The Jørgensen number of the Whitehead link

Hiroki Sato

佐藤 宏樹 (静岡大学理学部) *

ABSTRACT. In this paper we will sketch out the result obtained recently: the Jørgensen number of the Whitehead link is two. Furthermore we will represent points corresponding to the Whitehead link by using the cordinates introduced in Sato [7]. The details will appear in Sato [9].

1. In 1976 Jørgensen obtained the following important theorem called Jørgensen's inequality, which gives a necessary condition for a non-elementary Möbius transformation group $G=\langle A,B\rangle$ to be discrete.

THEOREM A (Jørgensen [1]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A,B):=|\mathrm{tr}^2(A)-4|+|\mathrm{tr}(ABA^{-1}B^{-1})-2|\geq 1.$$

The lower bound 1 *is best possible.*

DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number*

*Partly supported by the Grants-in-Aid for Co-operative Research as well as Scientific Research, the Ministry of Education, Science, Sports and Culture, Japan.

2000 *Mathematics Subject Classification.* Primary 32G15; Secondary 20H10, 30F40.

Key Words and Phrases. Jørgensen's inequality, Jørgensen number, Jørgensen groups, the Whitehead link.

$J(A,B)$ of the ordered pair (A,B) is defined as

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

We denote by Möb the set of all Möbius transformations. Throughout this paper we will always write elements of Möb as matrices with determinant 1. We recall that Möb ($= PSL(2,\mathbf{C})$) acts on the upper half space H^3 of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space. A subgroup G of Möb is said to be *elementary* if there exists a finite G-orbit in $\mathbf{R}^3$.

DEFINITION 2. Let G be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for G is defined as

$J(G) := \inf\{J(A,B) \mid A \text{ and } B \text{ generate } G\}.$

DEFINITION 3. A non-elementary two-generator subgroup G of Möb is a *Jørgensen group* if G is a discrete group with $J(G) = 1$.

THEOREM B (Jørgensen-Kiikka [2]). *Let $\langle A,B\rangle$ be a non-elementary discrete group with $J(A,B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A,B\rangle$ is a Jørgensen group such that A is parabolic, then we call it a *Jørgensen group of parabolic type*. Here we only consider Jørgensen groups of parabolic type.

2. Let $\langle A,B\rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $\mu \in \mathbf{C}$. We denote by $G_{\sigma,\mu}$ the marked group generated by A and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that (σ,μ) is the point representing a marked group $G_{\sigma,\mu}$ and that $G_{\sigma,\mu}$ is the marked group corresponding to a point (σ,μ).

In particular, we consider the case of $\mu = ik$ $(k \in \mathbf{R})$. Namely, we consider marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $k \in \mathbf{R}$.

3. Let C be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$.

THEOREM C (Sato [7]). *If a marked two-generator group $G_{\sigma,ik}$ is a Jørgensen group, then the point (σ, ik) representing $G_{\sigma,ik}$ lies on the cylinder C.*

By Theorem C we consider marked two-generator groups $G_{\sigma,\mu} = \langle A, B_{\mu,\sigma}\rangle$ with $\sigma = -ie^{i\theta}$ $(0 \le \theta < 2\pi)$ and $\mu = ik$ $(k \in \mathbf{R})$. For simplicity we set $B_{\theta,k} := B_{\sigma,ik}$ and $G_{\theta,k} = \langle A, B_{\sigma,ik}\rangle$ for $\sigma = -ie^{i\theta}$.

4. There are infinite number of Jørgensen groups (see Jørgensen-Lascurain-Pignataro [3], Sato [7]). The following familier groups are all Jørgensen groups: The modular group, the Picared group (Jørgensen-Lascurain-Pignataro [3], Sato [8, 9], Sato-Yamada [10]), the figure-eight knot group (Sato [7]), "the Gehring-Maskit group" (Sato [7]), where "the Gehring-Maskit group" is the group studied in Maskit [5]. Namely, we have the following theorem:

THEOREM D (Jørgensen-Lascurain-Pignataro [3], Sato [7, 8], Sato-Yamada [10]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{\theta,k} = \langle A, B_{\theta,k}\rangle$ be the group generated by A and $B_{\theta,k}$, where $0 \le \theta < 2\pi$ and $k \in \mathbf{R}$. *Then*

(i) $G_{\pi/2,0}$ *is a Jørgensen group.*

(ii) $G_{\pi/2,1/2}$ *is a Jørgensen group.*

(iii) $G_{\pi/6,\sqrt{3}/2}$ *is a Jørgensen group.*

(iv) $G_{0,\sqrt{3}/2}$ *is a Jørgensen group.*

REMARK (1) The groups $G_{\pi/2,0}$ $G_{\pi/2,1/2}$ $G_{\pi/6,\sqrt{3}/2}$ and $G_{0,\sqrt{3}/2}$ are conjugate to the modular group, the Picard group, the figure-eight knot group and "the Gehring-Maskit group", respectively.

(2) See Sato [7] for other Jørgensen groups of parabolic type.

5. Now it gives rise to the following problem.

PROBLEM. Is the Whitehead link a Jørgensen group ?

Here we can give the answer to the problem, that is, we have the following theorems.

THEOREM 1 (Sato [9]). *The Jørgensen number of the Whitehead link is two.*

COROLLARY (Sato [9]). The Whitehead link is not a Jørgensen group.

THEOREN 2 (Sato [9]). *The Whitehead link is conjugate to* the marked two-generator group $G_{\sigma,\mu}$ *where* $\sigma = \sqrt{2}e^{3\pi i/4}$ *and* $\mu = -1/2$.

6. The proofs of the theorems will appear elsewhere. Here we only give sketches of the proofs.

THEOREM E (cf. Wielenberg [11], Krushkal', Apanasov and Gusevskiĭ,[4]). *The Whitehead link G_W has the following presentation:*

$$G_W = \langle A, B \mid (A^{-1}BAB^{-1})(ABA^{-1}B^{-1})(AB^{-1}A^{-1}B)(A^{-1}B^{-1}AB) = 1\rangle,$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}.$$

PROPOSITION 1 *Let G_W be the Whitehead link defined in Theorem E. Then an element X of G_W has the following form:*

$$X = \begin{pmatrix} 1+(1-i)a & b_1+(1-i)b_2 \\ (1-i)c & 1+(1-i)d \end{pmatrix}.$$

where $a, b_1, b_2, c, d \in \mathbf{Z}+i\mathbf{Z}$, $a+d-b_1c+(1-i)(ad-b_2c)=0$.

PROPOSITION 2. *Let G_W be the Whitehead link defined in Theorem E and let $\langle X, Y\rangle$ be a non-elementary subgroup generated by X and Y, where $X, Y \in G_W$. Then the Jørgensen number of (X,Y) is greater than or equal to two: $J(X,Y) \geq 2$.*

PROPOSITION 3. *Let A, B be the matrices in Theorem E. Set $C = AB$. Then A and C generate the Whitehead link G_W and $J(A,C) = 2$.*

Theorem 1 follows from Propositons 2 and 3.

6. Next we will give a sketch of the proof of Theorem 2.

Let P be the regular ideal octahedorn in Ratcliffe [**6**, p.454]. Let the sides $S_A, S_B, S_C, S_D, S_{A'}, S_{B'}, S_{C'}$ and $S_{D'}$ be the sides of P. Let f_A, f_B, f_C and f_D be the side pairing transformations of S_A to $S_{A'}$, of S_B to $S_{B'}$, of S_C to $S_{C'}$, and of S_D to $S_{D'}$, respectively.

PROPOSITION 4. *Let f_A, f_B, f_C and f_D be the side pairing transformations defined in the above. Then f_A, f_B, f_C and f_D generate the Whitehead link $G_{W,R}$ in the sense of Ratcliffe.*

PROPOSITION 5. *Let*

$$G^*_{W,R} = \langle A^*, B^* \mid A^*(B^*)^{-2}A^*B^*(A^*)^{-1}(B^*)^{-1}$$
$$(A^*)^{-1}(B^*)^2(A^*)^{-1}(B^*)^{-1}A^*B^* = 1\rangle,$$

where

$$A^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B^* = \begin{pmatrix} 1/2+i/2 & 3/4+i/4 \\ -1+i & 1/2+i/2 \end{pmatrix}.$$

Then $G^*_{W,R}$ is conjugate to the Whitehead link $G_{W,R}$ in the sense of Ratcliffe.

(ii) $J(A^*, B^*) = 2$.

PROPOSITION 6. *The marked group $G^*_{W,R} = \langle A^*, B^* \rangle$ in Proposition 5 corresponds to the point $(-1+i, -1/2)$.*

Theorem 2 follows from Propositions 5 and 6,

References

[1] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976) 739-749.

[2] T. Jørgensen and M. Kiikka, *Some extreme discrete groups*, Ann. Acad. Sci. Fenn. **1** (1975), 245-248.

[3] T. Jørgensen, A. Lascurain and T. Pignataro, *Translation extentions of the classical modular group*, Complex Variable **19** (1992), 205-209.

[4] S. L. Krushkal', B. N. Apanasov and N. A. Gusevskiĭ, *Kleinian groups and uniformization in examples and problems*, Trans. math. Monographs **62**, Amer. Math. Soc., Providence, Rhode Island, 1986.

[5] B. Maskit, *Some special 2-generator Kleinian groups*, Proc. Amer. Math. Soc., **106** (1989), 175-186.

[6] J. R. Ratcliffe, *Foundations of hyperbolic manifolds*, GTM **149**, Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[7] H. Sato, *One-parameter families of extreme groups for Jørgensen's inequality*, Contemporary Math. **256** (The First Ahlfors - Bers Colloquium) edited by I. Kra and B. Maskit, 2000, 271-287.

[8] H. Sato, *Jørgensen groups and the Picard group*, to appear in the Proc. of The Third ISAAC International Conference, Academic Scientific Publ., 2002.

[9] H. Sato, *The Jørgensen number of the Whitehead link*, in preparation.

[10] H. Sato and R. Yamada, *Some extreme Kleinian groups for Jørgensen's inequality*, Rep. Fac. Sci. Shizuoka Univ. **27** (1993), 1-8.

[11] N. J. Wielenberg, The structure of certain subgroups of the Picard group, Math. Proc. Cambridge Philos. Soc. **84** (1978), 427-438.

Department of Mathematics

Faculty of Science

Shizuoka University

Ohya Shizuoka 422-8529

Japan

e-mail: smhsato@ipc.shizuoka.ac.jp