\begin{aligned} \gamma_k(j) &= F_k(\gamma_{k-1}(j-p) \cap U_{k-1}), \\ \gamma_k^{\pm}(j) &= F_k(\gamma_{k-1}^{\pm}(j-p) \cap U_{k-1}) \end{aligned} \tag{1}$$

for $k = 1, \ldots, N-1$. Let $S_k(j)$ (*resp.* $L_k(j)$) be the sectors in V_k between $\gamma_k(j-1)$ and $\gamma_k(j)$ (*resp.* $\gamma_k^+(j-1)$ and $\gamma_k^-(j)$).

Then, since the rotation number of x_0 for F_0^N is p_0/q_0, we can easily verify $F_0^N(\gamma_0(j) \cap F_0^{-N}(V_0)) = \gamma_0(j+Np_0)$. Therefore, by the assumption that (N, p_0, q_0, p) is admissible,

$$\begin{aligned} F_{N-1}(\gamma_{N-1}(j-p) \cap U_{N-1}) &= F_0^N(\gamma_0(j-Np) \cap F_0^{-N-1}(U_{N-1})) \\ &= \gamma_0(j - Np + Np_0) \\ &= \gamma_0(j). \end{aligned}$$

This equation also holds for $\gamma_k^{\pm}$ instead of γ_k. Therefore, the equation (1) holds for any $k \in \mathbb{Z}_N$.

Since O is repelling, it is linearlizable. Namely, there are a neighborhood O_k of x_k and a map $\psi_k : O_k \to \mathbb{C}$ for each k such that $\psi_k(x_k) = 0$ and $\psi_{k+1} \circ F_k(z) = \lambda_k \psi_k(z)$ on O'_k, where $\lambda_k = F'_k(x_k)$ and O'_k is the component of $F_k^{-1}(O_{k+1})$ containing x_k.

For each $j \in \mathbb{Z}_{Nq_0}$, the quotient space $(L_k(j) \cap O_k)/F_k^{Nq_0}$ is an annulus of finite modulus. So we denote the modulus of this quotient annulus by $\operatorname{mod} L_k(j)$. Since F_k maps $L_k(j) \cap O'_k)$ univalently to $L_{k+1}(j+p) \cap O_{k+1}$, we have $\operatorname{mod} L_k(j) = \operatorname{mod} L_{k+1}(j+p)$.

Now we deform the N-polynomial-like map $F : \mathcal{U} \to \mathcal{V}$ by a hybrid conjugacy so that we can identify N disks $V_0 \ldots V_{N-1}$ quasiconformally and define a quasiregular map on it.

Lemma 3.1. *There exists an N-polynomial-like map $\hat{F} = (\hat{F}_k : \hat{U}_k \to \hat{V}_{k+1})_{k \in \mathbb{Z}_N}$ hybrid equivalent to F such that the sector $\hat{L}_k(j)$ which corrseponds to $L_k(j)$ satisfies that*

$$\operatorname{mod} \hat{L}_k(j) = \operatorname{mod} \hat{L}_{k'}(j)$$

for any $k, k' \in Z_N$ and $j \in Z_{Nq_0}$.

Let $\hat{x}_k$, $\hat{\gamma}_k(j)$, $\hat{\gamma}_k^{\pm}(j)$, $\hat{S}_k(j)$, $\hat{O}_k$ and $\hat{O}'_k$ correspond to x_k, $\gamma_k(j)$, $\gamma_k^{\pm}(j)$, $S_k(j)$, O_k and O'_k respectively by the hybrid conjugacy in the above lemma.

Now we construct quasiconformal maps $\tau_k : \hat{V}_0 \to \hat{V}_k$ $(k \in \mathbb{Z}_N)$ to identify $\hat{V}_0, \ldots, \hat{V}_{N-1}$ together. First of all, take C^1 diffeomorphisms

$$\tilde{\tau}_k : \bigcup_j \hat{\gamma}_k(j) \to \bigcup_j \hat{\gamma}_{k+1}(j)$$

$$\hat{F}_{k+1} \circ \tilde{\tau}_k = \tilde{\tau}_{k+1} \circ \hat{F}_k \tag{2}$$

$$\tilde{\tau}_k(\hat{\gamma}_k(j)) = \hat{\gamma}_{k+1}(j) \tag{3}$$

and let $\tau_k = \tau_{k-1} \circ \cdots \circ \tau_0$ on $\bigcup \hat{\gamma}_k(j)$. Next, let $\tau_k|_{\hat{L}_0(j)} : \hat{L}_0(j) \to \hat{L}_k(j)$ be the conformal isomorphism which sends x_0 to x_k, $\hat{\gamma}_0^+(j-1)$ to $\hat{\gamma}_0^+(j-1)$, and $\hat{\gamma}_0^-(j)$ to $\hat{\gamma}_0^-(j)$.

The following lemma is due to Bielefeld [Bi, Lemma 6.4, 6.5].

Lemma 3.2. *We can extend τ_k quasiconformally to $\tau_k : \hat{V}_0 \to \hat{V}_k$ ($k \in \mathbf{Z}_N$).*

Let $V = \hat{V}_0$ and

$$U = \bigcup_{j \in \mathbf{Z}_q, k=0,\ldots,N-1} \tau_k^{-1}\left(\overline{\hat{S}_k(jN + kp)} \cap \hat{U}_k\right).$$

Define a quasiregular map $g : U \to V$ as follows. When $z \in \hat{S}_0(Nj + kp) \cap U$ for some $j \in \mathbf{Z}_{q_0}$, let

$$\tilde{g}(z) = \tau_{k+1}^{-1} \circ \hat{F}_k \circ \tau_k(z).$$

By (2), $\tilde{g}$ extends continuously on U.

Lemma 3.3.

1. $\tilde{g}\left(\bigcup(\hat{S}_0(j) \setminus \hat{L}_0(j)) \cap U\right) \subset \bigcup(\hat{S}_0(j) \setminus \hat{L}_0(j))$. *Namely, $E = \bigcup \hat{S}_0(j) \setminus \hat{L}_0(j)$ is forward invariant by $\tilde{g}$.*
2. $\tau_k \circ \tilde{g}^N \circ \tau_k^{-1}$ *is conformal on $\hat{S}_k(jN + kp) \setminus \hat{L}_k(jN + kp)$.*

Let σ_0 be the standard complex structure. On $\hat{S}_0(jN + kp) \setminus \hat{L}_0(jN + kp)$,

$$\begin{aligned} \sigma_0 &= (\tau_k \circ \tilde{g}^N \circ \tau_k^{-1})^*(\sigma_0) \\ &= (\tau_k^*)^{-1} \circ (\tilde{g}^N)^*(\tau_k^* \sigma_0). \end{aligned}$$

by the previous lemma. Therefore,

$$(\tilde{g}^N)^*(\tau_k^* \sigma_0) = \tau_k^* \sigma_0 \tag{4}$$

on $\hat{S}_0(jN + kp) \setminus \hat{L}_0(jN + kp)$.

So define an almost complex structure σ on V as follows:

$$\sigma = \begin{cases} (\tau_k \circ \tilde{g}^n)^* \sigma_0 & \text{on } \tilde{g}^{-n}(\hat{S}_0(Nj + kp)). \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Lemma 3.4. *σ is well-defined and it is really a complex structure.*

Proof. On $\hat{S}_0(Nj+kp) \setminus \hat{L}_0(jN+kp)$ $(1 \geq k < N)$,

$$\begin{aligned}\tilde{g}^*\sigma &= (\tau_k^{-1} \circ F_{k-1} \circ \tau_{k-1})^*(\tau_k^*\sigma_0) \\ &= \tau_{k-1}^*(F_{k-1}^*\sigma_0) \\ &= \tau_{k-1}^*\sigma_0 \\ &= \sigma.\end{aligned}$$

Therefore, together with (4), σ is invariant under $\tilde{g}$ on E. (Note that E is forward invariant by $\tilde{g}$.) Since $\sigma \neq \sigma_0$ only on $\bigcup \tilde{g}^{-n}(E)$, σ is well-defined.

Furthermore, $\tilde{g}$ is conformal except on $\tilde{g}^{-1}(E)$. So the maximal dilatation of σ on V is equal to that of σ on $\tilde{g}^{-1}(E)$, which is bounded. So σ is a complex structure. □

Therefore, there exists a quasiconformal mapping $h : V \to \mathbb{C}$ such that $h^*\sigma_0 = \sigma$. $\hat{g} = h \circ \tilde{g} \circ h$ is a polynomial-like map, so there exists a polynomial g hybrid equivalent to $\hat{g}$.

It is easy to check this g is a p-rotatory intertwining of F.

4 Uniqueness

In this section, we show that two p-rotatory intertwinings (g, x) and (g', x') of (F, O) are affinely conjugate.

4.1 Puzzles

Let (g, x) be a p-rotatory intertwining of an N-polynomial map (F, O) with marked periodic point of period N. Denote $\mathcal{K}$ by the filled Julia set of the renormalization $G = (g : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$ corresponding to F. Let $\omega_0, \dots, \omega_{Nq-1}$ be the landing angles of x ordered counterclockwise.

Let $\varphi : (\mathbb{C} \setminus \overline{\Delta}) \to (\mathbb{C} \setminus K(g))$ be the Böttcher coordinate of g. Fix $R > 0$ and small $\epsilon > 0$ so that sectors

$$\tilde{S}_{0,j} = \left\{ \varphi(r \exp(2\pi i \theta)) \,\middle|\, 1 < r < R,\ |\theta - \omega_j| < \epsilon \log r \right\}.$$

are mutually disjoint. Let $D_0 = \varphi(\{|z| < R\}) \cup K(g)$ and $D_n = g^{-n}(D_0)$ for $n \geq 0$. Let $\tilde{P}_{0,j}$ be the component of $D_0 \setminus \overline{\bigcup \tilde{S}_{0,j}}$ between $\tilde{S}_{0,j-1}$ and $\tilde{S}_{0,j}$. Let $S_{0,j} = \overline{\tilde{S}_{0,j}}$,

$P_{0,j} = \overline{\tilde{P}_{0,j}}$ and

$$\mathcal{P}_n = \left\{\text{the closures of components of } g^{-n}(\tilde{P}_{0,j}) \ (j \in \mathbf{Z}_{Nq})\right\}$$
$$\mathcal{S}_n = \left\{\text{the closures of components of } g^{-n}(\tilde{S}_{0,j}) \ (j \in \mathbf{Z}_{Nq})\right\}.$$

We call an element of $\mathcal{P}_n$ a *piece of depth n* and an element of $\mathcal{S}_n$ a *sector of depth n*.

Then $\mathcal{P}_n$ and $\mathcal{S}_n$ have the following properties. Let $n \geq 0$.

1. $\mathcal{P}_n \cup \mathcal{S}_n$ is a partition of $\overline{D_n}$.

2. For any $x \in K(g) \setminus \bigcup_j g^{-j}(x)$, there exists a unique piece $P_n(x)$ of depth n which contains x. In particular, $\mathcal{P}_n$ covers $K(g)$.

3. For any $P \in \mathcal{P}_{n+1}$, there exists some $P' \in \mathcal{P}_n$ with $P \subset P'$.

4. When $P \in \mathcal{P}_{n+1}$, we have $g(P) \in \mathcal{P}_n$.

5. When $S \in \mathcal{S}_{n+1}$, either there exists some $S' \in \mathcal{S}_n$ with $S = S' \cap D_{n+1}$, or there exists some $P \in \mathcal{P}_n$ with $S \subset \operatorname{int} P$.

6. For any $X \in \mathcal{P}_n \cup \mathcal{S}_n$, $\operatorname{int} X \cap g^{-n+1}(\mathcal{K}) \neq \emptyset$ or there exists a unique $y \in g^{-n}(x_0)$ with $y \in X$.

7. For any $P \in \mathcal{P}_n$, there exists a unique component E of $g^{-n}(\mathcal{K} \setminus g^{-n}(x_0))$ with $E \subset P$. This map $P \mapsto E$ is a bijection between $\mathcal{P}_n$ and $\{$components of $g^{-n}(\mathcal{K} \setminus g^{-n}(x_0)))\}$.

Theorem 4.1. *The set $K(g) \setminus \bigcup_{n>0} g^{-n}(\mathcal{K})$ has zero Lebesgue measure.*

For a later use, we give a canonical form of the renormalization G. Take small $r > 0$ and $\eta > 0$. For $j \in \mathbf{Z}_{Nq}$, let $\hat{P}_{0,j}$ be the union of $B(x,r)$ and the domain in $D_0 \setminus B(x,r)$ between $\mathcal{R}(g; \omega_{j-1} - \eta, R)$ and $\mathcal{R}(g; \omega_j + \eta, R)$. Let Q_j be the component of $g^{-1}(\hat{P}_{0,j})$ which is contained in $\hat{P}_{0,j}$. Let U_k and V_k are disks obtained by smoothing the boundary of $\bigcup_{j \in \mathbf{Z}_q} Q_{k+Nj}$ and $\bigcup_{j \in \mathbf{Z}_q} \hat{P}_{0,k+Nj}$. Then $G = (g : U_k \to V_{k+1})$ is a renormalization hybrid equivalent to F.

4.2 Proof of the uniqueness

Let (g, x) and (g', x') be two p-rotatary intertwinings of an N-polynomial map (F, O) with a marked periodic point of rotation number p_0/q. We use the notation in section 4.1 for g. For g', we attach a prime to each notation (*e.g.*, $\mathcal{K}', D'_n, \mathcal{P}'_n, \mathcal{S}'_n, \ldots$).

In this section, we show that g and g' are affinely conjugate. Since $K(g)$ and $K(g')$ are connected, we need only show that g and g' are hybrid equivalent. To do this, we first construct a standard hybrid conjugacy between renormalizations G and G', next by pulling back it repeatedly, we construct a quasiconformal conjugacy between g and g', and show it is actually a hybrid conjugacy.

Lemma 4.2. *There exists a quasiconformal map $\Phi_0 : \overline{D_0} \to \overline{D'_0}$ satisfies the following:*

- $\bar{\partial}\Phi_0 \equiv 0$ *on* $K(G)$.
- $\Phi_0 \circ g = g' \circ \Phi_0$ *on* $\bigcup(P_{1,j} \cup S_{0,j}) \cup \partial D_1$.

Proof. For each $k \in \mathbf{Z}_N$, take a C^1-diffeomorphism $\tilde{\Phi}_k : \overline{V_k \setminus U_k} \to \overline{V'_k \setminus U'_k}$ which satisfies the following:

1. $\tilde{\Phi}_k(\partial V_k) = \partial V'_k$ and $\tilde{\Phi}_k(\partial U_k) = \partial U'_k$.
2. For $j \in \mathbf{Z}_{Nq}$ with $P_{0,j} \subset V_k$ (equivalently, $j \equiv k \mod N$), we have $\tilde{\Phi}_k(\partial(P_{0,j} \setminus U_k)) = \partial(P'_{0,j} \setminus U'_k)$ and $\tilde{\Phi}_k(P_{0,j} \setminus U_k) = P'_{0,j} \setminus U'_k$.
3. For $z \in \partial U_k$, $\Phi_{k+1}(g(z)) = g'(\Phi_k(z))$.

As in [DH], we can extend $\tilde{\Phi}_k$ to a diffeomorphism on $\overline{V_k} \setminus K_k(G)$ to $\overline{V'_k} \setminus K_k(G')$ by the equation $\tilde{\Phi}_k(g(z)) = g'(\Phi_k(z))$. Furthremore, since G and G' are hybrid equivalent (they are both hybrid equivalent to F), this $\tilde{\Phi}_k$ extends to a hybrid conjugacy of G to G'. (To do this, we should use [DH, Proposition 6]. So we need to check $[\tilde{\Phi}_0, \psi, g^n, (g')^n] = 0$ in $\mathbf{Z}_{deg(G^n)}$ where ψ is a given hybrid conjugacy of G and G' considered as classical polynomial-like maps. But it is trivial because of the property 2 above.)

Now we define Φ_0 first on $\bigcup S_{0,j}$. For each $S_{0,j}$, define a quasiconformal map $\Phi_0|_{S_{0,j}} : S_{0,j} \to S'_{0,j}$ so that

$$\begin{aligned} \Phi_0 \circ g &= g' \circ \Phi_0, \\ \Phi_0|_{\mathcal{R}(g;\omega_j,R,-\epsilon)} &= \tilde{\Phi}_{j-1}|_{\mathcal{R}(g;\omega_j,R,-\epsilon)}, \\ \Phi_0|_{\mathcal{R}(g;\omega_j,R,+\epsilon)} &= \tilde{\Phi}_j|_{\mathcal{R}(g;\omega_j,R,\epsilon)}, \end{aligned}$$

and

$$\begin{aligned} \Phi_0|_{\partial S_{j-1} \cap \partial D_0} &= \tilde{\Phi}_{j-1} \quad \text{on a neighborhood of } \varphi(R\exp(2\pi i(\omega_j - \epsilon \log R))), \\ \Phi_0|_{\partial S_j \cap \partial D_0} &= \tilde{\Phi}_j \quad \text{on a neighborhood of } \varphi(R\exp(2\pi i(\omega_j + \epsilon \log R))). \end{aligned}$$

Let $\hat{\Phi}_k : \overline{V_k \setminus U_k} \to \overline{V'_k \setminus U'_k}$ be a C^1-diffeomorphism such that for $k, k' \in \mathbf{Z}_N$ and $j, j' \in \mathbf{Z}_{Nq}$ with $j \equiv k \mod N$,

- $\hat{\Phi}_k = \tilde{\Phi}_k$ on $\partial(V_k \setminus U_k)$.
- $g' \circ \hat{\Phi}_k(z) = \tilde{\Phi}_{k'} \circ g(z)$ when z lies in $P_{0,j} \cap \partial D_1 \cap g^{-1}(P_{0,j'})$.
- $g' \circ \hat{\Phi}_k(z) = \Phi_0 \circ g(z)$ when z lies in $P_{0,j} \cap \partial D_1 \cap g^{-1}(S_{0,j})$.
- $\hat{\Phi}_k = \tilde{\Phi}_k$ on $\partial(V_k \setminus U_k) \cap \partial P_j$.

As in the case of $\tilde{\Phi}_k$, we can extend $\hat{\Phi}_k$ quasiconformally to V_k and obtain hybrid equivalence between G and G'.

Now let $\Phi_0 = \hat{\Phi}_k$ on P_j where $k \equiv j \mod N$. It is easy to check this Φ_0 has the desired properties. □

Then we define $\Phi_n : \overline{D_0} \to \overline{D_0'}$ inductively. Suppose Φ_n is defined and satisfies:

- $\bar{\partial}\Phi_n \equiv 0$ on $g^{-n}(\mathcal{K})$.
- $\Phi_n \circ g = g' \circ \Phi_n$ on $\bigcup g^{-n}(P_{1,j} \cup S_{0,j}) \cup \partial(D_1 \setminus D_{n+1})$.

First of all, let $\Phi_{n+1}|_{\overline{D_0} \setminus \overline{D_{n+1}}} = \Phi_n$. Let $P \in \mathcal{P}_{n+1}$. When $\operatorname{int} P \cap g^{-n}(\mathcal{K}) \neq \emptyset$, define $\Phi_{n+1}|_P = \Phi_n$. Otherwise, by the property 6 in p. 9, there exists a unique $y \in g^{-n}(x) \in P$. Let $P' \cap \mathcal{P}_{n+1}$ be the piece of depth $n+1$ which combinatorially corresponds to P', *i.e.* which satisfies that $\Phi_n(g(P)) = g(P')$ and $\Phi_n(y) \in P'$ (when y is not a critical point, such P' is unique. When y is a critical point, P' is determined by the cyclic order at y to make Φ_n continuous). Then, since $C(g) \subset \mathcal{K}$, $g|_P$ is conformal and so is $g'|_{P'}$. So define

$$\Phi_{n+1}|P = (g'|_{P'})^{-1} \circ \Phi_n \circ g : P \to P'.$$

(In other words, $\Phi_{n+1}|_{\overline{D_{n+1}}}$ is defined by lifting Φ_n by the branched covering g and g'.)

Then Φ_{n+1} also satisfies the property above. First, we show the continuity of Φ_{n+1}. By the construction, Φ_{n+1} is continuous on and outside $\overline{D_{n+1}}$. Furthermore, for $z \in \partial D_{n+1}$,

$$\begin{aligned}\Phi_{n+1}(z) &= (g'|_{P'})^{-1} \circ \Phi_n \circ g(z)\\ &= (g'|_{P'})^{-1} \circ g' \circ \Phi_n(z)\\ &= \Phi_n(z)\end{aligned}$$

by the second property above for Φ_n. So Φ_{n+1} is continuous.

For every $X \in \mathcal{P}_{n+1} \cap \mathcal{S}_{n+1}$, $\Phi_{n+1}|_X$ is a quasiconformal homeomorphism from X to corresponding piece or sector for g' and so is $\Phi_{n+1}|_{\overline{D_0} \setminus D_{n+1}} = \Phi_n$. Hence Φ_{n+1} is a quasiconformal homeomorphism. By the construction, it is clear that $\bar{\partial}\Phi_{n+1} \equiv 0$ on $g^{-n-1}(\mathcal{K})$.

It is also clear that $g' \circ \Phi_{n+1} = \Phi_{n+1} \circ g$ on $E_{n+1} = \bigcup g^{-n-1}(P_{1,j} \cup S_{0,j})$. Let $z \in \partial D_{n+2} \setminus E_{n+1}$. Then z lies in some $P \in \mathcal{P}_{n+1}$ with $\operatorname{int} P \cap g^{-n}(\mathcal{K}) = \emptyset$. Therefore,

$$\begin{aligned} g' \circ \Phi_{n+1}(z) &= g' \circ (g'|_{P'})^{-1} \circ \Phi_n \circ g(z) \\ &= \Phi_n \circ g(z). \end{aligned}$$

Since $g(z) \in \partial D_{n+1}$, we have $\Phi_n(g(z)) = \Phi_{n+1}(g(z))$ and the second property holds for Φ_{n+1}.

Since all Φ_n are quasiconformal with same dilatation ratio, it is equicontinuous. Furthermore, $\Phi_n = \Phi_{n+1}$ except on $D_{n+1} \setminus g^{-n}(\mathcal{K})$. Therefore, $\Phi = \lim \Phi_n$ exists and is quasiconformal. Also, it satisfies that $\bar{\partial}\Phi \equiv 0$ on $\bigcup g^{-n}(\mathcal{K})$ and that $g' \circ \Phi = \Phi \circ g$. Since $K(g) \setminus \bigcup g^{-n}(\mathcal{K})$ has zero Lebesgue measure, Φ is a hybrid conjugacy between g and g'.

Therefore, a p-rotatory intertwining of (F, O) is unique up to affine conjugacy.

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