数理解析研究所講究録 1276 巻 2002 年 94-102

ON A CERTAIN FORMULA OF TRIGONOMETRIC SUM

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1. Introduction

The following formula is known,

$$\sum_{j=1}^{n}(-1)^{j}\frac{(2n)!}{(n-j)!(n+j)!}\cos j\theta=2^{2n-1}\sin^{2n}(\frac{\theta}{2})-\frac{1}{2}\binom{2n}{n}.$$

and the curve of the trigonometric sum can be guessed from the right hand side of itself. On the contrary, we could not guess how like a curve the following trigonometric sum does figure,

$$\sum_{j=1}^{n}(-1)^{j}\frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta,$$

and the appropriate formula to guess the curve of the above trigonometric sum by itself has not been known. This author prove in this paper that

$$\sum_{j=1}^{n}(-1)^{j}\frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta=-\left(c_{0}^{n}+c_{1}^{n}y^{2}+\cdots+c_{n-1}^{n}y^{2(n-1)}\right)\sin\theta, \qquad (1)$$

where c_0^n, c_1^n, $\cdots$, c_{n-1}^n are positive constants and $y=\sin(\theta/2)$, and $c_0^n=(2n-2)!/\{(n-1)!\}^2$, $c_k^n=4c_{k-1}^{n-1}$, $(k=1,\cdots,n-1)$ hold. The author thinks that the above formula for trigonometric sum is an interesting result and useful in applied mathematics.

2. CHARACTERISTIC FUNCTIONS AND POLYNOMIALS

It is known in the complex function theory that the following relation holds,

$$\Gamma(1-ix)\Gamma(1+ix)=\frac{\pi ix}{\sin\pi ix}=\frac{1}{\prod_{n=1}^{\infty}(1+x^{2}/n^{2})},\quad -\infty<x<\infty. \qquad (2)$$

From the infinite divisibility of the Cauchy distribution, the author is interested in infinite divisibility of the normed product of Cauchy densities such as the following form,

$$f(x) = \frac{c}{\Pi_{j=1}^{n}(x^2 + a_j^2)}, \quad -\infty < x < \infty, \tag{3}$$

where $0 < a_1 < a_2 < \cdots < a_n$ and c is a normalized constant. The density function $f(x)$ is an approximation of the above right hand side of (2) in the sense of weak limit. Consider a characteristic function of the density fucntion (3). We obtain the following

Lemma 1. *It holds that*

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} e^{itx} \frac{c}{\Pi_{j=1}^{n}(x^2 + a_j^2)} dx \\ &= \pi c \sum_{j=1}^{n} \frac{\exp(-a_j|t|)}{a_j \Pi_{l=1, l\neq j}^{n}(-a_j^2 + a_l^2)}, \quad -\infty < t < \infty. \end{aligned} \tag{4}$$

Proof. For simplicity, let us consider a characteristic function for the case $n = 3$;

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx = \int_{-\infty}^{\infty} e^{itx} \frac{c}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2)} dx.$$

By residue analysis we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{imx} \frac{1}{x^2 + a^2} dx &= 2\pi i \, res\{\frac{\exp(imz)}{z^2 + a^2}, ai\} \\ &= 2\pi i [\frac{\exp(imz)}{z + ia}]_{z=ai} = 2\pi i \frac{\exp(-ma)}{2ai}, \\ (m > 0, \, a > 0) & \end{aligned}$$

and for positive $t = m$, we obtain

$$\begin{aligned} & 2\pi i \quad limit_{z \to ia_j}(z - ia_j) \exp(itz) \cdot \frac{c}{(z^2 + a_1^2)(z^2 + a_2^2)(z^2 + a_3^2)} \\ = & \quad 2\pi i \, res\{\exp(itz) \cdot \frac{c}{(z^2 + a_1^2)(z^2 + a_2^2)(z^2 + a_3^2)}, a_j i\} \\ = & \quad 2\pi i [\frac{ce^{-ta_j}}{2a_j i} \frac{1}{\Pi_{l=1, l\neq j}^{3}(-a_j^2 + a_l^2)}] \\ = & \quad \frac{\pi c e^{-ta_j}}{a_j} \frac{1}{\Pi_{l=1, l\neq j}^{3}(-a_j^2 + a_l^2)}. \end{aligned}$$

From the above we obtain a characteristic function such as the following form,

$$\phi(t) = \pi c \{ \frac{e^{-|t|a_1}}{a_1 \Pi_{l=2}^{3}(-a_1^2 + a_l^2)} + \frac{e^{-|t|a_2}}{a_2 \Pi_{l=1, l\neq 2}^{3}(-a_2^2 + a_l^2)} + \frac{e^{-|t|a_3}}{a_3 \Pi_{l=1, l\neq 3}^{3}(-a_3^2 + a_l^2)} \}.$$

In the same way, we have

$$\phi(t) = \frac{\pi c}{a_n^2 - a_1^2}\Big(\sum_{j=1}^{n-1} \frac{\exp(-a_j|t|)}{a_j \Pi_{l=1,l\neq j}^{n-1}(-a_j^2 + a_l^2)} - \sum_{j=2}^{n} \frac{\exp(-a_j|t|)}{a_j \Pi_{l=2,l\neq j}^{n}(-a_j^2 + a_l^2)}\Big). \quad (5)$$

Note that $a_1 = 1$, $a_2 = 2, ..., a_n = n$ for the case of (2), and we set $x = \exp(-|t|)$, then we obtain a polynomial as the following form,

$$\phi(t) = \pi c x \sum_{j=1}^{n} \frac{x^{j-1}}{a_j \Pi_{l=1,l\neq j}^{n}(-a_j^2 + a_l^2)}, \quad 0 \le x \le 1.$$

From this polynomial we can consider a complex valued polynomial on the whole complex plane,

$$Q_n(z) = \pi c z \sum_{j=1}^{n} \frac{z^{j-1}}{a_j \Pi_{l=1,l\neq j}^{n}(-a_j^2 + a_l^2)},$$

and so, we have

$$P_{n-1}(z) = (-1)^{n-1} a_n \Pi_{l=1}^{n-1}(-a_n^2 + a_l^2) \sum_{j=1}^{n} \frac{z^{j-1}}{a_j \Pi_{l=1,l\neq j}^{n}(-a_j^2 + a_l^2)}.$$

We obtain the following

Lemma 2. *The polynomial can be written in the following form,*

$$P_{n-1}(z) = \sum_{j=1}^{n}(-1)^{j-1} \frac{(2n)!}{(n+j)!(n-j)!} \frac{j}{n} z^{j-1}, \quad (6)$$

and

$$\frac{(2n)!}{(n+j)!(n-j)!} \frac{j}{n}$$

are natural numbers.

Lemma 3. *It holds that*

$$\sum_{j=n-r}^{n} \frac{1}{j\ \Pi_{l=n-r,l\neq j}^{n}(-j^2 + l^2)} = \frac{(2r-1)!}{r\{(r-1)!\}^2(2n-1)(2n-2)\cdots(2n-2r+1)(n-r)n}. \quad (7)$$

Proof. We have the following equality from (4) and (5),

$$\sum_{j=1}^{n}\frac{1}{a_j\Pi_{l=1,l\neq j}^{n}(-a_j^2+a_l^2)} = \frac{1}{a_n^2-a_1^2}\Big(\sum_{j=1}^{n-1}\frac{1}{a_j\Pi_{l=1,l\neq j}^{n-1}(-a_j^2+a_l^2)} - \sum_{j=2}^{n}\frac{1}{a_j\Pi_{l=2,l\neq j}^{n}(-a_j^2+a_l^2)}\Big). \tag{8}$$

In general it holds that

$$\begin{aligned}&\sum_{j=r}^{n}\frac{1}{a_j\Pi_{l=r,l\neq j}^{n}(-a_j^2+a_l^2)}\\ =\ &\frac{1}{a_n^2-a_r^2}\Big(\sum_{j=r}^{n-1}\frac{1}{a_j\Pi_{l=r,l\neq j}^{n-1}(-a_j^2+a_l^2)} - \sum_{j=r+1}^{n}\frac{1}{a_j\Pi_{l=r+1,l\neq j}^{n}(-a_j^2+a_l^2)}\Big).\end{aligned} \tag{9}$$

We will show the formula of lemma by induction. If $r=n-1$ we have

$$\begin{aligned}&\sum_{j=n-1}^{n}\frac{1}{a_j\Pi_{l=n-1,l\neq j}^{n}(-a_j^2+a_l^2)}\\ =\ &\frac{1}{a_n^2-a_{n-1}^2}\Big(\sum_{j=n-1}^{n-1}\frac{1}{a_j\Pi_{l=n-1,l\neq j}^{n-1}(-a_j^2+a_l^2)} - \sum_{j=n}^{n}\frac{1}{a_j\Pi_{l=n,l\neq j}^{n}(-a_j^2+a_l^2)}\Big)\\ =\ &\frac{1}{n^2-(n-1)^2}\Big(\frac{1}{(n-1)}-\frac{1}{n}\Big)\\ =\ &\frac{1}{(2n-1)(n-1)n}.\end{aligned} \tag{10}$$

If $r=n-2$ we have

$$\begin{aligned}&\sum_{j=n-2}^{n}\frac{1}{a_j\Pi_{l=n-2,l\neq j}^{n}(-a_j^2+a_l^2)}\\ =\ &\frac{1}{a_n^2-a_{n-2}^2}\Big(\sum_{j=n-2}^{n-1}\frac{1}{a_j\Pi_{l=n-2,l\neq j}^{n-1}(-a_j^2+a_l^2)} - \sum_{j=n-1}^{n}\frac{1}{a_j\Pi_{l=n-1,l\neq j}^{n}(-a_j^2+a_l^2)}\Big)\\ =\ &\frac{1}{n^2-(n-2)^2}\Big(\frac{1}{(2n-3)(n-2)(n-1)}-\frac{1}{(2n-1)(n-1)n}\Big)\\ =\ &\frac{3!}{2(2n-1)(2n-2)(2n-3)(n-2)n}.\end{aligned} \tag{11}$$

Suppose that (7) holds and we will show that (7) holds for $r+1$. We see that

$$\begin{aligned}
&\sum_{j=n-r-1}^{n} \frac{1}{a_j \prod_{l=n-r-1,l\neq j}^{n}(-a_j^2+a_l^2)} \\
= \ & \frac{1}{a_n^2 - a_{n-r-1}^2}\Big(\sum_{j=n-r-1}^{n-1} \frac{1}{a_j \prod_{l=n-r-1,l\neq j}^{n-1}(-a_j^2+a_l^2)} \\
&\qquad - \sum_{j=n-r}^{n} \frac{1}{a_j \prod_{l=n-r,l\neq j}^{n}(-a_j^2+a_l^2)}\Big) \\
= \ & \frac{1}{n^2-(n-r-1)^2} \\
\cdot \ & \Big(\frac{(2r-1)!}{r\{(r-1)!\}^2(2n-3)(2n-4)\cdots(2n-2r-1)(n-1-r)(n-1)} \\
- \ & \frac{(2r-1)!}{r\{(r-1)!\}^2(2n-1)(2n-2)\cdots(2n-2r+1)(n-r)n}\Big) \\
= \ & \frac{(2r-1)!}{(2n-r-1)(r+1)r\{(r-1)!\}^2(2n-3)(2n-4)\cdots(2n-2r+1)} \\
\cdot \ & \Big(\frac{1}{(2n-2r)(2n-2r-1)(n-1-r)(n-1)} - \frac{1}{(2n-1)(2n-2)(n-r)n}\Big) \\
= \ & \frac{(2r-1)!}{(2n-r-1)(r+1)r\{(r-1)!\}^2(2n-3)(2n-4)\cdots(2n-2r+1)} \\
\cdot \ & \frac{1}{(2n-2r)(n-1)} \\
\cdot \ & \Big(\frac{1}{(2n-2r-1)(n-1-r)} - \frac{1}{(2n-1)n}\Big) \\
= \ & \frac{(2r-1)!}{(2n-r-1)(r+1)r\{(r-1)!\}^2(2n-3)(2n-4)\cdots(2n-2r+1)} \\
\cdot \ & \frac{1}{(2n-2r)(n-1)}\Big(\frac{(2n-r-1)(2r+1)}{(2n-2r-1)(n-1-r)(2n-1)n}\Big) \\
= \ & \frac{(2r+1)!}{(r+1)\{r!\}^2(2n-1)\cdots(2n-2r)(2n-2r-1)(n-r-1)n}. \qquad (12)
\end{aligned}$$

This shows that the equality (7) holds for $r+1$. q.e.d.

3. A FORMULA OF TRIGONOMETRIC SUM

In this section we shall discuss about a formula for trigonometric sum coming from the polynomials $P_{n-1}(z)$. Let us set

$$f(z) = \frac{1}{n}\sum_{j=1}^{n}(-1)^j \frac{(2n)!}{(n-j)!(n+j)!} z^j.$$

Then we have $f'(z) = -P_{n-1}(z)$. If we set $C:\ z = e^{i\theta}\ \ (0 \le \theta \le 2\pi)$ we have

$$f(e^{i\theta}) = \frac{1}{n}\Big(\sum_{j=1}^{n}(-1)^j \frac{(2n)!}{(n-j)!(n+j)!}\cos j\theta + i\sum_{j=1}^{n}(-1)^j \frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta\Big).$$

Let us set

$$nV(\theta) = \sum_{j=1}^{n}(-1)^j \frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta. \qquad (13)$$

Let $y = \sin(\theta/2)$.
The case of $n = 1$: Then $nV(\theta) = -\sin\theta$.
The case of $n = 2$: Then $nV(\theta) = -(2 + 4y^2)\sin\theta$.
The case of $n = 3$: Then

$$nV(\theta) = -(6 + 8y^2 + 16y^4)\sin\theta.$$

The case of $n = 4$: Then

$$nV(\theta) = -(20 + 24y^2 + 32y^4 + 64y^6)\sin\theta.$$

The case of $n = 5$: Then

$$nV(\theta) = -(70 + 80y^2 + 96y^4 + 128y^6 + 256y^8)\sin\theta.$$

The case of $n = 6$: Then

$$nV(\theta) = -(252 + 280y^2 + 320y^4 + 384y^6 + 512y^8 + 1024y^{10})\sin\theta.$$

The case of $n = 7$: Then

$$\begin{aligned} nV(\theta) = -(&924 + 1008y^2 + 1120y^4 + 1280y^6 \\ &+1536y^8 + 2048y^{10} + 4096y^{12})\sin\theta. \end{aligned}$$

The case of $n = 11$: Then

$$\begin{aligned} nV(\theta) &= -(184756 + 194480y^2 + 205920y^4 + 219648y^6 \\ &+ 236544y^8 + 258048y^{10} + 286720y^{12} + 327680y^{14} \\ &+ 393216y^{16} + 524288y^{18} + 1048576y^{20})\sin\theta. \end{aligned}$$

The author describes the proof in what follows.

Theorem 1. *It holds that*

$$\begin{aligned} &\sum_{j=1}^{n}(-1)^j \frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta \\ = &-\sin\theta \sum_{r=0}^{n-1} \frac{(2n-2r-2)!}{\{(n-r-1)!\}^2} 2^{2r} (\sin\frac{\theta}{2})^{2r}. \end{aligned} \qquad (14)$$

Proof. It is known that if we set $y = \sin\frac{\theta}{2}$ then

$$\frac{\sin j\theta}{\sin\theta} = \sum_{r=0}^{j-1}(-1)^r \frac{j\,2^{2r}}{(2r+1)!}[\Pi_{k=1}^{r}(j^2-k^2)]y^{2r}.$$

By this equality we have

$$-\frac{nV(\theta)}{\sin\theta} = \sum_{j=1}^{n}(-1)^{j-1}\frac{(2n)!}{(n-j)!(n+j)!}\frac{\sin j\theta}{\sin\theta} \tag{15}$$

$$= n(2n-1)!\sum_{j=1}^{n}\frac{1}{j(-1)^{(j-1)}\Pi_{l=1}^{j-1}(j^2-l^2)\Pi_{l=j+1}^{n}(l^2-j^2)} \tag{16}$$

$$\cdot\sum_{r=0}^{j-1}(-1)^r\frac{2^{2r}}{(2r+1)!}[\Pi_{k=1}^{r}(j^2-k^2)]y^{2r}.$$

Summing the terms of $y^{2(j-1)}$, we have

$$\begin{aligned}
& \frac{1}{j(-1)^{(j-1)}\Pi_{l=1}^{j-1}(j^2-l^2)\Pi_{l=j+1}^{n}(l^2-j^2)} \\
& \cdot(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!}[\Pi_{k=1}^{j-1}(j^2-k^2)]y^{2(j-1)} \\
+ & \frac{1}{(j+1)(-1)^{j}\Pi_{l=1}^{j}((j+1)^2-l^2)\Pi_{l=j+2}^{n}(l^2-(j+1)^2)} \\
& \cdot(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!}[\Pi_{k=1}^{j-1}((j+1)^2-k^2)]y^{2(j-1)} \\
+ & \frac{1}{(j+2)(-1)^{j+1}\Pi_{l=1}^{j+1}((j+2)^2-l^2)\Pi_{l=j+3}^{n}(l^2-(j+2)^2)} \\
& \cdot(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!}[\Pi_{k=1}^{j-1}((j+2)^2-k^2)]y^{2(j-1)} \\
+ & \cdots \\
+ & \frac{1}{(n-1)(-1)^{n-2}\Pi_{l=1}^{n-2}((n-1)^2-l^2)\Pi_{l=n}^{n}(l^2-(n-1)^2)} \\
& \cdot(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!}[\Pi_{k=1}^{j-1}((n-1)^2-k^2)]y^{2(j-1)} \\
+ & \frac{1}{n(-1)^{n-1}\Pi_{l=1}^{n-1}(n^2-l^2)} \\
& \cdot(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!}[\Pi_{k=1}^{j-1}(n^2-k^2)]y^{2(j-1)}.
\end{aligned} \tag{17}$$

We see that

$$(2n-1)!\Big\{\frac{1}{j(-1)^{j-1}\Pi_{l=j+1}^{n}(l^2-j^2)}$$

$$
\begin{aligned}
&+ \quad \frac{1}{(j+1)(-1)^j((j+1)^2-j^2)\Pi_{l=j+2}^{n}(l^2-(j+1)^2)} \\
&+ \quad \frac{1}{(j+2)(-1)^{j+1}\Pi_{l=j}^{j+1}(j+2)^2-l^2)\Pi_{l=j+3}^{n}(l^2-(j+2)^2)} \\
&+ \quad \cdots \\
&+ \quad \frac{1}{(n-1)(-1)^{n-2}\Pi_{l=j}^{n-2}((n-1)^2-l^2)(n^2-(n-1)^2)} \\
&+ \quad \frac{1}{n(-1)^{n-1}\Pi_{l=j}^{n-1}(n^2-l^2)}\Big\}(-1)^{j-1}\frac{2^{2(j-1)}}{(2(j-1)+1)!} \\
&= \quad (2n-1)!\Big\{\frac{1}{j\Pi_{l=j+1}^{n}(-j^2+l^2)} \\
&+ \quad \frac{1}{(j+1)(j^2-(j+1)^2)\Pi_{l=j+2}^{n}(l^2-(j+1)^2)} \\
&+ \quad \frac{1}{(j+2)\Pi_{l=j}^{j+1}(l^2-(j+2)^2)\Pi_{l=j+3}^{n}(l^2-(j+2)^2)} \\
&+ \quad \cdots \\
&+ \quad \frac{1}{(n-1)\Pi_{l=j}^{n-2}(l^2-(n-1)^2)(n^2-(n-1)^2)} \\
&+ \quad \frac{1}{n\Pi_{l=j}^{n-1}(-n^2+l^2)}\Big\}\frac{2^{2(j-1)}}{(2(j-1)+1)!} \\
&= \quad (2n-1)!\frac{(2(n-j)-1)!2^{2(j-1)}}{(n-j)\{(n-j-1)!\}^2} \\
&\quad \cdot\frac{1}{(2n-1)(2n-2)\cdots(2n-2(n-j)+1)jn(2j-1)!} \\
&= \quad \frac{(2j)!(2(n-j)-1)!2^{2j-1}}{(n-j)!(n-j-1)!(2j)!n} \\
&= \quad \frac{[2(n-j)]!2^{2(j-1)}}{(n-j)!(n-j)!n} \qquad (18)
\end{aligned}
$$

and if $j=r+1$, $(j=1,\ 2,\cdots,n)$ we lastly have

$$\text{the term of } y^{2r} = \frac{[2(n-r-1)]!\ 2^{2r}}{(n-r-1)!(n-r-1)!}y^{2r}. \qquad (19)$$

q.e.d.

By the above formula we see that

$$nV(\theta)=\sum_{j=1}^{n}(-1)^j\frac{(2n)!}{(n-j)!(n+j)!}\sin j\theta$$

is negative on $0<\theta<\pi$ for all n, and we obtain

$$nV'(0)=\sum_{j=1}^{n}(-1)^j\frac{(2n)!\ j}{(n-j)!(n+j)!}=-\frac{(2n-2)!}{(n-1)!(n-1)!}. \qquad (20)$$

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, New York, Dover, 1970.

[2] L. Bondesson, On the infinite divisibility of the half-Cauchy and other decreasing densities and probability functions on the nonnegative line, *Scand. Acturial J.*, (1985), 225-247.

[3] E. Grosswald, The Student t-distribution of any degree of freedom is infinitely divisible, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* 36 (1976), 103–109.

[4] D. H. Kelker, Infinite divisibility and variance mixtures of the normal distribution, *Ann. Math. Statist.*, 42 (1971), 802–808.

[5] G. K. Kristiansen, A proof of Steutel's conjecture, *The Annals of Probability,* 22 (1994), 442–452.

[6] G. Sansone & J. Gerretsen, Lectures on the theory of functions of a complex variable, 1. Holomorphic functions, P. Noordhoff-Groningen, 1960

[7] F. W. Steutel, Preservation of infinite divisibility under mixing and related topics, *Math. Centre Tracts, Math. Centre, Amsterdam,* 33. 1970

[8] K. Takano, On a family of polynomials with zeros outside of the unit disk, To appear in *International J. Applied Mathematics.*

[9] K. Takano, On analytic poperty of some probability distributions and Bessel functions, *Proceedings of international conference on mathematics and its applications in the new millennium,* Universiti Putra Malysia Press, Depart. Math., Universiti Putra Malaysia, July 2000, 63-71.

[10] O. Thorin, On the infinite divisibility of the Pareto distribution, *Scand. Acturial. J.*, (1977), 31–40.

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