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COMPACT DIFFERENCES OF TWO COMPOSITION OPERATORS

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1. INTRODUCTION

Throughout this article, we let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ its boundary. Let dm denote the normalized Lebesgue measure on $\partial\mathbb{D}$. We denote the classical Hardy space by H^p for $0 < p \leq \infty$. Let $\mathcal{S}(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. Every $\varphi \in \mathcal{S}(\mathbb{D})$ induces through composition a linear *composition operator* C_φ. Thus C_φ is defined by

$$C_\varphi f = f \circ \varphi$$

for analytic function f on $\mathbb{D}$. By the Littlewood's subordination theorem, C_φ is a bounded operator on H^2.

Many authors have investigated some properties of composition operators and tried to characterize such properties of the operators C_φ using functional analytic properties of its symbol φ . Here we will give a report on the problem when the difference of two composition operators would be compact on H^2. For a general information on composition operators, see [4],[15] and [17: Chaper 10].

2. THE DEVELOPMENT

The work originates from the following result of E. Berkson([1]).

[E. Berkson(1981)] Let $\varphi \in \mathcal{S}(\mathbb{D})$ such that $m(E(\varphi)) > 0$, where $E(\varphi) = \{|\varphi| = 1\}$. If $\|C_\varphi - C_\psi\|^2 < m(E(\varphi))/2$ for $\psi \in \mathcal{S}(\mathbb{D})$, then $\varphi = \psi$.

This result makes a topological statement about the space $\mathcal{C}(H^2)$ of composition operators on H^2, endowed with the operator norm metric. Indeed this says that the identity operator is isolated in $\mathcal{C}(H^2)$.

A. Siskakis (1986) asked if every non-compact composition operator had to be isolated in the space $\mathcal{C}(H^2)$. Then it was begun to explore the ground that lies between the compactness and the isolation in $\mathcal{C}(H^2)$, and the question above had a negative answer later ([16]).

B.D. MacCluer ([9]) gave a sufficient condition on φ for the component containing the composition operator C_φ to be the singleton $\{C_\varphi\}$.

An analytic map $\varphi \in \mathcal{S}(\mathbb{D})$ is said to have an angular derivative at a point $\zeta \in \partial\mathbb{D}$ if there exists $w \in \partial\mathbb{D}$ so that the non-tangential limit

$$\lim_{z\to\zeta} \frac{\varphi(z) - w}{z - \zeta}$$

[B.D. MacCluer (1989)] If φ has a finite angular derivative on a set of positive measure, then C_φ is isolated in $\mathcal{C}(H^2)$.

J.H. Shapiro and C. Sundberg ([16]) explored these territory and gave a number of conjectures:

1. *Characterize the components of* $\mathcal{C}(H^2)$.
2. *Which composition operators are isolated in* $\mathcal{C}(H^2)$*?*
3. *Which composition differences are compact on* H^2*?*

They supposed that two composition operators may belong to the same component of $\mathcal{C}(H^2)$ if and only if they differ by a compact. They offered some sort of joint Nevanlinna counting functions figuring into the problem.

They gave the following result to the isolation problem.

[J.H. Shapiro and C. Sundberg (1990)] If $\varphi \in \mathcal{S}(\mathbb{D})$ satisfies

$$\int \log(1 - |\varphi|)dm > -\infty$$

then C_φ is not isolated in $\mathcal{C}(H^2)$.

It is well known that the condition above characterizes the non- extreme point of the unit ball of H^∞ ([5]). So by Berkson's result and this we can reduce that if φ is an exposed point of the unit ball of H^∞, then C_φ is isolated in $\mathcal{C}(H^2)$ and that if C_φ is isolated in $\mathcal{C}(H^2)$, φ is an extreme point of the unit ball of H^∞.

Moreover this hinges a following sufficient condition for the difference to be compact.

[J.H. Shapiro and C. Sundberg (1990)] If, for $\varphi, \psi \in \mathcal{S}(\mathbb{D})$,

$$\int \frac{|\varphi - \psi|}{(\min\{1 - |\varphi|, 1 - |\psi|\})^3} dm < \infty,$$

then $C_\varphi - C_\psi$ is compact on H^2.

H. Hunziker, H. Jarchow and V. Mascioni([7]) defined the following metric in $\mathcal{C}(H^2)$ and called the topology induced by this the Hilbert-Schmidt topology: for $\varphi, \psi \in \mathcal{S}(\mathbb{D})$,

$$d(\varphi, \psi) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\varphi - \psi}{1 - \overline{\varphi}\psi} \right|^2 \frac{1 - |\varphi|^2 |\psi|^2}{(1 - |\varphi|^2)(1 - |\psi|^2)} d\theta \right)^{1/2}.$$

And they gave the result.

[H. Hunziker, H. Jarchow and V. Mascioni(1990)] For $\varphi \in \mathcal{S}(\mathbb{D})$, the following are equivalent:

(i) φ is an extreme point of the unit ball of H^∞;

(ii) φ is isolated in $(\mathcal{S}(\mathbb{D}), d)$;

(iii) C_φ is isolated in $\mathcal{S}(\mathbb{D})$.

3. NEW RESULTS

Recently some authors have attacked these problems using new tools. In this section we summarize them.

In 1997, J.A. Cima and A.L. Matheson ([3]) characterized the essential norm $\| \ \|_e$ of composition operators using the notion of Aleksandrov measures.

For $\varphi \in \mathcal{S}(\mathbb{D})$ and $\lambda \in \mathbb{D}$, there exists a positive measure μ_λ on $\partial\mathbb{D}$ such that

$$\begin{aligned} \mathrm{Re}\frac{\lambda + \varphi(z)}{\lambda - \varphi(z)} &= \frac{1 - |\varphi(z)|^2}{|\lambda - \varphi(z)|^2} \\ &= \int P(\zeta, z) d\mu_\lambda(\zeta), \end{aligned}$$

where $P(\cdot, z)$ is the Poisson kernel for z,

$$P(\zeta, z) = \frac{1 - |z|^2}{|\zeta - z|^2}.$$

Then μ_λ is called the Aleksandrov measure with the function φ. Denote the absolutely continuous part and the sigular part of μ_λ by $\mu_\lambda^{a,c}$ and μ_λ^s respectively.

[J.A. Cima and A.L. Matheson(1997)]

$$\|C_\varphi\|_e^2 = \sup\{\|\mu_\lambda^s\| : \lambda \in \partial\mathbb{D}\}.$$

This result has the immediate corollary: C_φ is compact on H^2 if and only if for all $\lambda \in \partial\mathbb{D}$ μ_λ is absolutely continous with respect to the Lebesgue measure dm.

Then J.E. Shapiro ([12]) considered the compact difference using this notion.

[J.E. Shapiro(1998)] If $C_\varphi - C_\psi$ is compact on H^2 for $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, $\mu_\lambda^s = \nu_\lambda^s$ for all $\lambda \in \partial\mathbb{D}$.

He conjectured whether its converse would be true.

But it does not seem to be easy to calculate Aleksandrov measure with respect to any self-map of $\mathbb{D}$.

[Example 1] Let $\varphi(z) = sz + (1-s)z$ for $0 < s < 1$. Let μ_λ be the Aleksandrov measure with the function φ.

Then we have

$$\begin{aligned}\|\mu_\lambda^s\| &= \|\mu_\lambda\| - \|\mu_\lambda^{a,c}\| \\ &= \frac{1-|\varphi(0)|^2}{|\lambda - \varphi(0)|^2} - \int \frac{1-|\varphi(\zeta)|^2}{|\lambda-\varphi(\zeta)|^2} dm(\zeta).\end{aligned}$$

Putting $\lambda = 1$, we have the first term of the right side is $(2-s)/s$ and the second term is $(1-s)/s$. So $\|\mu_1^s\| = 1/s > 0$. Consequently C_φ is not compact on H^2.

These measures have played an interesting role in the study of the de Branges-Rovnyak space. J.E. Shapiro has provided the study of relative angular derivatives ([13], [14]).

T.E Goeber, Jr. ([6]) connected this problem with the compactness of composition operators between different Hardy spaces.

Let $0 < q < p < \infty$. Then C_φ is always bounded from H^p to H^q for $\varphi \in \mathcal{S}(\mathbb{D})$. He characterized the essential norm of differences of two composition operators from H^p to H^q.

[T.E Goeber, Jr.(2001)] For $0 < q < p < \infty$, $\|C_\varphi - C_\psi\|_e = 0$ if and only if C_φ and C_ψ are compact from H^p to H^q.

And he offered the following conjecture : Let $0 < q < p < \infty$. Is it true that C_φ, C_ψ are in the same component of the space of composition operators from H^p to H^q if and only if C_φ, C_ψ are compact from H^p to H^q?

Indeed this result inspires us to consider one question:

[Question] What is the space X of analytic functions on $\mathbb{D}$ satisfying that $C_\varphi - C_\psi : X \to H^2$ is compact if and only if $C_\varphi - C_\psi : H^2 \to H^2$ is compact?

When B.D. MacCluer, S. Ohno and R. Zhao ([11]) reduce the problem of compact difference to the H^∞ case, they obtain the result: $C_\varphi - C_\psi$:

$H^\infty \to H^\infty$ is compact if and only if $C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ is compact, where $\mathcal{B}$ is the Bloch space.

So we can suppose the Bloch space as a candidate of the answer to the problem above. But we can find out the interesting result due to E.G. Kwon ([8]):

[E.G. Kwon (1996)] For $\varphi \in \mathcal{S}(\mathbb{D})$, $C_\varphi : \mathcal{B} \to H^2$ is compact if and only if φ is not an extreme point of the unit ball of H^∞, that is,

$$\int \log(1 - |\varphi|) dm > -\infty.$$

We here see again the condition of the non-extreme point of the unit ball of H^∞, which appears in the problem of the hypercyclicity of composition operators ([2]). This condition seems to be interesting and mysterious.

We have the following equivalence.

[Proposition] For $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, the following are equivalent:

(i) $C_\varphi - C_\psi : \mathcal{B} \to H^2$ is bounded;

(ii) $C_\varphi - C_\psi : \mathcal{B} \to H^2$ is compact;

(iii) $C_\varphi - C_\psi : \mathcal{B}_o \to H^2$ is bounded;

(iv) $C_\varphi - C_\psi : \mathcal{B}_o \to H^2$ is compact,

where $\mathcal{B}_o$ is the little Bloch space.

About the compact difference, we can find out two examples in [4]: Example 9.1 at p.336 says that for $\varphi(z) = (z+1)/2$ and $\psi(z) = \varphi(z)+t(z-1)^3$, $C_\varphi - C_\psi$ is compact on H^2 . On the other hand, Exercises 9.3.3 at p.344 gives that for $\varphi(z) = (z+1)/2$ and $\psi(z) = \varphi(z) + t(z-1)^2$, $C_\varphi - C_\psi$ is not compact on H^2 . What exists between these two examples? We have calculated but not completed.

Recently it is reported by B.D. MacCluer ([10]) that J. Moorhouse answers this as follows.

[Example 2] Let $\varphi(z) = sz + 1 - s$ and $\psi(z) = \varphi(z) + t(z-1)^b$ for fixed real numbers s and t such that $0 < s < 1$ and $\psi(\mathbb{D}) \subset \mathbb{D}$. Notice that $|t|$ is so small. For a positive number b,

(i) In the case $0 < b \leq 2$, $C_\varphi - C_\psi$ is not Hilbert-Schmidt on H^2 .

(ii) In the case $2 < b < 5/2$, $C_\varphi - C_\psi$ is compact on H^2 .

(iii) In the case $5/2 < b$, $C_\varphi - C_\psi$ is Hilbert-Schmidt on H^2 .

In the case of the Bergman space $L_a^2 = L_a^2(\mathbb{D}, dA)$ where dA is the normalized Area measure on $\mathbb{D}$, we have the following incomplete result.

[Example 3] Under the same assumption as Example 2,

(i) If $0 < b \leq 2$, $C_\varphi - C_\psi$ is not compact on L_a^2 .

(ii) If $3 < b$, $C_\varphi - C_\psi$ is compact on L_a^2 .

We will add the outline of the proof: (i) At first suppose $0 < b < 2$. For any $\lambda \in \mathbb{D}$, let $k_\lambda(z) = (1-|\lambda|^2)/(1-\bar{\lambda}z)^2$. And then $k_\lambda \in L_a^2$, $\|k_\lambda\| = 1$ and k_λ converges to 0 weakly in L_a^2 as $|\lambda| \to 1$. Then

$$\begin{aligned}
&(*)\|(C_\varphi - C_\psi)^* k_\lambda\|^2 \\
&= \left(\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}\right)^2 + \left(\frac{1-|\lambda|^2}{1-|\psi(\lambda)|^2}\right)^2 - 2\mathrm{Re}\left(\frac{1-|\lambda|^2}{1-\overline{\varphi(\lambda)}\psi(\lambda)}\right)^2 \\
&\geq \left(\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}\right)^2 - 2\left|\frac{1-|\lambda|^2}{1-\overline{\varphi(\lambda)}\psi(\lambda)}\right|^2 .
\end{aligned}$$

We also consider for a sequence $\{\lambda_n\}$of points approaching 1 along the circle $|1-\lambda_n|^2 = 1-|\lambda_n|^2$. Then we have

$$\|(C_\varphi - C_\psi)^* k_{\lambda_n}\|^2 \geq \frac{1}{(2-s)^2 s^2} - \frac{2(1-|\lambda_n|^2)^{2-b}}{|(2-s)s(1-|\lambda_n|^2)^{1-b/2} - |t\varphi(\lambda_n)||^2}.$$

Consequently

$$\lim\{\|(C_\varphi - C_\psi)^* k_{\lambda_n}\|_2^2 : |\lambda_n| \to 1, |1-\lambda_n|^2 = 1-|\lambda_n|^2\} \geq \frac{1}{(2-s)^2 s^2},$$

that is, $C_\varphi - C_\psi$ is not compact on L_a^2.

Secondly suppose $b = 2$. For a sequence of points approaching 1 along the circle $|1-\lambda|^2 = 1-|\lambda|^2$, we can calculate the right side of the equation (*) and show that $C_\varphi - C_\psi$ is not compact on L_a^2.

(ii) For a function $f \in L_a^2$, we have

$$
\begin{aligned}
&(C_\varphi - C_\psi)f(z) \\
&= \int f(w)\left\{\frac{1}{(1-\varphi(z)\bar{w})^2} - \frac{1}{(1-\psi(z)\bar{w})^2}\right\} dA(w) \\
&= \int f(w)\left(\frac{1}{1-\varphi(z)\bar{w}} - \frac{1}{1-\psi(z)\bar{w}}\right) \\
&\quad \times \left(\frac{1}{1-\varphi(z)\bar{w}} + \frac{1}{1-\psi(z)\bar{w}}\right) dA(w)
\end{aligned}
$$

So

$$
\begin{aligned}
&|(C_\varphi - C_\psi)f(z)|^2 \\
&\leq \int |f(w)|^2 \left|\frac{1}{1-\varphi(z)\bar{w}} - \frac{1}{1-\psi(z)\bar{w}}\right|^2 dA(w) \\
&\quad \times \int \left|\frac{1}{1-\varphi(z)\bar{w}} + \frac{1}{1-\psi(z)\bar{w}}\right|^2 dA(w) \\
&\leq \int |f(w)|^2 \left|\frac{\varphi(z)-\psi(z)}{(1-\varphi(z)\bar{w})(1-\psi(z)\bar{w})}\right|^2 dA(w) \\
&\quad \times 2\left\{\int \left|\frac{1}{1-\varphi(z)\bar{w}}\right|^2 dA(w) + \int \left|\frac{1}{1-\psi(z)\bar{w}}\right|^2 dA(w)\right\} \\
&\leq C \int |f(w)|^2 dA(w) |t||z-1|^{2(b-4)} dA(w) \\
&\quad \times \left(\log\frac{1}{1-|\varphi(z)|^2} + \log\frac{1}{1-|\varphi(z)|^2}\right)
\end{aligned}
$$

where C is a constant.

Using the facts that $\log 1/(1-|z|^2) \in L^p$ for $0 < p$ and $1/(z-1) \in L_a^p$ for $0 < p < 2$, we can show $C_\varphi - C_\psi$ is compact on L_a^2 for $3 < b$.

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