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Algebras of Infinite Dimensional Holomorphic Functions and Application to Probability

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Abstract. In this paper, we study a new class of nuclear algebras of entire functionals of exponential growth and several variables. Then using the convolution calculus we develop the theory of operators defined on this algebras. In particular we define the exponential of some operators which permits to solve some quantum stochastic differential equations.

1 Introduction

In the last years new classes of spaces of generalized and test functions were introduced by many authors, see e.g., [2], [6], [7], [16]. Let $\mathcal{N}$ be a complex Fréchet nuclear space with topology given by an increasing family of Hilbertian norms $\{|\cdot|_n,\ n \in \mathbb{N}\}$. It is well known that $\mathcal{N}$ may be represented as $\mathcal{N} = \cap_{n\in\mathbb{N}}\mathcal{N}_n$, where the Hilbert space $\mathcal{N}_n$ is the completion of $\mathcal{N}$ with respect to $|\cdot|_n$. By the general duality theory $\mathcal{N}'$ is given by $\mathcal{N}' = \cup_{n\in\mathbb{N}}\mathcal{N}_{-n}$, where $\mathcal{N}_{-n} = \mathcal{N}_n'$ is the topological dual of $\mathcal{N}_n$. Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous convex strictly increasing function such that

$$\lim_{x\to\infty}\frac{\theta(x)}{x} = \infty, \quad \theta(0) = 0. \tag{1}$$

Such functions are called Young functions. For a Young function θ we define

$$\theta^*(x) = \sup_{t>0}(tx - \theta(t)) \tag{2}$$

This is called the polar function associated to θ. It is known that θ^* is again a Young function and $(\theta^*)^* = \theta$. For every $p \in \mathbf{Z}$ and $m > 0$, we denote by $Exp(\mathcal{N}_p, \theta, m)$ the space of entire functions f on the complex Hilbert space $\mathcal{N}_p$ such that

$$\|f\|_{\theta,p,m} := \sup_{z\in\mathcal{N}_p}|f(z)|e^{-\theta(m|z|_p)} < +\infty \tag{3}$$

We fix a Young function θ. Then

$$\{\mathcal{F}_{\theta,m}(\mathcal{N}_{-p}) := Exp(\mathcal{N}_{-p}, \theta, m); p \in \mathbb{N}, m > 0\}$$

becomes a projective system of Banach spaces and we put

$$\mathcal{F}_\theta(\mathcal{N}') = \operatorname*{proj\,lim}_{p\to\infty; m\downarrow 0} Exp(\mathcal{N}_{-p}, \theta, m) \tag{4}$$

which is called the space of entire functions on $\mathcal{N}'$ with an θ-exponential growth of minimal type. On the other hand $\{Exp(\mathcal{N}_p, \theta, m); p \in \mathbb{N}, m > 0\}$ becomes an inductive system of Banach spaces and we put

$$\mathcal{G}_\theta(\mathcal{N}) = \underset{p\to\infty; m\to\infty}{\operatorname{ind\,lim}} Exp(\mathcal{N}_p, \theta, m). \tag{5}$$

This is called the space of entire functions on $\mathcal{N}$ with θ-exponential growth of arbitrary type. Then $\mathcal{F}_\theta(\mathcal{N}')$ equipped with the projective limit topology is our test function space. The corresponding topological dual, equipped with the inductive limit topology, is denoted by $\mathcal{F}_\theta^*(\mathcal{N}')$ which is the generalized function space, see [7] for more details. In particular, if $\mathcal{N} = S_{\mathbb{C}}(\mathbb{R})$ (the complexification of the Schwartz test function space $S(\mathbb{R})$) and $\theta(x) = x^2$, then $\mathcal{F}_\theta(\mathcal{N}')$ is nothing than the analytic version of the Kubo-Takenaka test functions space and the corresponding topological dual is the Hida distribution space, see e.g.,[9]. The test function space of Kondratiev-Streit type $(S)_\beta$, $\beta \in [0, 1)$ are obtained choosing $\theta(x) = x^{\frac{2}{1+\beta}}$, see [13], [14], [18], [20].

More recently, a two-variable version of the above spaces was introduced, see [10]. In fact for arbitrary $k \in \mathbb{N}$, we can replace the nuclear space $\mathcal{N}$ by a Cartesian product $\mathcal{N}_1 \times \ldots \times \mathcal{N}_k$, and θ by $(\theta_1, \ldots, \theta_k)$ where θ_i are Young functions and $\mathcal{N}_i$ is a complex nuclear Fréchet space, $1 \le i \le k$. Then it is possible to extend all the results obtained in [7] in the mulivariable case. In particular, the Laplace transform $\mathcal{L}$ is a topological isomorphism between the generalized function space $\mathcal{F}_\theta^*(\mathcal{N}_1' \times \ldots \times \mathcal{N}_k')$ and $\mathcal{G}_{\theta^*}(\mathcal{N}_1 \times \ldots \times \mathcal{N}_k)$, where $\mathcal{G}_{\theta^*}(\mathcal{N}_1 \times \ldots \times \mathcal{N}_k)$ is the space of entire functions on $\mathcal{N}_1 \times \ldots \times \mathcal{N}_k$ which verify some exponential growth condition similar to (3) with respect to $\theta^* = (\theta_1^*, \ldots, \theta_k^*)$, where θ_i^* is the polar function corresponding to θ_i. Another important result in [4] and [5] is the characterization theorem for convergent sequences of distributions in $\mathcal{F}_\theta^*(\mathcal{N}_1' \times \ldots \times \mathcal{N}_k')$. Using this result, we can directly define for any given continuous stochastic process $X(t) \in \mathcal{F}_\theta^*(\mathcal{N}_1' \times \ldots \times \mathcal{N}_k')$ the integral

$$\int_0^t X(s)ds = \mathcal{L}^{-1} \int_0^t \mathcal{L}X(s)ds. \tag{6}$$

Very useful in applications is the convolution product on $\mathcal{F}_\theta^*(\mathcal{N}')$, see [3], [5] and [8] for details. In fact, we define the convolution of two distributions $\Phi, \Psi \in \mathcal{F}_\theta^*(\mathcal{N}')$ by

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi \cdot \mathcal{L}\Psi), \tag{7}$$

which is well defined because $\mathcal{G}_{\theta^*}(\mathcal{N})$ is an algebra under pointwise multiplication. We can define for any generalized function $\Phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ the convolution exponential of Φ denoted by $\exp^* \Phi$ as a generalized function on $\mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{N}')$. Note that for a generalized function $\Phi \in (S)'_\beta$ the Wick exponential of Φ denoted by $\exp^\diamond \Phi$ does not belong to $(S)'_\beta$, but it belongs to a bigger space of distributions $(S)^{-1}$ called Kondratiev distribution space, see [12].

In this paper, we do not restrict ourselves to the theory of gaussian (white noise) and non-gaussian analysis studied for example in [1], [9], [12], [13] and [14] but we develop a general infinite dimensional analysis. First, we give a decomposition of convolution operators from $\mathcal{F}_\theta(\mathcal{N}')$ into itself, into a sum of holomorphic derivation operators. Second, we establish a topological isomorphism between the space $\mathcal{L}(\mathcal{F}_\theta(\mathcal{N}'), \mathcal{F}_\theta(\mathcal{N}'))$ of operators and the space $\mathcal{F}_\theta(\mathcal{N}')\widehat{\otimes}\mathcal{G}_{\theta^*}(\mathcal{N})$ of holomorphic functions. Next, we develop a new convolution calculus over $\mathcal{L}(\mathcal{F}_\theta(\mathcal{N}'), \mathcal{F}_\theta(\mathcal{N}'))$ and we give a sense to the expression $e^T := \sum_{n\geq 0} \frac{T^n}{n!}$ for some class of operators T. Finally, as an application of this theory we solve some linear quantum stochastic differential equations.

2 Preliminaries

For any $n \in \mathbb{N}$ we denote by $\mathcal{N}^{\odot n}$ the n-th symmetric tensor product of $\mathcal{N}$ equipped with the π-topology and by $N_p^{\odot n}$ the n-th symmetric Hilbertian tensor product of N_p. We will preserve the notation $|.|_p$ and $|.|_{-p}$ for the norms on $\mathcal{N}_p^{\odot n}$ and $\mathcal{N}_{-p}^{\odot n}$ respectively. We denote by $\langle ., . \rangle$ the $\mathbf{C}$-bilinear form on $\mathcal{N}'\times\mathcal{N}$ connected to the inner product $\langle .|. \rangle$ of $H = \mathcal{N}_0$, i.e.,

$$\langle z, \xi \rangle = \langle \bar{z} | \xi \rangle \,, \quad z \in H, \quad \xi \in \mathcal{N}.$$

By definition $f \in \mathcal{F}_\theta(\mathcal{N}')$ and $g \in \mathcal{G}_\theta(\mathcal{N})$ admit the Taylor expansions:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \langle z^{\otimes n}, f_n \rangle, \qquad z \in \mathcal{N}', \quad f_n \in \mathcal{N}^{\odot n}, \\ g(\xi) &= \sum_{n=0}^{\infty} \langle g_n, \xi^{\otimes n} \rangle, \qquad \xi \in \mathcal{N}, \quad g_n \in (\mathcal{N}^{\odot n})', \end{aligned} \tag{8}$$

where we used the common symbol $\langle \cdot, \cdot \rangle$ for the canonical bilinear form on $(\mathcal{N}^{\odot n})' \times \mathcal{N}^{\odot n}$ for all n. In order to characterize $\mathcal{F}_\theta(\mathcal{N}')$ and $\mathcal{G}_\theta(\mathcal{N})$ in terms of the Taylor expansions, we introduce weighted Fock spaces $F_{\theta,m}(N_p)$ and $G_{\theta,m}(N_{-p})$. First we define a sequence $\{\theta_n\}$ by

$$\theta_n = \inf_{r>0} \frac{\exp \theta(r)}{r^n}, \qquad n = 0, 1, 2, \cdots .$$

Suppose a pair $p \in \mathbb{N}$, $m > 0$ is given. Then, for $\vec{f} = (f_n)_{n=0}^{\infty}$ with $f_n \in N_p^{\odot n}$ we put

$$\| \vec{f} \|_{\theta,p,m}^2 = \sum_{n=0}^{\infty} \theta_n^{-2} m^{-n} \, |f_n|_p^2,$$

and for $\vec{\Phi} = (\Phi_n)_{n=0}^{\infty}$ with $\Phi_n \in N_{-p}^{\odot n}$,

$$\| \vec{\Phi} \|_{\theta,-p,m}^2 = \sum_{n=0}^{\infty} (n!\theta_n)^2 m^n |\Phi_n|_{-p}^2.$$

Accordingly, we put

$$\begin{aligned} F_{\theta,m}(N_p) &= \left\{ \vec{f} = (f_n) \, ; \, f_n \in \mathcal{N}_p^{\odot n}, \, \|\vec{f}\|_{\theta,p,m}^2 < \infty \right\}, \\ G_{\theta,m}(N_{-p}) &= \left\{ \vec{\Phi} = (\Phi_n) \, ; \, \Phi_n \in \mathcal{N}_{-p}^{\odot n}, \| \vec{\Phi} \|_{\theta,-p,m}^2 < \infty \right\}. \end{aligned} \tag{9}$$

Finally, we define

$$
\begin{aligned}
F_\theta(\mathcal{N}) &= \operatorname*{proj\,lim}_{p\to\infty;m\downarrow 0} F_{\theta,m}(\mathcal{N}_p),\\
G_\theta(\mathcal{N}') &= \operatorname*{ind\,lim}_{p\to\infty;m\to\infty} G_{\theta,m}(\mathcal{N}_{-p}).
\end{aligned}
\tag{10}
$$

It is easily verified that $F_\theta(\mathcal{N})$ becomes a nuclear Fréchet space. By definition, $F_\theta(\mathcal{N})$ and $G_\theta(\mathcal{N}')$ are dual each other, namely, the strong dual of $F_\theta(\mathcal{N})$ is identified with $G_\theta(\mathcal{N}')$ through the canonical bilinear form:

$$
\langle\!\langle \overrightarrow{\Phi}, \overrightarrow{f} \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle \Phi_n, f_n \rangle. \tag{11}
$$

The *Taylor series map* $\mathcal{T}$ (at zero) associates to any entire function the sequence of coefficients. For example, if the Taylor expansion of $f \in \mathcal{F}_\theta(\mathcal{N}')$ is given as in (8), the Taylor series map is defined by $\mathcal{T}f = \overrightarrow{f} = (f_n)$. In particular, for every $z \in \mathcal{N}'$, the Dirac mass δ_z defined by

$$
\langle\!\langle \delta_z, \varphi \rangle\!\rangle := \varphi(z), \tag{12}
$$

belongs to $\mathcal{F}_\theta^*(\mathcal{N}')$. Moreover, δ_z coincide with the distribution associated to the formal series $\overrightarrow{\delta_z} := (\frac{z^{\otimes n}}{n!})_{n\geq 0}$.

Theorem 1 ([7]) *The Taylor series map $\mathcal{T}$ gives two topological isomorphisms: $\mathcal{F}_\theta(\mathcal{N}') \to F_\theta(\mathcal{N})$ and $\mathcal{G}_{\theta^*}(\mathcal{N}) \to G_\theta(\mathcal{N}')$.*

3 Application to White Noise Analysis

For some functions θ, the spaces $\mathcal{F}_\theta(\mathcal{N}')$ and $\mathcal{G}_\theta(\mathcal{N})$ play an important role in the theory of Gaussian and non Gaussian analysis (Poisson, Lévy,...). In fact let

$$
X \subset H \subset X' \tag{13}
$$

be a real Fréchet nuclear triplet. Let γ be the standard Gaussian measure on $(X', \mathcal{B})$ where $\mathcal{B}$ is the σ-Borelian algebra on X', determined via the Bochner-Minlos theorem by the characteristic function:

$$
C(\xi) = \int_{X'} \exp i\langle x, \xi \rangle d\gamma(x) = \exp\Big(-\frac{1}{2}\|\xi\|_0^2\Big) \tag{14}
$$

and $\|\xi\|_0^2 = (\xi, \xi)_H$ is the Hilbertian norm in the space H. By complexification of the triplet (13) we obtain

$$
\mathcal{N} \subset Z \subset \mathcal{N}',
$$

where $\mathcal{N} = X + iX$ and $Z = H + iH$. Suppose that $\lim_{x\to+\infty} \frac{\theta(x)}{x^2} < \infty$. Then $\mathcal{F}_\theta(\mathcal{N}')$ can by densely topologically embedded in the Hilbert space $L^2(X', \gamma)$ and we can construct the following Gelfand Triplet

$$
\mathcal{F}_\theta(\mathcal{N}') \subset L^2(X', \gamma) \subset \mathcal{F}_\theta^*(\mathcal{N}'). \tag{15}
$$

3.1 S-Transform

Let θ be a Young function. Denote by $\mathcal{F}_\theta^*(\mathcal{N}')$ the strong dual of the test functions space $\mathcal{F}_\theta(\mathcal{N}')$. From condition (1) we deduce that for every $\xi \in \mathcal{N}$, the exponential function e_ξ defined by

$$e_\xi(z) = e^{\langle z,\xi\rangle}, \qquad z \in \mathcal{N}', \tag{16}$$

belongs to the space $\mathcal{F}_\theta(\mathcal{N}')$. The *Laplace transform* $\mathcal{L}$ of a distribution $\phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ is defined by

$$\mathcal{L}(\phi)(\xi) = \widehat{\phi}(\xi) = \langle\!\langle \phi, e_\xi \rangle\!\rangle, \qquad \xi \in \mathcal{N}. \tag{17}$$

By composition of the Taylor series map with the Laplace transform, we deduce that $\phi \in \mathcal{F}_\theta^*(\mathcal{N}')$ if and only if there exists a unique formal series $\vec{\phi} = (\phi_n)_{n\geq 0} \in G_\theta\ (\mathcal{N})$ such that

$$\widehat{\phi}(\xi) = \sum_{n\geq 0} \langle \xi^{\otimes n}, \phi_n \rangle.$$

Then, the action of the distribution ϕ on a test function $\varphi(z) = \sum_{n\geq 0}\langle z^{\otimes n}, \varphi_n\rangle$ is given by

$$\langle\!\langle \phi, \varphi \rangle\!\rangle = \sum_{n\geq 0} n! \langle \phi_n, \varphi_n \rangle. \tag{18}$$

In the White Noise Analysis we use the S-transform

$$S(\phi)(\xi) := \mathcal{L}\phi(\xi) \exp\Big(-\frac{1}{2}\xi^2\Big), \qquad \xi \in \mathcal{N}, \quad \phi \in \mathcal{F}_\theta^*(\mathcal{N}'). \tag{19}$$

Let now be given k nuclear gaussian spaces

$$(X_j \subset H_j \subset X_j', \gamma)$$

and $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be a multivariable Young function, i.e., $\theta_1, \theta_2, ..., \theta_k$ are k given Young functions and denote by

$$X = \prod_{1\leq j\leq k} X_j \quad \text{and} \quad \mathcal{N} = \prod_{1\leq j\leq k} \mathcal{N}_j,$$

where $\mathcal{N}_j = X_j + iX_j$ and $Z_j = H_j + iH_j$. Setting $\gamma^{\otimes k} = \gamma \otimes \gamma \otimes \cdots \otimes \gamma$ the k-fold tensor product of the standard gaussian measure. The next result gives a characterization of new Gelfand triplet.

Theorem 2 *If we suppose that for every* $1 \leq j \leq k$,

$$\lim_{x\to\infty} \frac{\theta_j(x)}{x^2} < \infty,$$

then $\mathcal{F}_\theta(\mathcal{N}')$ can be densely topologically embedded in the space $L^2(X', \gamma^{\otimes k})$, and we can construct the following Gelfand triplet,

$$\mathcal{F}_\theta(\mathcal{N}') \subset L^2(X', \gamma^{\otimes k}) \subset \mathcal{F}_\theta^*(\mathcal{N}').$$

Moreover the chaotic transform (S-Transform) realizes a topological isomorphism of nuclear triplets :

$$\begin{array}{ccc} \mathcal{F}_\theta(\mathcal{N}') \subset & L^2(X', \gamma^{\otimes k}) & \subset \mathcal{F}_\theta^*(\mathcal{N}') \\ \downarrow & \downarrow I_S & \downarrow S \\ \mathcal{F}_\theta(\mathcal{N}') \subset & Fock(Z^k) & \subset \mathcal{G}_{\theta^*}(\mathcal{N}) \end{array}$$

where I_S is the Wiener-Itô-Segal isometry and $Fock(Z^k)$ is the bosonic Fock space on Z^k and

$$\theta^* = (\theta_1, \theta_2, ..., \theta_k)^* = (\theta_1^*, \theta_2^*, ..., \theta_k^*).$$

3.2 Relation of this theorem with previous results

1. If $k = 1$ we obtain the results of [7]. In particular if $\theta(x) = \frac{x^\alpha}{\alpha}, \alpha > 1$ then $\theta^*(x) = \frac{x^{\alpha'}}{\alpha'}$ with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, and we obtain in this case the usual space of entire functions of exponential type, see e.g., [18], [19] and [20]. For every $\vec{f} \in F_\theta(\mathcal{N})$ we have:

$$\forall\, m, p \geq 0 : \quad ||\vec{f}||_{\theta,m,p}^2 = \Sigma(n!)^{2/\alpha} m^{-n} |f_n|_p^2 < \infty.$$

   ($\frac{2}{\alpha} = 1 + \beta$, in the notations of [12].) If $\alpha = 2$ and X is the Schwartz space $S(\mathbb{R})$, the space $F_{x^2}^*(S(\mathbb{R}))$ is the Hida distributions space [9].

2. The Potthoff-Streit characterization theorem, see [21], is a particular case of the general topological isomorphism: $\mathcal{F}_\theta^*(\mathcal{N}') \to \mathcal{G}_{\theta^*}(\mathcal{N})$ where $k = 1, \theta(t) = t^2$ and $X = S(\mathbb{R})$.

3. In the particular case where $k = 1$ and $\mathcal{N}$ is an arbitrarily Banach complex space B and $\theta(t) = t^\alpha$, $\alpha \geq 1$, the spaces $\mathcal{F}_\theta(\mathcal{N}'), F_\theta(\mathcal{N}), \mathcal{G}_\varphi(\mathcal{N}), G_\varphi(\mathcal{N}')$ are introduced first by the author in [17], and the analog of Theorems 1 is given in this case.

4. In [6] Cochran-Kuo-Sengupta introduce the "CKS" space of distributions $[\nu]_\alpha^*$ where $\alpha = (\alpha_n)_{n\in\mathbf{N}}$ is a positive sequence and

$$G_\alpha(t) = \sum_{n\geq 0} \alpha(n) \frac{t^n}{n!}$$

   is an analytic function. If we put $\theta^*(t) = \operatorname{Log}(G_\alpha(t^2))$ then $[\nu]_\alpha^* = F_\theta'(\mathcal{N})$. The hypothesis of the analycity of the function $G_\alpha(t)$ in [6] is not necessary in our case, moreover we here obtain explicitly the space test functions and also a characterization theorem for this space.

4 Convolution Calculus

In the next we develop a new convolution calculus over generalized functionals space $\mathcal{F}^*_\theta(\mathcal{N}')$. Unlike the Wick Calculus studied by many authors, see [9], [14], [15], [13] and [20], the convolution calculus is developed independently of the Gaussian Analysis. In fact for $\phi \in \mathcal{F}^*_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ the convolution of ϕ and φ is defined by

$$(\phi * \varphi)(z) := \langle\langle \phi, \tau_{-z}\varphi \rangle\rangle, \qquad z \in \mathcal{N}', \tag{20}$$

where τ_{-z} is the translation operator, i.e., $\tau_{-z}\varphi(x) = \varphi(z+x)$, $x \in \mathcal{N}'$ and for every $z \in \mathcal{N}'$, the linear operator τ_{-z} is continuous from $\mathcal{F}_\theta(\mathcal{N}')$ into itself. A direct calculation shows that $\phi * \varphi \in \mathcal{F}_\theta(\mathcal{N}')$. Let $\phi_1, \phi_2 \in \mathcal{F}^*_\theta(\mathcal{N}')$, we define the convolution product of ϕ_1 and ϕ_2, denoted by $\phi_1 * \phi_2$, by

$$\langle\langle \phi_1 * \phi_2, \varphi \rangle\rangle := [\phi_1 * (\phi_2 * \varphi)](0), \qquad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

Moreover, $\forall\ \phi_1, \phi_2 \in \mathcal{F}^*_\theta(\mathcal{N}')$ we have

$$\widehat{\phi_1 * \phi_2} = \widehat{\phi}_1\widehat{\phi}_2. \tag{21}$$

4.1 Convolution operators

In infinite dimensional complex analysis, a convolution operator on the test space $\mathcal{F}_\theta(\mathcal{N}')$ denoted for simplicity by $\mathcal{F}_\theta$ is a continuous linear operator from $\mathcal{F}_\theta$ into itself which commutes with translation operators. It was proved in [3] and [8] that T is a convolution operator on $\mathcal{F}_\theta$ if and only if there exists $\phi_T \in \mathcal{F}^*_\theta$ such that

$$T\varphi = \phi_T * \varphi\ , \quad \forall\ \varphi \in \mathcal{F}_\theta. \tag{22}$$

Moreover, if the distribution ϕ_T is given by $\vec{\phi}_T = (\phi_m)_{m\geq 0} \in G_\theta$ and $\varphi(z) = \sum_{n\geq 0}\langle z^{\otimes n}, \varphi_n\rangle \in \mathcal{F}_\theta$ then

$$\phi_T * \varphi(z) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(n+m)!}{n!}\langle z^{\otimes n}, \langle \phi_m, \varphi_{m+n}\rangle_m\rangle. \tag{23}$$

where $\langle \phi_m, \varphi_{m+n}\rangle_m$ denotes the right contraction of ϕ_m and φ_{m+n} of order m, see [14]. In particular, we have

$$T(e_\xi)(z) = \phi_T * e_\xi(z) = \widehat{\phi}(\xi)e_\xi(z).$$

Let θ be a Young function, $y \in \mathcal{N}'$ and $\varphi(z) = \sum_{n\geq 0}\langle z^{\otimes n}, \varphi_n\rangle \in \mathcal{F}_\theta$. We define the holomorphic derivative of φ at the point $z \in \mathcal{N}'$ in a direction y by

$$D_y\varphi(z) := \sum_{n\geq 0}(n+1)\langle z^{\otimes n}, \langle y, \varphi_{n+1}\rangle_1\rangle.$$

Lemma 3 *The operator D_y is continuous from $\mathcal{F}_\theta$ into itself. Moreover, for every $\varphi \in \mathcal{F}_\theta$, $p \in \mathbb{N}$ and $m > 0$ we have*

$$\|\overrightarrow{D_y\varphi}\|_{\theta,p,m} \leq \sqrt{m}\,\theta_1 |y|_{-p_y}\|\vec{\varphi}\|_{\theta,p_y\vee p,\frac{m}{16}},$$

where $p_y = \min\{p \in \mathbb{N}, y \in N_{-p}\}$ and $p_y \vee p = \max(p_y, p)$.

PROOF. By definition of the norm $\|.\|_{\theta,p,m}$ defined on the space F_θ of formal series, we have

$$\begin{aligned}
\|\overrightarrow{D_y\varphi}\|_{\theta,p,m} &= \left(\sum_{n\geq 0}(n+1)^2\theta_n^{-2}m^{-n}|\langle y,\varphi_{n+1}\rangle_1|_p^2\right)^{\frac{1}{2}}\\
&\leq |y|_{-p_y}\left(\sum_{n\geq 0}(n+1)^2\theta_n^{-2}m^{-n}|\varphi_{n+1}|_{p\vee p_y}^2\right)^{\frac{1}{2}}\\
&\leq \sqrt{m}\,|y|_{-p_y}\left(\sum_{n\geq 0}\theta_{n+1}^{-2}\left(\frac{m}{16}\right)^{-n-1}|\varphi_{n+1}|_{p\vee p_y}^2\left[\frac{(n+1)\theta_{n+1}}{2^{2n+2}\theta_n}\right]^2\right)^{\frac{1}{2}}\\
&\leq \sqrt{m}\,|y|\sup_{n\geq 1}\left[\frac{\theta_{n+1}}{2^{n+1}\theta_n}\right]\|\vec{\varphi}\|_{\theta,p\vee p_y,\frac{m}{16}}.
\end{aligned}$$

Finally, the desired inequality follows immediately using the fact that $2^{-l-k}\theta_l\theta_k\leq\theta_{l+k}\leq 2^{l+k}\theta_l\theta_k$, $\forall\, l,k\in\mathbb{N}\setminus\{0\}$. ■

For each $m\in\mathbb{N}$ the m-linear operator $D: N'\times\cdots\times N'\to\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$ defined by

$$(y_1,\ldots,y_m)\mapsto D_{y_1}\ldots D_{y_m}$$

is symmetric and continuous, hence it can be continuously extended to $\mathcal{N}'^{\widehat{\otimes}m}$, i.e., $D:\phi_m\in\mathcal{N}'^{\widehat{\otimes}m}\mapsto D_{\phi_m}\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$. The action of the operator D_{ϕ_m} on a test function $\varphi(z)=\sum_{n\geq 0}\langle z^{\otimes n},\varphi_n\rangle$ is given by

$$D_{\phi_m}(\varphi)(z)=\sum_{n\geq 0}\frac{(n+m)!}{n!}\langle z^{\otimes n},\langle\phi_m,\varphi_{n+m}\rangle_m\rangle. \tag{24}$$

Then, in view of (22), (23) and (24), we give an expansion of convolution operators in terms of holomorphic derivation operators.

Proposition 4 *Let $T\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$. Then T is a convolution operator if and only if there exists $\vec{\phi}=(\phi_m)_{m\geq 0}\in G_\theta$ such that*

$$T=\sum_{m\geq 0}D_{\phi_m}.$$

Let $T_\phi=\sum_{m\geq 0}D_{\phi_m}$ be a convolution operator and $n\in\mathbb{N}$. Then equality (22) shows that

$$T_\phi^n:=\underbrace{T_\phi\circ\ldots\circ T_\phi}_{n}=T_{\phi^{*n}}. \tag{25}$$

In particular,

$$T_\phi^n(e_\xi)(z)=T_{\phi^{*n}}(e_\xi)(z)=\left(\widehat{\phi}(\xi)\right)^n e_\xi(z),\quad z\in\mathcal{N}',\quad \xi\in\mathcal{N}.$$

4.2 Symbols of operators

We denote by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ the space of continuous linear operators from $\mathcal{F}_\theta$ into itself, equipped with the topology of bounded convergence. In this section we define the symbol map on the space $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$. Then we give an expansion of such operators in terms of multiplication and derivation operators.

Definition 5 *Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$, the symbol $\sigma(T)$ of the operator T is a $\mathbb{C}$-valued function defined by*

$$\sigma(T)(z,\xi) := e^{-\langle z,\xi\rangle} T(e_\xi)(z), \qquad z \in \mathcal{N}' \,, \;\; \xi \in \mathcal{N}.$$

Similar definitions of symbols have been introduced in various contexts, see [10], [11], [14], [15], and [19]. In the general theory [22], if we take two nuclear Fréchet spaces $\mathcal{X}$ and $\mathcal{D}$ then the canonical correspondence $T \longleftrightarrow K^T$ given by

$$\langle Tu, v\rangle \;=\; \langle K^T, u\otimes v\rangle, \quad u \in \mathcal{X}, \quad v \in \mathcal{D}',$$

yields a topological isomorphism between the spaces $\mathcal{L}(\mathcal{X},\mathcal{D})$ and $\mathcal{X}'\widehat{\otimes}\mathcal{D}$. In particular if we take $\mathcal{X} = \mathcal{D} = \mathcal{F}_\theta$ which is a nuclear Fréchet space, then we get

$$\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta) \cong \mathcal{F}_\theta^* \widehat{\otimes} \mathcal{F}_\theta. \tag{26}$$

So, the symbol $\sigma(T)$ of an operator T can be regarded as the Laplace transform of the kernel K^T

$$\sigma(T)(z,\xi) = K^T(e_\xi \otimes \delta_z), \qquad z \in N', \quad \xi \in \mathcal{N}. \tag{27}$$

Moreover, with the help of equalities (12), (26), (27) and Theorem 1, we obtain the following theorem.

Theorem 6 *The symbol map yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ and $\mathcal{F}_\theta \widehat{\otimes} \mathcal{G}_\theta^*$. More precisely, we have the following isomorphisms*

$$\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta) \quad \xrightarrow{\sigma} \quad \mathcal{F}_\theta \widehat{\otimes} \mathcal{G}_\theta^* \quad \xrightarrow{S.T} \quad F_\theta \widehat{\otimes} G_\theta,$$

$$T \mapsto \sigma(T)(z,\xi) = \sum_{l,m\geq 0} \langle K_{l,m}, z^{\otimes l} \otimes \xi^{\otimes m}\rangle \mapsto \overrightarrow{K} = (K_{l,m})_{l,m\geq 0}.$$

Example 7 1) The symbol of a convolution operator $T_\phi = \sum_{m\geq 0} D_{\phi_m}$ is given by

$$\sigma(T_\phi)(z,\xi) = e^{-\langle z,\xi\rangle} \sum_{m\geq 0} D_{\phi_m}(e_\xi)(z) = \sum_{m\geq 0} \langle \phi_m, \xi^{\otimes m}\rangle = \widehat{\phi}(\xi).$$

Hence, the operator T_ϕ can be expressed in an obvious way by

$$T_\phi = \sum_{m\geq 0} D_{\phi_m} := \sum_{m\geq 0} \langle \phi_m, D^{\otimes m}\rangle = \sigma(T_\phi)(z, D), \qquad z \in \mathcal{N}'.$$

2) If we denote by M_f the multiplication operator by the test function f, its symbol is given by

$$\sigma(M_f)(z,\xi) = e^{-\langle z,\xi\rangle}(fe_\xi)(z) = e^{-\langle z,\xi\rangle}f(z)e_\xi(z) = f(z).$$

By the same argument the multiplication operator is also expressed by $M_f = \sigma(M_f)(z,D)$. We note that the symbol of a convolution (resp. multiplication) operator $\sigma(T)(z,\xi)$ depends only on ξ (resp. z).

Let $\overrightarrow{K} \in F_\theta \widehat{\otimes} G_\theta$ and assume that $\overrightarrow{K} = \vec{f} \otimes \vec{\phi} = (f_l \otimes \phi_m)_{l,m\geq 0}$. Then the operator T associated to $\overrightarrow{K}$ satisfies

$$T = M_f T_\phi, \tag{28}$$

where $f(z) = \sum_{l\geq 0}\langle z^{\otimes l}, f_l\rangle$ and T_ϕ is the convolution operator associated to the distribution ϕ given by $\vec{\phi}$. Moreover, we have

$$T = M_f T_\phi = \sigma(M_f)(z,D)\sigma(T_\phi)(z,D) = \sigma(T)(z,D).$$

Thus, using the density of $F_\theta \otimes G_\theta$ in $F_\theta \widehat{\otimes} G_\theta$, we obtain the following result.

Proposition 8 *The vector space generated by operators of type (28) is dense in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$.*

4.3 Convolution product of operators

Let T_1, T_2 two operators in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$; the convolution product of T_1 and T_2, denoted by $T_1 * T_2$, is uniquely determined by

$$\sigma(T_1 * T_2) = \sigma(T_1)\sigma(T_2).$$

If the operators T_1 and T_2 are of type (28), *i.e.*, $T_1 = M_{f_1}T_{\phi_1}$ and $T_2 = M_{f_2}T_{\phi_2}$, then

$$T_1 * T_2 = M_{f_1 f_2} T_{\phi_1 * \phi_2}.$$

In particular, if $T = M_f T_\phi$, then for every $n \in \mathbb{N}$ we have

$$T^{*n} = M_{f^n} T_{\phi^{*n}}. \tag{29}$$

Let T_ϕ (resp. M_f) be a convolution (resp. multiplication) operator. Then for every $n \in \mathbb{N}$

$$T_\phi^{*n} = T_{\phi^{*n}} = T_\phi^n \ \text{ and } \ M_f^{*n} = M_{f^n} = M_f^n.$$

Lemma 9 *Let γ_1, γ_2 two Young functions and (F_n) a sequence belonging to $\mathcal{F}_{\gamma_1} \widehat{\otimes} \mathcal{G}_{\gamma_2}$. Then (F_n) converges in $\mathcal{F}_{\gamma_1} \widehat{\otimes} \mathcal{G}_{\gamma_2}$ if and only if*

1. *(F_n) is bounded in $\mathcal{F}_{\gamma_1} \widehat{\otimes} \mathcal{G}_{\gamma_2}$.*

2. *(F_n) converges simply.*

PROOF. The proof is based on the nuclearity of the spaces $\mathcal{F}_\theta$ and $\mathcal{G}_{\theta^*}$. A similar proof is established with more details in [4]. ■

Proposition 10 *Let $T \in \mathcal{L}_\theta$. Then the operator $e^{*T} := \sum_{n\geq 0} \frac{T^{*n}}{n!}$ belongs to $\mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}, \mathcal{F}_{e^\theta})$*

PROOF. Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ and put $S_n = \sum_{k=0}^n \frac{T^{*k}}{k!}$. Then using Lemma 9 we show that $\sigma(S_n)$ converges in $\mathcal{F}_{e^\theta}\widehat{\otimes}\mathcal{G}_{e^{\theta^*}}$ to $e^{\sigma(T)}$, from which the assertion follows. ■

Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ and consider the linear differential equation

$$\frac{dE}{dt} = TE\,, \quad E(0) = I.$$

Then the solution is given informally by : $E(t) = e^{tT}$, $t \in I\!R$. In the particular case, where T is a convolution or a multiplication operator; the solution $E(t) = e^{tT}$ is well defined since $e^T = e^{*T}$. If T is not a convolution or a multiplication operator then the following theorem gives a sufficient condition on T to insure the existence of its exponential e^T.

Theorem 11 *Let $\overrightarrow{K} = (K_{l,m}) \in F_\theta\widehat{\otimes}G_\theta$ satisfying $\langle K_{l,m}, K_{l',m'}\rangle_k = 0$ for every $m, l' \geq 1$, $m', l \geq 0$ and $1 \leq k \leq m \wedge l'$ and denote by T the operator associated to $\overrightarrow{K}$. Then*

$$T^n = T^{*n}, \ \forall n \in \mathbb{N}.$$

Moreover, $e^T = e^{*T} \in \mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}, \mathcal{F}_{e^\theta})$.

PROOF. Using Proposition 8, it will be sufficient to assume that $K_{l,m} = (f_l \otimes \phi_m)$, *i.e.*,

$$T = M_f T_\phi = \sum_{l,m\geq 0} M_{f_l} D_{\phi_m},$$

where $f_l(z) = \langle z^{\otimes l}, f_l\rangle$. Assume that $f_l = \eta^{\otimes l}$, $\eta \in \mathcal{N}$ and $\phi_m = y^{\otimes m}$, $y \in \mathcal{N}'$. Then it is easy to see that

$$D_{\phi_m} M_{f_l} = M_{f_l} D_{\phi_m} + \sum_{k=1}^{m\wedge l} k! C_l^k C_m^k \langle y, \eta\rangle^k M_{f_{l-k}} D_{\phi_{m-k}},$$

an equality on $\mathcal{F}_\theta$. The assumption $\langle K_{l,m}, K_{l',m'}\rangle_k = 0$ implies that $\langle y, \eta\rangle = 0$. Then

$$D_{\phi_m} M_{f_l} = M_{f_l} D_{\phi_m}. \tag{30}$$

Thus, using the density of the vector space generated by $\{\eta^{\otimes l},\ \eta \in N\}$ in the space $N^{\odot l}$ and the density of the vector space generated by $\{y^{\otimes m},\ y \in N'\}$ in $N'^{\odot m}$, we can extend equality (30) to every $f_l \in \mathcal{N}^{\odot l}$ and $\phi_m \in \mathcal{N}'^{\odot m}$ such that $\langle \phi_m, f_l\rangle_k = 0$, $\forall\ 1 \leq k \leq l \wedge m$. Hence, we obtain

$$M_f T_\phi = \sum_{l,m\geq 0} M_{f_l} D_{\phi_m} = \sum_{l,m\geq 0} D_{\phi_m} M_{f_l} = T_\phi M_f.$$

Using equalities (25) and (29), for every $n \in \mathbb{N}$ we have

$$T^n = (M_f T_\phi)^n = (M_f)^n (T_\phi)^n = M_{f^n} T_{\phi^{*n}} = T^{*n}.$$

This completes the proof. ■

5 Applications to Quantum Stochastic Differential Equations

A one parameter quantum stochastic process with values in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ is a family of operators $\{E_t,\ t \in [0,T]\} \subset \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ such that the map $t \mapsto E_t$ is continuous. For such a quantum process E_t we set

$$E_n = \frac{t}{n}\sum_{k=0}^{n-1} E_{\frac{tk}{n}}, \qquad n \in \mathbb{N}\backslash\{0\}, \quad t \in [0,T].$$

Then we prove using Lemma 9 that the sequence (E_n) converges in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$. We denote its limit by

$$\int_0^t E_s ds := \lim_{n \to +\infty} E_n \ \text{ in } \ \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta).$$

Moreover, we have

$$\sigma\left(\int_0^t E_s ds\right) = \int_0^t \sigma(E_s) ds, \qquad \forall t \in [0,T].$$

Theorem 12 *Let $t \in [0,T] \mapsto f(t) \in \mathcal{F}_\theta$ and $t \in [0,T] \mapsto \phi(t) \in \mathcal{F}_\theta^*$ be two continuous processes and set $L_t = M_{f(t)} T_{\phi(t)}$. Then the linear differential equation*

$$\frac{dE_t}{dt} = M_{f(t)} E_t T_{\phi(t)}, \qquad E_0 = I, \tag{31}$$

has a unique solution $E_t \in \mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}, \mathcal{F}_{e^\theta})$ given by

$$E_t = e^{*\left(\int_0^t L_s ds\right)}.$$

Proof. Applying the symbol map to equation (31) to get

$$\frac{d\sigma(E_t)}{dt} = \sigma(L_t)\sigma(E_t)\ ,\ \ \sigma(I) = 1.$$

Then $\sigma(E_t) = e^{\int_0^t \sigma(L_s) ds}$ which is equivalent to $E_t = e^{*\left(\int_0^t L_s ds\right)}$. Finally, we conclude by Proposition 10 that $E_t \in \mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}, \mathcal{F}_{e^\theta})$. ∎

Theorem 13 *Let L_t be a quantum stochastic process with values in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ such that*

$$\sigma\left(\int_0^t L_s ds\right)(z,\xi) = \sum_{l,m \geq 0} \langle K_{l,m}(t), z^{\otimes l} \otimes \eta^{\otimes m}\rangle,$$

and assume that for every $t \in [0,T]$, $m', l \geq 0$ and $m, l' \geq 1$ we have $\langle K_{l,m}(t), K_{l',m'}(t)\rangle_k = 0$, $\forall 1 \leq k \leq m \wedge l'$. Then the following differential equation

$$\frac{dE}{dt} = L_t E, \qquad E(0) = I,$$

has a unique solution in $\mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}, \mathcal{F}_{e^\theta})$ given by

$$E(t) = e^{\int_0^t L_s ds}.$$

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